DHAGE ITERATION METHOD FOR PBVPS OF NONLINEAR FIRST ORDER IMPULSIVE DIFFERENTIAL EQUATIONS

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Abstract. In this paper we prove a couple of existence and approximation theorems for the PBVPs of first order nonlinear impulsive differential equations under certain mixed partial Lipschitz and partial compactness type conditions. Our main results are based on Dhage iteration method embodied in the hybrid fixed point principles of Dhage (2014) involving the sum of two monotone nondecreasing operators in a partially ordered Banach space. Our abstract main result is also illustrated by indicating a numerical example. We claim that the results of this paper are new and complement the work of Li *et al* [22] and Nieto [23, 24] on PBVPs of nonlinear impulsive differential equations.

1. Introduction

The existence theory for nonlinear impulsive differential equations has received much attention during the last decade, but the study of the approximation of the solutions to such equations is relatively rare in the literature. The dynamical systems, which involve the jumps or discontinuities at finite number of points are modeled on the nonlinear impulsive differential equations. The exhaustive account on the topic appear in in the research monographs of Bainov and Simeonov [1], Samoilenko and Perestyuk [26], Lakshmikantham *et al* [25] and the references therein. The existence theorems so far discussed in the literature for such nonlinear impulsive differential equations involve either the use of Lipschitz or compactness type condition on the nonlinearities involved in the equations. Very recently, the present author in [12] initiated the study of IVPs of nonlinear first order impulsive differential equations via a new Dhage iteration method which does not involve the use of usual continuity and classical arguments of analysis and topology. Here in the present study, we continue the same line of arguments for PBVPs of nonlinear impulsive differential equations. A few details concerning the importance of impulsive PBVPs in the study of dynamic systems are given in Nieto [23, 24] and references therein. Note that in the framework of present iteration method for impulsive PBVPs we do not need the usual Lipschitz and compactness conditions of the nonlinearity but require weaker partial Lipschitz and partial compactness type

conditions for proving the existence as well as approximation theorems for such equations. However, the existence and approximation results are obtained under certain additional monotonic conditions. We claim that the results of this paper are new to the literature on nonlinear impulsive differential equations.

Let $\mathbb{R}$ be the real line and let $J = [0, T]$ be a closed and bounded interval in $\mathbb{R}$. Let $t_0, \ldots t_{p+1}$ be the points in J such that $0 = t_0 < t_1 < \cdots, < t_p < t_{p+1} = T$ and let $J' = J \setminus \{t_1, \ldots, t_p\}$. Denote $J_j = (t_j, t_{j+1}) \subset J$ for $j = 1, 2, \ldots, p$. By $X = C(J, \mathbb{R})$ and $L^1(J, \mathbb{R})$ we denote respectively the spaces of continuous and Lebesgue integrable real-valued functions defined on J.

Now, given a function $h \in L^1(J, \mathbb{R})$, $h > 0$ on J, consider the periodic boundary value problem (in short PBVP) for the first order impulsive differential equation (in short IDE)

$$\left.\begin{aligned}
x'(t) + h(t)x(t) &= f(t, x(t)), \quad t \in J \setminus \{t_1, \ldots, t_p\}, \\
x(t_j^+) - x(t_j^-) &= \mathcal{I}_j(x(t_j)), \, j = 1, \ldots, p, \\
x(0) &= x(T),
\end{aligned}\right\} \tag{1}$$

where, the limits $x(t_j^+)$ and $x(t_j^-)$ are respectively the right and left limit of x at $t = t_j$ such that $x(t_j) = x(t_j^-)$, $\mathcal{I}_j \in C(\mathbb{R}, \mathbb{R})$, $\mathcal{I}_j(x(t_j))$ are the impulsive effects at the points $t = t_j$, $j = 1, \ldots, p$ and $f : J \times \mathbb{R} \to \mathbb{R}$ is such that f is continuous on $J' = J - \{t_1, ..., t_p\}$, and there exist the limits

$$\lim_{t \to t_j^-} f(t, u) = f(t_j, u) \quad \text{and} \quad \lim_{t \to t_{j+}} f(t, u), \, u \in \mathbb{R},$$

for each $j = 1, \ldots, p$.

