Electronic Journal of Mathematical Analysis and Applications
Vol. 8(2) July 2020, pp. 346-351
ISSN: 2090-729X(online)
http://math-frac.org/Journals/EJMAA/

GENERALIZING TRAN'S CONJECTURE

R. BØGVAD, I. NDIKUBWAYO AND B. SHAPIRO

Abstract. A conjecture of Khang Tran [6] claims that for an arbitrary pair of polynomials $A(z)$ and $B(z)$, every zero of every polynomial in the sequence $\{P_n(z)\}_{n=1}^{\infty}$ satisfying the three-term recurrence relation of length k

$$P_n(z) + B(z)P_{n-1}(z) + A(z)P_{n-k}(z) = 0$$

with the standard initial conditions $P_0(z) = 1$, $P_{-1}(z) = \cdots = P_{-k+1}(z) = 0$ which is not a zero of $A(z)$ lies on the real (semi)-algebraic curve $\mathcal{C} \subset \mathbb{C}$ given by

$$\Im\left(\frac{B^k(z)}{A(z)}\right) = 0 \quad \text{and} \quad 0 \le (-1)^k \Re\left(\frac{B^k(z)}{A(z)}\right) \le \frac{k^k}{(k-1)^{k-1}}.$$

In this short note, we show that for the recurrence relation (generalizing the latter recurrence of Tran) given by

$$P_n(z) + B(z)P_{n-\ell}(z) + A(z)P_{n-k}(z) = 0,$$

with coprime k and ℓ and the same standard initial conditions as above, every root of $P_n(z)$ which is not a zero of $A(z)B(z)$ belongs to the real algebraic curve $\mathcal{C}_{\ell,k}$ given by

$$\Im\left(\frac{B^k(z)}{A^\ell(z)}\right) = 0.$$

1. Basic notions and main result

Linear recurrence relations with various types of coefficients have been studied for more than a century and appear in different contexts throughout the whole body of mathematics. A natural univariate set-up in this area is as follows.

Let $\{P_n(z)\}$ be a polynomial sequence satisfying a finite linear recurrence relation of the form

$$P_n(z) + Q_1(z)P_{n-1}(z) + Q_2(z)P_{n-2}(z) + \cdots + Q_k(z)P_{n-k}(z) = 0, \qquad (1)$$

with polynomials coefficients $Q_1(z), \ldots, Q_k(z)$ and initial polynomial conditions of the form $P_0(z) = p_0(z)$, $P_{-1}(z) = p_{-1}(z), \ldots,$ and $P_{-k+1}(z) = p_{-k+1}(z)$.

Problem 1 In the above notation, describe the (asymptotic) behavior of the roots for polynomials in $\{P_n(z)\}$.

2010 *Mathematics Subject Classification.* 12D10, 26C10, 30C15.
Key words and phrases. recurrence, polynomial sequence, generating function, lattice paths.
Submitted Dec. 31, 2019. Revised March 5, 2020.

A major result related to Problem 1 has been proven in [1], [2]. It states that independently of the initial conditions, the sequence $\{\mu_n\}$ of the root-counting measures of $\{P_n(z)\}$ converges in the weak sense to the measure $\mu_{\overline{Q}}$ supported on $\Gamma_{\overline{Q}}$, where $\overline{Q} = (Q_1(z), Q_2(z), \ldots, Q_k(z))$ and $\Gamma_{\overline{Q}}$ is defined as follows. Consider the symbol equation of (1) given by

$$t^k + Q_1(z)t^{k-1} + \cdots + Q_k(z) = 0. \tag{2}$$

For a given fixed $z \in \mathbb{C}$, let $t_1(z) \geq t_2(z) \geq \cdots \geq t_k(z)$ be the k-tuple of the absolute values of all (not necessarily distinct) roots of (2) in the non-increasing order. Finally, define $\Gamma_{\overline{Q}}$ as:

$$\Gamma_{\overline{Q}} := \{z \in \mathbb{C} \mid t_1(z) = t_2(z)\}. \tag{3}$$

The density of $\mu_{\overline{Q}}$ can be also determined using (2) and (3), but we will not need this expression below.

