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NEW INEQUALITIES FOR THE FUNCTION $y = t \ln t$

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Abstract. The main aim of this note, which can be viewed as a certain addendum to the paper [1], is to propose several new inequalities for the function $y = t \ln t$. We consider the local behaviour of this function near the point $t = 1$, as well as the global behaviour of this function on the intervals $[1, \infty)$ and $(0, 1]$.

1. Introduction

The reading of paper [1] by C. Chesneau and Y. J. Bagul has strongly infuenced us to write this note. In Theorem 2, we give new abstract local bounds for the function $y = t \ln t$ near the point $t = 1$. The obtained inequalities can be used to improve the main results of paper [1], Proposition 1 and Proposition 2. We also present an interesting result with regards to these propositions, which claims that there is no rational real function which intermediates the functions $\ln(1 + x)$ and $f(x)/\sqrt{x + 1}$ for $x \geq 0$ ($x \in (-1, 0]$); here and hereafter,

$$f(x) := \pi + \frac{1}{2}(4 + \pi)x - 2(x + 2) \arctan \sqrt{x + 1}, \quad x \geq -1.$$

The following inequalities are well known (see also [3, Problem 3.6.19, p. 274] and [4]):

$$\ln(1 + x) \leq \frac{x}{\sqrt{x + 1}}, \quad x \geq 0, \quad \ln(1 + x) \leq \frac{x(2 + x)}{2(1 + x)}, \quad x \geq 0, \tag{1}$$

$$\ln(1 + x) \leq \frac{x(6 + x)}{2(3 + 2x)}, \quad x \geq 0 \quad \text{and} \quad \ln(1 + x) \leq \frac{(x + 2)\left[(x + 1)^3 - 1\right]}{3(1 + x)\left[(x + 1)^2 + 1\right]}, \quad x \geq 0. \tag{2}$$

Taken together, the first inequality in (1) and the second inequality in (2) are known in the existing literature as Karamata's inequality [2]. As clarified in [1], all these inequalities are weaker than the inequality:

$$\ln(1 + x) \leq \frac{f(x)}{\sqrt{x + 1}}, \quad x \geq 0. \tag{3}$$

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This inequality has been proved in [1, Proposition 1]. In [1, Proposition 2], the authors have proved that

$$\ln(1+x) \geq \frac{f(x)}{\sqrt{x+1}}, \quad x \in (-1,0], \tag{4}$$

as well.

Our approach leans heavily on the use of substitution $t = \sqrt{x+1}$. Then the inequalities (3) and (4) become

$$2\ln t \leq \frac{f\big(t^2-1\big)}{t}, \quad t \geq 1 \text{ and } 2\ln t \geq \frac{f\big(t^2-1\big)}{t}, \quad t \in (0,1],$$

i.e.,

$$2t\ln t \leq f\big(t^2-1\big), \quad t \geq 1 \text{ and } 2t\ln t \geq f\big(t^2-1\big), \quad t \in (0,1]. \tag{5}$$

We can prove (5) in the following way. Notice that

$$\left[\ln t - \left(\frac{1}{2}(4+\pi)t - 2t\arctan t - 2\right)\right]''(t) = -\big(t^2-1\big)^2 t^{-2}\big(t^2+1\big)^{-2}, \quad t > 0.$$

Using an elementary argumentation, this estimate implies

$$\ln t \leq \frac{1}{2}(4+\pi)t - 2t\arctan t - 2, \quad t > 0.$$

Define $R(t) := 2t\ln t - f(t^2-1)$, $t > 0$. Since $R'(t) = 2(1+\ln t) - (4+\pi)t + 4t\arctan t + 2$, $t > 0$, the previous inequality yields $R'(t) \leq 0$, $t > 0$ and (5). Moreover, by taking the limit of function $R(\cdot)$ as $t \to 0+$, we get that $2t\ln t - f(t^2-1) \in (2-(\pi/2), 0]$ for $t \in (0,1]$.

In this paper, we will first generalize the inequalities in (5) by considering the local behaviour of the function $y = t\ln t$ near the point $t = 1$. We will use the following simple lemmae, which is known from the elementary courses of mathematical analysis:

Lemma 1 Suppose $t_0 \in \mathbb{R}$, $a > 0$, $n \in \mathbb{N}$ and function $f : (t_0 - a, t_0 + a) \to \mathbb{R}$ is $2n$-times differentiable. If $f^{(i)}(t_0) = 0$ for all $i = 1, \cdots, 2n-1$ and $f^{(2n)}(t_0) > 0$ ($f^{(2n)}(t_0) < 0$), then the function $y = f(t)$ has a local minimum (maximum) at $t = t_0$.