By a *impulsive solution* of the IDE (1) we mean a function $x \in PC^1(J, \mathbb{R})$ that satisfies the differential equation and the conditions in (1), where $PC^1(J, \mathbb{R})$ is the space of piecewise continuously differentiable real-valued functions defined on J.

The special case of the IDE (1), when $\lambda = 1$ has already been discussed in Li *et. al* [22] and Nieto [23, 24] for existence and uniqueness theorems under Lipschitz and compactness type conditions of the nonlinear function f using Schauder and Banach fixed point theorems respectively. The existence and uniqueness theorems for the IDE (1) may also be proved using the hybrid fixed point theorems given in Dhage [9] and references therein. The approximation of the solution for IDE (1) may be proved via Dhage iterative technique, but in that case we require existence of both upper as well as lower solutions together with a certain comparison principle. Here in the present study, we relax the above stringent conditions and discuss the IDE (1) for existence and approximation of impulsive solution under mixed partial Lipschitz and partial compactness type conditions via Dhage iteration method based on a hbrid fixed point theorems of Dhage [6]. We claim that the results of this paper are new to the literature.

2. AUXILIARY RESULTS

Throughout this paper, unless otherwise mentioned, let $(E, \preceq, \|\cdot\|)$ denote a partially ordered normed linear space. Two elements x and y in E are said to be **comparable** if either the relation $x \preceq y$ or $y \preceq x$ holds. A non-empty subset C of E is called a **chain** or **totally ordered** if all the elements of C are comparable. It is known that E is **regular** if $\{x_n\}$ is a nondecreasing (resp. nonincreasing)

sequence in E such that $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. The conditions guaranteeing the regularity of E may be found in Guo and Lakshmikantham [20], Heikkilä and Lakshmikantham [21] and the references therein.

We need the following definitions (see Dhage [3, 4, 5, 6] and the references therein) in what follows.

A mapping $\mathcal{T} : E \to E$ is called **isotone** or **monotone nondecreasing** if it preserves the order relation $\preceq$, that is, if $x \preceq y$, then $\mathcal{T}x \preceq \mathcal{T}y$ for all $x, y \in E$. Similarly, $\mathcal{T}$ is called **monotone nonincreasing** if $x \preceq y$ implies $\mathcal{T}x \succeq \mathcal{T}y$ for all $x, y \in E$. Finally, $\mathcal{T}$ is called **monotonic** or simply **monotone** if it is either monotone nondecreasing or monotone nonincreasing on E. A mapping $\mathcal{T} : E \to E$ is called **partially continuous** at a point $a \in E$ if for given $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\mathcal{T}x - \mathcal{T}a\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathcal{T}$ is called partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathcal{T}$ is partially continuous on E, then it is continuous on every chain C contained in E and vice-versa. A non-empty subset S of the partially ordered metric space E is called **partially bounded** if every chain C in S is bounded. A mapping $\mathcal{T}$ on a partially ordered metric space E into itself is called **partially bounded** if $\mathcal{T}(E)$ is a partially bounded subset of E.

A non-empty subset S of the partially ordered metric space E is called **partially compact** if every chain C in S is a compact subset of E. A mapping $\mathcal{T} : E \to E$ is called **partially compact** if every chain C in $\mathcal{T}(E)$ is a relatively compact subset of E.

Remark 2.1. Suppose that $\mathcal{T}$ is a monotone operator on E into itself. Then $\mathcal{T}$ is a partially bounded or partially compact on E if $\mathcal{T}(C)$ is a bounded or compact subset of E for each chain C in E.

Definition 2.1 (Dhage [3, 4, 5, 6])**.** The order relation $\preceq$ and the metric d on a non-empty set E are said to be $\mathcal{D}$-**compatible** if $\{x_n\}$ is a monotone sequence, that is, monotone nondecreasing or monotone nonincreasing sequence in E and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x^* implies that the original sequence $\{x_n\}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be $\mathcal{D}$-compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are $\mathcal{D}$-compatible. A subset S of E is called a **Janhavi set** if the order relation $\preceq$ and the metric d or the norm $\|\cdot\|$ are $\mathcal{D}$-compatible in it. In particular, if $S = E$, then E is called a **Janhavi metric space** or **Janhavi Banach space**.