In the majority of the situations, the roots of $P_n(z)$ only tend to the Beraha-Kahane-Weiss curve $\Gamma_{\overline{Q}}$ when $n \to \infty$ and it is difficult to say something about their location for finite n. The only general exception from this rule is probably the case when all polynomials in $\{P_n(z)\}$ are real-rooted which is often discussed in the literature and important for applications.

However in [6] Khang Tran was able to find a non-trivial situation in which the roots of $P_n(z)$ are not necessarily real and (almost all of them) still lie on the limiting curve $\Gamma_{\overline{Q}}$ for all n. Observe that this property is destroyed by small generic deformations of coefficients/initial polynomials.

Namely, Conjecture 6 of [6] claims the following.

Conjecture 1 For an arbitrary pair of polynomials $A(z)$ and $B(z)$, all zeros of every polynomial in the sequence $\{P_n(z)\}_{n=1}^{\infty}$ satisfying the three-term recurrence relation of length k

$$P_n(z) + B(z)P_{n-1}(z) + A(z)P_{n-k}(z) = 0$$

with the standard initial conditions $P_0(z) = 1$, $P_{-1}(z) = \cdots = P_{-k+1}(z) = 0$ which do satisfy $A(z) \neq 0$ lie on the real (semi)-algebraic curve $\mathcal{C} \subset \mathbb{C}$ given by

$$\Im\left(\frac{B^k(z)}{A(z)}\right) = 0 \quad \text{and} \quad 0 \leq (-1)^k \Re\left(\frac{B^k(z)}{A(z)}\right) \leq \frac{k^k}{(k-1)^{k-1}}. \tag{4}$$

Moreover, these roots become dense in $\mathcal{C}$ when $n \to \infty$.

One can check that in this specific case, the latter curve $\mathbb{C}$ given by (4) is exactly the Beraha-Kahane-Weiss curve $\Gamma_{\overline{Q}}$. In [6] Conjecture 1 was proven for $k = 2, 3, 4$ and in [7] Conjecture 1 was proven for arbitrary k, but only for polynomials $P_n(z)$ with sufficiently large n. Several other aspects of this problem are discussed in [3], [8], [9]. The purpose of this short note is to generalize and settle the first part of Conjecture 1.

We note that in the case when k and ℓ are not coprime, say $k = dk', \ell = d\ell'$, $\gcd(k', \ell') = 1$ and

$$\sum_{n=0}^{\infty} P_n(z)t^n = \frac{1}{1 + B(z)t^{\ell'} + A(z)t^{k'}},$$

we obtain

$$\sum_{n=0}^{\infty} Q_n(z)t^n \quad = \quad \frac{1}{1+B(z)t^{\ell}+A(z)t^k} = \frac{1}{1+B(z)t^{d\ell'}+A(z)t^{dk'}} = \sum_{n=0}^{\infty} P_n(z)t^{dn}.$$

Thus it suffices to study the zero distribution in the case $\gcd(k,\ell)=1$.

Theorem 1 For an arbitrary pair of polynomials $A(z)$ and $B(z)$, all zeros of every polynomial in the sequence $\{P_n(z)\}_{n=1}^{\infty}$ satisfying the three-term recurrence relation of length k

$$P_n(z)+B(z)P_{n-\ell}(z)+A(z)P_{n-k}(z)=0 \tag{5}$$

with coprime k and ℓ and with the standard initial conditions $P_0(z)=1$, $P_{-1}(z)=\cdots=P_{-k+1}(z)=0$ which satisfy the condition $A(z)B(z)\neq 0$ lie on the real algebraic curve given by

$$\Im\left(\frac{B^k(z)}{A^{\ell}(z)}\right)=0. \tag{6}$$

Initial results in this direction together with the inequality determining which part of the curve given by (6) contain the roots of $\{P_n(z)\}$ can be found in a recent paper [4] of the second author.