Lemma 2 We have

$$(\arctan x)^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(1+x^2)^{n/2}} \sin(n\pi/2 - n\arctan x), \quad x \in \mathbb{R},\ n \in \mathbb{N}.$$

After that, we will prove the following result with regards to [1, Proposition 1, Proposition 2]:

Theorem 1

(i) There do not exist real polynomials $P(\cdot)$ and $Q(\cdot)$ such that $Q(x) \neq 0$ for $x \geq 0$ and

$$\ln(1+x) \leq \frac{P(x)}{Q(x)} \leq \frac{f(x)}{\sqrt{x+1}}, \quad x \geq 0. \tag{6}$$

(ii) There do not exist real polynomials $P(\cdot)$ and $Q(\cdot)$ such that $Q(x) \neq 0$ for $x \in (-1,0]$ and

$$\ln(1+x) \geq \frac{P(x)}{Q(x)} \geq \frac{f(x)}{\sqrt{x+1}}, \quad x \in (-1,0]. \tag{7}$$

2. The main results and their proofs

We start this section by stating the following result:

Theorem 2 Suppose that $a \in (0,1)$, $P : (1-a, 1+a) \to \mathbb{R}$ is a function and $P(1) = 0$. Then the following holds:

(i) If $P'(1) \geq 2$ and there exists an odd natural number n such that $P(\cdot)$ is $(n+2)$-times differentiable, $P^{(n+2)}(1) + 2(-1)^{n+1} n! > 0$ and

$$P^{(j)}(1) + 2(-1)^{j+1}(j-2)! = 0 \quad \text{for all} \quad j = 2, 3, \cdots, n+1,$$

then there exists a real number $\zeta \in (0, a]$ such that

$$2t \ln t \leq P(t), \quad t \in [1, 1+\zeta] \quad \text{and} \quad 2t \ln t \geq P(t), \quad t \in [1-\zeta, 1]. \tag{8}$$

(ii) Assume that there exists an even natural number $n \geq 6$ such that $P(\cdot)$ is $(n+1)$-times differentiable, $P^{(n+1)}(1) + 2(-1)^n (n-1)! > 0$ and

$$P^{(j)}(1) + 2(-1)^{j+1}(j-2)! = 0 \quad \text{for all} \quad j = 1, 2, \cdots, n.$$

Then there exists a real number $\eta \in (0, a]$ such that

$$2t \ln t \leq P(t) \leq f\big(t^2 - 1\big), \quad t \in [1, 1+\eta]$$
$$\text{and} \quad 2t \ln t \geq P(t) \geq f\big(t^2 - 1\big), \quad t \in [1-\eta, 1]. \tag{9}$$

(iii) Assume that there exists an even natural number $n \geq 6$ such that $P(\cdot)$ is $(n+1)$-times differentiable,

$$P^{(j)}(1) + 2(-1)^{j+1}(j-2)! = 0 \quad \text{for all} \quad j = 1, 2, 3, 4, \tag{10}$$

$$P^{(n+1)}(1) + 4\left[\frac{(-1)^n n!}{2^{(n+1)/2}} \sin((n+1)\pi/4) + \frac{(-1)^{n+1} n!}{2^{n/2}} \sin(n\pi/4)\right] < 0$$

and, for every $j = 5, 6, \cdots, n$,

$$P^{(j)}(1) + 4\left[\frac{(-1)^{j-1}(j-1)!}{2^{j/2}} \sin(j\pi/4) + \frac{(-1)^j (j-1)!}{2^{(j-1)/2}} \sin((j-1)\pi/4)\right] = 0. \tag{11}$$

Then there exists a real number $\eta \in (0, a]$ such that (9) holds.

(iv) If $P(\cdot)$ is five times differentiable, (10) holds and $P^{(v)}(1) \in (-12, -8)$, then there exists a real number $\eta \in (0, a]$ such that (9) holds.

Proof. Define $G(t) := P(t) - 2t \ln t$, $t > 0$. Then, for every real number $t > 0$, we have $G'(t) = P'(t) - 2(1 + \ln t)$, $G''(t) = P''(t) - (2/t)$ and $G^{(n)}(t) = P^{(n)}(t) + 2(-1)^{n+1}(n-2)! \cdot t^{1-n}$, $n \geq 3$. The assumptions made in (i) imply that $G'(1) \geq 0$, $(G')^{(j)}(1) = 0$ for $1 \leq j \leq n$ and $(G')^{(n+1)}(1) > 0$. Applying Lemma 1, we get that the function $t \mapsto G'(t)$ has a local minimum at $t = 1$. Since $G'(1) \geq 0$, we get that the function $t \mapsto G'(t)$ is non-negative in an open neighborhood of point $t = 1$, so that the mapping $t \mapsto G(t)$ is increasing in an open neighborhood of point $t = 1$. This finishes the proof of (i). For the proof of (ii), define $Q(t) := P(t) - f(t^2 - 1)$, $t > 0$. Then a simple computation yields that, for every real number $t > 0$, we have $Q'(t) = P'(t) - (4+\pi)t + 4t \arctan t + 2$ and $Q''(t) = P''(t) - (4+\pi) + 4 \arctan t + \frac{4t}{t^2+1}$.