Clearly, the set $\mathbb{R}$ of real numbers with usual order relation $\leq$ and the norm defined by the absolute value function $|\cdot|$ has this property. Similarly, the finite dimensional Euclidean space $\mathbb{R}^n$ with usual componentwise order relation and the standard norm possesses the compatibility property and so is a Janhavi set. Consequently, $(\mathbb{R}^n, \|\cdot\|, \leq)$ is a **Janhavi Banach space**.

Definition 2.2 (Dhage [2, 3, 9])**.** An upper semi-continuous and monotone nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathcal{D}$-function provided $\psi(0) = 0$. A monotone operator $\mathcal{T} : E \to E$ is called nonlinear partial $\mathcal{D}$-contraction if there exists a $\mathcal{D}$-function ψ such that

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \psi\big(\|x - y\|\big) \tag{2}$$

for all comparable elements $x, y \in E$, where $0 < \psi(r) < r$ for $r > 0$. In particular, if $\psi(r) = k\,r$, $k > 0$, $\mathcal{T}$ is called a partial Lipschitz operator with a Lipschitz constant k and moreover, if $0 < k < 1$, $\mathcal{T}$ is called a linear partial contraction on E with the contraction constant k.

The **Dhage iteration method** or **Dhage iteration principle** embodied in the following applicable hybrid fixed point theorems of Dhage [3, 4, 5, 6] in a partially ordered complete normed linear space is used as a key tool for our work contained in this paper. The details of the Dhage iteration principle or method appear in Dhage [3] and the references therein. Similarly, for applications of Dhage iteration principle the readers are referred to Dhage and Dhage [13, 14, 15, 16], Dhage *et.al* [17, 18], Dhage and Otrocol [19] and references therein.

Theorem 2.1 (Dhage [6])**.** *Let $(E, \preceq, \|\cdot\|)$ be a partially ordered Banach space and let $\mathcal{T} : E \to E$ be a monotone nondecreasing and nonlinear partial $\mathcal{D}$-contraction. Suppose that there exists an element $x_0 \in E$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$. If $\mathcal{T}$ is continuous or E is regular, then $\mathcal{T}$ has a unique comparable fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges monotonically to x^*. Moreover, the fixed point x^* is unique if every pair of elements in E has a lower bound or an upper bound.*

Theorem 2.2 (Dhage [5, 6])**.** *Let $(E, \preceq, \|\cdot\|)$ be a regular partially ordered complete normed linear space and let every compact chain C in E be Janhavi set. Let $\mathcal{A}, \mathcal{B} : E \to E$ be two monotone nondecreasing operators such that*

(a) *$\mathcal{A}$ is partially bounded and nonlinear partial $\mathcal{D}$-contraction,*
(b) *$\mathcal{B}$ is partially continuous and partially compact, and*
(c) *there exists an element $x_0 \in E$ such that $x_0 \preceq \mathcal{A}x_0 + \mathcal{B}x_0$ or $x_0 \succeq \mathcal{A}x_0 + \mathcal{B}x_0$.*

Then the hybrid operator equation $\mathcal{A}x + \mathcal{B}x = x$ has a solution x^* in E and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = \mathcal{A}x_n + \mathcal{B}x_n,\ n=0,1,\ldots,$ converges monotonically to x^*.

Remark 2.2. The condition that every compact chain of E is Janhavi set holds if every partially compact subset of E possesses the compatibility property with respect to the order relation $\preceq$ and the norm $\|\cdot\|$ in it.

Remark 2.3. We remark that hypothesis (a) of Theorem 2.2 implies that the operator $\mathcal{A}$ is partially continuous and consequently both the operators $\mathcal{A}$ and $\mathcal{B}$ in the above theorem are partially continuous on E. The regularity of E in above Theorems 2.1 and 2.2 may be replaced with a stronger continuity condition respectively of the operators $\mathcal{T}$ and $\mathcal{A}$ and $\mathcal{B}$ on E.