2. Proofs

Lemma 2 In notation of Theorem 1,

$$P_n(z)=\sum_{\substack{i\geq 0, j\geq 0\\ i\ell+jk=n}} (-1)^{i+j}\binom{i+j}{i}A^j(z)B^i(z). \tag{7}$$

Proof. Equation (5) together with the standard initial conditions imply that

$$\begin{aligned}\sum_{n=0}^{\infty} P_n(z)t^n &= \frac{1}{1+B(z)t^{\ell}+A(z)t^k}\\ &= 1-(B(z)t^{\ell}+A(z)t^k)+(B(z)t^{\ell}+A(z)t^k)^2-\cdots\end{aligned}$$

Comparing the coefficients of t^n on both sides of the above equation gives (7).

Let $L:=L_{\ell,k,n}=\{(i,j)\in\mathbb{Z}^2_{\geq 0}: i\ell+jk=n\}$. Since the gradient of $x\ell+ky=n$ is negative, the set L is a (possibly empty) finite set, say with s elements. In general, the linear Diophantine equation $x\ell+ky=n$ (with $gcd(k,\ell)=1$) has integer solutions of the form $x=x_0+ku$ and $y=y_0-\ell u$ where $x_0,y_0,u\in\mathbb{Z}$. We can choose an x_0 and y_0 in a such a way that $L=\{(i_u,j_u):=(x_0+ku,y_0-\ell u)\in\mathbb{Z}^2_{\geq 0}: u=1,\ldots,s\}$. For any $1\leq u\leq s$, we have $i_u-i_1=k(u-1)$ and $j_1-j_u=\ell(u-1)$.

Let

$$G_{\ell,k,n}(\tau):=\sum_{u=1}^{s}\binom{i_u+j_u}{i_u}\tau^{u-1}. \tag{8}$$

Then $G_{\ell,k,n}(\tau)$ is the generating function for number of north and east lattice paths from the origin $(0,0)$ to the point $(i_u,j_u)\in L$. The proof of the next result can be found in [10]. See Conjecture 1 and its proof in Section 2 of the the same paper. (Weaker statements in the same direction can be found in [5].)

Theorem A In the above notation, for any given positive integers ℓ, k, n, $G_{\ell,k,n}(\tau)$ as a polynomial in τ has only negative roots.

Example 1 Take $\ell = 2,\ k = 3,\ n = 21$. Then $L = \{(0,7),\ (3,5),\ (6,3),\ (9,1)\}$. The generating polynomial is

$$G_{2,3,21}(\tau) = \binom{0+7}{0} + \binom{3+5}{3}\tau + \binom{6+3}{6}\tau^2 + \binom{9+1}{9}\tau^3 = 1 + 56\tau + 84\tau^2 + 10\tau^3.$$

The roots of this polynomial are approximately equal to $-7.67175, -0.70989$ and -0.0183618.

Proof of Theorem 1 For fixed k, ℓ and n with $gcd(k,\ell) = 1$ and $k > \ell$, as above set $L = \{(i_u, j_u) := (x_0 + ku, y_0 - \ell u) \in \mathbb{Z}^2_{\geq 0} : u = 1, \ldots, s\}$ where (x_0, y_0) is a solution to the equation $i\ell + jk = n$.

From Lemma 2, we have

$$\begin{aligned}
P_n(z) &= \sum_{u=1}^{s} (-1)^{i_u + j_u} \binom{i_u + j_u}{i_u} A^{j_u}(z) B^{i_u}(z) \\
&= A^{j_1}(z) B^{i_1}(z) (-1)^{i_1 + j_1} \sum_{u=1}^{s} (-1)^{i_u + j_u - (i_1 + j_1)} \binom{i_u + j_u}{i_u} A^{j_u - j_1}(z) B^{i_u - i_1}(z) \\
&= (-1)^{i_1 + j_1} A^{j_1}(z) B^{i_1}(z) \sum_{u=1}^{s} (-1)^{(u-1)(k-\ell)} \binom{i_u + j_u}{i_u} \frac{B^{(u-1)k}(z)}{A^{(u-1)\ell}(z)} \\
&= (-1)^{i_1 + j_1} A^{j_1}(z) B^{i_1}(z) \sum_{u=1}^{s} \binom{i_u + j_u}{i_u} \left((-1)^{(k-\ell)} \frac{B^k(z)}{A^\ell(z)} \right)^{u-1} \\
&= \pm B^{i_1}(z) A^{j_1}(z) G_{\ell,k,n}\left((-1)^{(k-\ell)} \frac{B^k(z)}{A^\ell(z)} \right).
\end{aligned}$$