Using Leibniz rule and Lemma 2, for every real number $t > 0$ and for every natural number $n \geq 3$, we can show that

$$\begin{aligned} Q^{(n)}(t) &= P^{(n)}(t) + 4\big[\cdot \arctan \cdot\big]^{(n-1)}(t) \\ &= P^{(n)}(t) + 4\Bigg[t\frac{(-1)^{n-1}(n-1)!}{(1+t^2)^{n/2}}\sin(n\pi/2 - n\arctan t) \\ &+ \frac{(-1)^n(n-1)!}{(1+t^2)^{(n-1)/2}}\sin((n-1)\pi/2 - (n-1)\arctan t)\Bigg]. \end{aligned}$$

Arguing as in the proof of (i), we have that $(Q')^{(j)}(1) = 0$ for $j = 0, 1, 2, 3$ and $(Q')^{(4)}(1) < 0$; hence, the function $t \mapsto Q'(t)$ has a local maximum at $t = 1$ and the mapping $t \mapsto Q(t)$ is decreasing in an open neighborhood of point $t = 1$. Similarly, $(G')^{(j)}(1) = 0$ for $j = 0, 1, 2, \cdot\cdot\cdot, n-1$ and $(G')^{(n)}(1) > 0$; hence, the function $t \mapsto G'(t)$ has a local minimum at $t = 1$ and the mapping $t \mapsto G(t)$ is increasing in an open neighborhood of point $t = 1$. This completes the proof of (ii). The proof of (iii) can be deduced similarly, by interchanging the roles of $G(t)$ and $Q(t)$. If the assumptions of (iv) holds, then we can apply Lemma 1, with $n = 2$, in order to see that the function $t \mapsto G'(t)$ has a local minimum at $t = 1$, as well as the function $t \mapsto G'(t)$ is non-negative in an open neighborhood of point $t = 1$; hence, the mapping $t \mapsto G(t)$ is increasing in an open neighborhood of point $t = 1$. Similarly, we can show that the mapping $t \mapsto Q(t)$ is decreasing in an open neighborhood of point $t = 1$. The proof of the theorem is thereby complete.

Remark 1 Define $H(t) := f(t^2 - 1)$, $t \in \mathbb{R}$. Concerning the conditions used in Theorem 2, it is worth noting that the function $H(\cdot)$ satisfies $H(1) = 0$, $H'(1) = H''(1) = 2$, $H'''(1) = -2$, $H^{(iv)}(1) = 4$ and $H^{(v)}(1) = -8$. This implies that the values of terms appearing at the right hand sides of (10) and (11) coincide for $j = 1, 2, 3, 4$ and differ for $j = 5$ (observe that $G^{(v)}(1) = P^{(v)}(1) + 12$).

Remark 2 The parts (ii)-(iv) of Theorem 2 ensure the existence of a large class of elementary functions for which we can further refine the inequalities in (5) locally around the point $t = 1$. Compared with the function $H(\cdot)$, the most simplest example of function which provides a better estimate describing the local behaviour of function $y = t\ln t$ around the point $t = 1$ is given by the function $t \mapsto H(t) - \epsilon(t-1)^5$, $t > 0$, where $\epsilon \in (0, 1/30)$.

Concerning the global behaviour of function $y = t\ln t$, $t > 0$, it is clear that the inequalities in (5) give some very uninteresting estimates with regards to the asymptotic behaviour of function $y = t\ln t$ when $t \to +\infty$ or $t \to 0+$; on the other hand, the importance of estimate (5) lies in the fact that it gives some bounds for the behaviour of function $y = t\ln t$ on any compact interval $[a, b]$, where $0 < a < 1 < b$. It is clear that there exists a large class of infinitely differentiable functions $P : (0, \infty) \to \mathbb{R}$ such that

$$\begin{aligned} 2t\ln t \leq P(t) \leq f\big(t^2 - 1\big), \quad t \geq 1 \\ \text{and} \quad 2t\ln t \geq P(t) \geq f\big(t^2 - 1\big), \quad t \in (0, 1]. \end{aligned} \tag{12}$$

Finding new elementary functions $P(\cdot)$ for which the equation (12) holds is without scope of this paper.