3. Existence and Approximation Theorem

Let $X_j = C(J_j, \mathbb{R})$ and $X_j^1 = C^1(J_j, \mathbb{R})$ denote respectively the classes of continuous and continuously differentiable real-valued functions defined on the intervals $J_j = (t_{j-1}, t_j)$ for each $j = 1, 2, ..., p$. Denote by $PC(J, \mathbb{R})$ the space of piecewise continuous real-valued functions on J defined by

$$PC(J, \mathbb{R}) = \left\{ x \in X_j \mid x(t_j^-) \text{ and } x(t_j^+) \text{ exist for } j = 1, \ldots, p; \text{ and } x(t_j^-) = x(t_j) \right\}. \tag{3}$$

Similarly, we define the space $PC^1(J, \mathbb{R})$ of piecewise continuously differentiable functions on J by

$$PC^1(J, \mathbb{R}) = \Big\{ x \in PC(J, \mathbb{R}) \mid x|_{(t_{j-1}, t_j)} \in C^1(t_{j-1}, t_j),$$
$$\text{and} \ \ x'(t_j^-), \ x'(t_j^+) \text{ exist for } j = 1, \ldots, p \Big\}. \tag{4}$$

Define the supremum norms $\|\cdot\|_{PC}$ and $\|\cdot\|_{PC^1}$ in the Banach spaces $PC(J, \mathbb{R})$ and $PC^1(J, \mathbb{R})$ respectively by

$$\|x\|_{PC} = \sup_{t \in J} |x(t)| \tag{5}$$

and

$$\|x\|_{PC^1} = \sup_{t \in J} |x(t)| + \sup_{t \in J} |x'(t)|. \tag{6}$$

We define the order cone K in $PC(J, \mathbb{R})$ by

$$K = \big\{ x \in PC(J, \mathbb{R}) \mid x(t) \geq 0 \text{ for all } t \in J \big\}, \tag{7}$$

which is obviously a normal cone in $PC(J, \mathbb{R})$. Now, define the order relation $\preceq$ in $PC(J, \mathbb{R})$ by

$$x \preceq y \iff y - x \in K \tag{8}$$

which is equivalent to

$$x \preceq y \iff x(t) \leq y(t) \text{ for all } t \in J.$$

Clearly, $\big(PC(J, \mathbb{R}), K\big)$ becomes a regular ordered Banach space with respect to the above norm and order relation in $PC(J, \mathbb{R})$ and every compact chain C in $PC(J, \mathbb{R})$ is Janhavi set in view of the following lemmas proved in Dhage [11].

Lemma 3.1 (Dhage [8, 10, 11]). *Every ordered Banach space (E, K) is regular.*

Lemma 3.2 (Dhage [8, 10, 11]). *Every partially compact subset S of an ordered Banach space (E, K) is a Janhavi set in E.*

We need the following definition in what follows.

Definition 3.1. A function $u \in PC^1(J, \mathbb{R})$ is said to be a lower impulsive solution of the IDE (1) if it satisfies

$$\left. \begin{array}{c} u'(t) + h(t)u(t) \leq f(t, u(t)), \ \ t \in J \setminus \{t_1, \ldots, t_p\}, \\ u(t_j^+) - u(t_j^-) \leq \mathcal{I}_j(u(t_j)), \, j = 1, \ldots, p \\ u(0) \leq x(T), \end{array} \right\}$$

for $j = 1, 2, ..., p$. Similarly, a function $v \in PC^1(J, \mathbb{R})$ is called an upper impulsive solution of the IDE (1) if the above inequality is satisfied with reverse sign.

We consider the following set of assumptions in what follows:

(H$_1$) The impulsive functions $\mathcal{I}_j \in C(\mathbb{R}, \mathbb{R})$ are bounded with bounds $M_{\mathcal{I}_j}$ for each $j = 1, \ldots, p.$,

(H$_2$) There exists a constants $L_{\mathcal{I}_j} > 0$ such that

$$0 \leq \mathcal{I}_j x - \mathcal{I}_j y \leq L_{\mathcal{I}_j} \big(x - y \big)$$

for all $x, y \in \mathbb{R}$, $x \geq y$, where $j = 1, \ldots, p$.

(H$_3$) The function f is bounded on $J \times \mathbb{R}$ with bound M_f.

(H$_4$) $f(t, x)$ is nondecreasing in x for each $t \in J$.

(H_5) There exists a constant $L_f > 0$ such that

$$0 \leq f(t, x) - f(t, y) \leq L_f(x - y)$$

for all $t \in J$ and $x, y \in \mathbb{R}$, $x \geq y$.