The last equality follows from (8). If $z_0 \in \mathbb{C}$ is such that $P_n(z_0) = 0$ and $A(z_0)B(z_0) \neq 0$, then it follows that

$$G_{\ell,k,n}\left((-1)^{(k-\ell)} \frac{B^k(z_0)}{A^\ell(z_0)} \right) = 0.$$

By Theorem 2, we have

$$\Im\left((-1)^{(k-\ell)} \frac{B^k(z_0)}{A^\ell(z_0)} \right) = 0,$$

which completes the proof.

Example 2 Let $A(z) = z^3 + z + 1, B(z) = z^2 - 2z + 7, \ell = 2, k = 3$, and $n = 21$. Then using Example 1, we get

$$\begin{aligned}
P_{21}(z) &= -A^7\, G_{2,3,21}\left(-\frac{B^3}{A^2} \right) \\
&= -(z^3 + z + 1)^7 \Big(1 - 56 \frac{(z^3 + z + 1)^3}{(z^2 - 2z + 7)^2} + 84 \Big(\frac{(z^3 + z + 1)^3}{(z^2 - 2z + 7)^2} \Big)^2 - 10 \Big(\frac{(z^3 + z + 1)^3}{(z^2 - 2z + 7)^2} \Big)^3 \Big).
\end{aligned}$$

On simplification, we have

$$P_{21}(z) = 393672761 - 646754633z + 667797557z^2 + 98239806z^3 - 1206661925z^4 + 2171467228z^5 - 2529964192z^6 + 2246607369z^7 - 1625784860z^8 + 969712412z^9 - 486724329z^{10} + 201422869z^{11} - 68243275z^{12} + 17375116z^{13} - 2717833z^{14} - 196756z^{15} + 295748z^{16} - 114667z^{17} + 27963z^{18} - 4619z^{19} + 492z^{20} - 19z^{21}.$$

This polynomial $P_{21}(z)$ is the same as one given by (5) using Mathematica for the same choice of parameters.

In Figure 1 below, we illustrate Theorem 1 for the above generated polynomial $P_{21}(z)$.

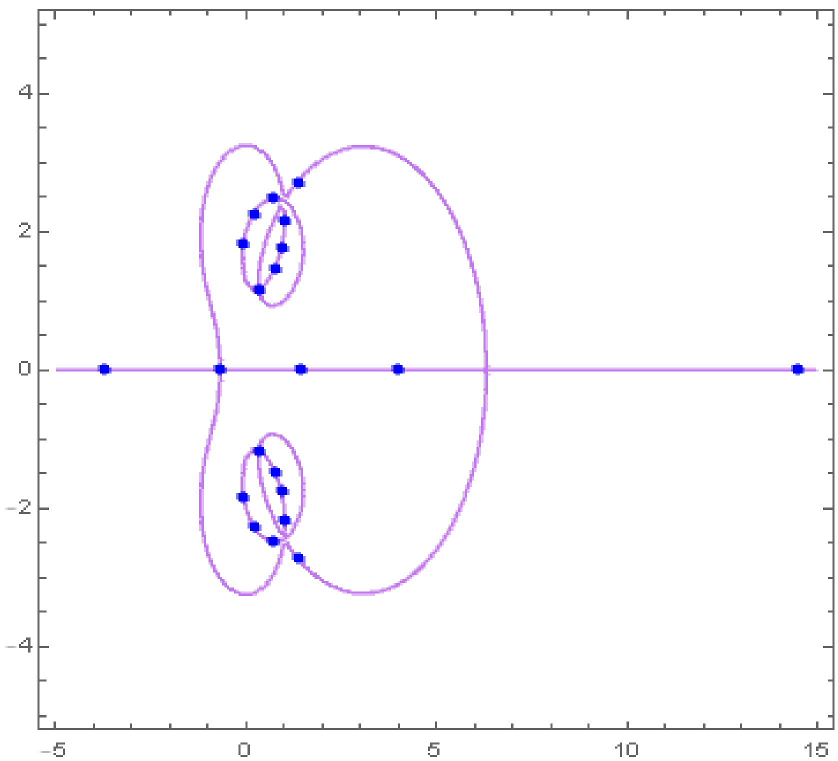

FIGURE 1. For $A(z) = z^3 + z + 1$ and $B(z) = z^2 - 2z + 7$, the graph of $\Im\left(\frac{B^3(z)}{A^2(z)}\right) = 0$ and the zeros of $P_{21}(z)$.