We close the paper by giving the proof of Theorem 1:

Proof of Theorem 1. Suppose that (6) holds for some real polynomials $P(\cdot)$ and $Q(\cdot)$ such that $Q(x) \neq 0$ for $x \geq 0$. Without loss of generality, we may assume that $Q(x) > 0$, $x \geq 0$. Using the substitution $t = \sqrt{x+1}$, we get that

$$2 \ln t \leq \frac{P(t^2-1)}{Q(t^2-1)} \leq \frac{f(t^2-1)}{t}, \quad t \geq 1.$$

If $P(t) = \sum_{j=0}^{n} a_j t^j$ and $Q(t) = \sum_{j=0}^{m} b_j t^j$ for some non-negative integers $m,\ n$ and some real numbers $a_j,\ b_j$ ($a_n b_m \neq 0$; clearly, we cannot have $P(x) \equiv 0$), we get

$$t \sum_{j=0}^{n} a_j \left(t^2-1\right)^j \leq f\left(t^2-1\right) \sum_{j=0}^{m} b_j \left(t^2-1\right)^j, \quad t \geq 1 \tag{13}$$

and

$$\sum_{j=0}^{n} a_j \left(t^2-1\right)^j \geq 2 \ln t \sum_{j=0}^{m} b_j \left(t^2-1\right)^j, \quad t \geq 1. \tag{14}$$

Since $f(t^2-1) \sim (2-(\pi/2))t^2$, $t \to +\infty$, the estimate (13) implies $n \leq m$. The positivity of polynomial $Q(\cdot)$ on the non-negative real axis implies $b_m > 0$ so that (14) gives $a_n > 0$. Considering the asymptotic behaviour of terms appearing in (14), we get that the inequality $n < m$ cannot be satisfied so that $m = n$. Dividing the both sides of (14) with t^{2n} and letting $t \to +\infty$ in the obtained expression, we get that $a_n/2b_n \geq +\infty$, which is a contradiction. This completes the proof of (i). To prove (ii), suppose that the estimates

$$\ln(1+x) \geq \frac{P_0(x)}{Q_0(x)} \geq \frac{f(x)}{\sqrt{x+1}}, \quad x \in (-1, 0]$$

hold for some real polynomials $P_0(\cdot)$ and $Q_0(\cdot)$ such that $Q_0(x) \neq 0$ for $x \in (-1, 0]$. Then (7) holds for some real polynomials $P(\cdot)$ and $Q(\cdot)$ such that $Q(x) > 0$ for $x \in (-1, 0]$. Letting $x \to -1-$ in (7), we get that $Q(-1) = 0$. If $P(x) = \sum_{j=0}^{n} a_j x^j$ and $Q(x) = \sum_{j=0}^{m} b_j x^j$ for some non-negative integers $m,\ n$ and some real numbers $a_j,\ b_j$ ($a_n b_m \neq 0$; again, we cannot have $P(x) \equiv 0$), this implies

$$\ln(1+x) \cdot \sum_{j=0}^{m} b_j x^j \geq \sum_{j=0}^{n} a_j x^j \geq \frac{f(x)}{\sqrt{x+1}} \sum_{j=0}^{m} b_j x^j, \quad x \in (-1, 0]. \tag{15}$$

Letting $x \to 0-$ in this expression, we get that $a_0 = 0$ so that $n \geq 1$ and $x|P(x)$. Define $P_1(x) := P(x)/x$ and $Q_1(x) := Q(x)/(x+1)$. Then $P_1(x)$ and $Q_1(x)$ are real polynomials, $Q_1(x) > 0$ for $x \in (-1, 0]$ and after multiplication with $\frac{x+1}{xQ(x)} \leq 0$ the estimate (15) implies

$$\frac{x+1}{x} \ln(1+x) \leq \frac{P_1(x)}{Q_1(x)} \leq \sqrt{x+1} \frac{f(x)}{x}, \quad x \in (-1, 0). \tag{16}$$

Letting $x \to -1-$ in this expression, we get that $\lim_{x \to -1-} \frac{P_1(x)}{Q_1(x)} = 0$, which implies $P_1(-1) = 0$. Since $P_1(x)$ is a non-zero polynomial, we get that $x+1|P_1(x)$. Multiplying the equation (16) with $\frac{x}{x+1} \leq 0$, we get

$$\ln(1+x) \geq \frac{P_1(x)}{Q_1(x)} \geq \frac{f(x)}{\sqrt{x+1}}, \quad x \in (-1, 0).$$

Letting $x \to 0-$, we get

$$\ln(1+x) \geq \frac{P_1(x)}{Q_1(x)} \geq \frac{f(x)}{\sqrt{x+1}}, \quad x \in (-1,0].$$

Repeating this procedure, we get that for every natural number k we have $(x+1)^k | Q(x)$, which is a contradiction.

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