3. Final remarks

1. The most important question related to this note is to find the analog of the inequality in (4) describing on which part of the real algebraic curve given by (6) the roots of $\{P_n(z)\}$ are located and become dense when $n \to \infty$. Some special cases are considered in [4].

2. Choosing an initial k-tuple $IN = \{P_0(z), \dots, P_{-k+1}(z)\}$, find other examples/classes of pairs $(\overline{Q}, IN)$ where (almost all) zeros of $\{P_n(z)\}$ lie on a fixed curve in the complex plane which is different from an affine line. One can try to find such examples for relations of order 4.

3. Theorem 2 apparently has a multivariate generalization when one considers a multivariate polynomial generating function for multinomial coefficients whose indices belong to a hyperplane given by an equation $x_1\ell_1 + x_2\ell_2 + \cdots + x_u\ell_u = n$. It is very tempting to find this generalization and check whether it leads to further implications related to Problem 1 and its version *à* la Tran.

Acknowledgements. The second author acknowledges the financial support provided by Sida Phase-IV bilateral program with Makerere University 2015-2020 under project 316 'Capacity building in Mathematics and its applications'. The third author wants to acknowledge the financial support of his research provided by the Swedish Research Council grant 2016-04416.

References

[1] S. Beraha, J. Kahane, N. J. Weiss, *Limits of zeroes of recursively defined polynomials*, Proc. Nat. Acad. USA **72** (1975), 4209.

[2] S. Beraha, J. Kahane, N. J. Weiss, *Limits of zeros of recursively defined families of polynomials*, in "Studies in Foundations and Combinatorics", pp. 213–232, Advances in Mathematics Supplementary Studies Vol. 1, ed. G.-C. Rota, Academic Press, New York, 1978.

[3] T. Forgacz, K. Tran, *Hyperbolic polynomials and linear-type generating functions*, arXiv:1810.0152.

[4] I. Ndikubwayo, *Around a conjecture of K. Tran*, arXiv:1910.00278, Electronic Journal of Mathematical Analysis and Applications (EJMAA). Vol. 8(2) July. (2020) 16–37.

[5] X.-T. Su, Y. Wang, *On unimodality problems in Pascal's triangle.* The Electronic Journal of Combinatorics (2008) Volume: 15, Issue: 1, page Research Paper R113, 12 p.-Research Paper R113, 12 p.

[6] K. Tran, *Connections between discriminants and the root distribution of polynomials with rational generating function*, J. Math. Anal. Appl. 410 (2014) 330–340.

[7] K. Tran, *The root distribution of polynomials with a three-term recurrence*, J. Math. Anal. Appl. 421 (2015) 878–892.

[8] K. Tran, A. Zumba, *Zeros of polynomials with four-term recurrence.* Involve, a Journal of Mathematics 11(2018), No. 3, 501–518.

[9] K. Tran, A. Zumba, *Zeros of polynomials with four-term recurrence and linear coefficients*, arXiv:1808.07133.

[10] Y. Yu, *Confirming two conjectures of Su and Wang on binomial coefficients*, Advances in Applied Mathematics 43 (2009) 317–322.

R. Bøgvad
Department of Mathematics, Stockholm University, Stockholm, Sweden
Email address: `rikard@math.su.se`

I. Ndikubwayo
Department of Mathematics, Stockholm University, Stockholm, Sweden
Email address: `innocent@math.su.se, ndikubwayo@cns.mak.ac.ug`

B. Shapiro
Department of Mathematics, Stockholm University, Stockholm, Sweden
Email address: `shapiro@math.su.se`