(H_6) The IDE (1) has a lower impulsive solution $u \in PC^1(J, \mathbb{R})$.

Below we state a useful result of Nieto [23, 24](see also Dhage [2] and references therein) concerning the Green's function of first order ordinary periodic boundary value problems.

Lemma 3.3 (Nieto [23]). *Given* $\sigma \in L^1(J, \mathbb{R})$, *a function* $x \in PC^1(J, \mathbb{R})$ *is a impulsive solution to the IDE*

$$\left. \begin{aligned} x'(t) + h(t)x(t) &= \sigma(t), \ \ t \in J \setminus \{t_1, \ldots, t_p\}, \\ x(t_j^+) - x(t_j^-) &= \mathcal{I}_j(x(t_j)), \ j = 1, \ldots, p, \\ x(0) &= x(T), \end{aligned} \right\} \tag{9}$$

if and only if it is an impulsive solution of the impulsive integral equation

$$x(t) = \sum_{j=1}^{p} G_h(t, t_j)\mathcal{I}_j(x(t_j)) + \int_0^T G_h(t, s)\sigma(s)\, ds, \ t \in J, \tag{10}$$

where the Green's function G_h *is given by*

$$G_h(t, s) = \begin{cases} \dfrac{e^{-[H(T) - H(s)]}}{1 - e^{-H(T)}} & \text{if } \ 0 \leq s \leq t \leq T, \\[4mm] \dfrac{e^{-[H(T) + H(t) - H(s)]}}{1 - e^{-H(T)}} & \text{if } \ 0 \leq t \leq s \leq T, \end{cases} \tag{11}$$

and $H(t) = \displaystyle\int_0^t h(s)\, ds > 0$.

Notice that the Green's function G_h is nonnegative and bounded on $J \times J$ and therefore, the number

$$M_{G_h} := \max\left\{ |G_h(t, s)| \, : \, t, s \in [0, T] \right\}$$

exists for all $h \in L^1(I, \mathbb{R}_+)$. For the sake of convenience, we write $G_h(t, s) = G(t, s)$ and $M_{G_h} = M_G$.

Another useful result for establishing the main results is as follows.

Lemma 3.4. *Given* $\sigma \in L^1(J, \mathbb{R})$, *if there is a function* $u \in PC^1(J, \mathbb{R})$ *satisfying the impulsive differential inequality*

$$\left. \begin{aligned} u'(t) + h(t)u(t) &\leq \sigma(t), \ \ t \in J \setminus \{t_1, \ldots, t_p\}, \\ u(t_j^+) - u(t_j^-) &\leq \mathcal{I}_j(u(t_j)), \ j = 1, \ldots, p, \\ u(0) &\leq u(T), \end{aligned} \right\} \tag{12}$$

then it satisfies the impulsive integral inequality

$$u(t) \leq \sum_{j=1}^{p} G(t, t_j)\mathcal{I}_j(u(t_j)) + \int_0^T G(t, s)\sigma(s)\, ds, \ t \in J, \tag{13}$$

where the Green's function G *is defined by the expression* (11) *on* $J \times J$.

Proof. First note that the integral in $H(t)$ is a continuous and nonnegative real-valued function on J. Therefore, we have $H(t) > 0$ on J provided h is not an identically zero function. Otherwise $H(t) \equiv 0$ on J. Moreover, by continuity of integral function H on J, we have that $H(t^-) = H(t) = H(t^+)$ for all $t \in J$.

First suppose that u satisfies the impulsive differential inequality (12) on J, that is, u is an impulsive lower solution of the IDE (7) on J. Then, we have

$$
\left.
\begin{aligned}
\left(e^{H(t)} u(t) \right)' &\leq e^{H(t)} \sigma(t), \ \ t \in J \setminus \{t_1, \ldots, t_p\}, \\
u(t_j^+) - u(t_j^-) &\leq \mathcal{I}_j(u(t_j)), \ j = 1, \ldots, p, \\
u(0) &\leq u(T),
\end{aligned}
\right\}
\tag{14}
$$

for $j = 1, 2, \ldots, p$. Let $y(t) = e^{H(t)} u(t)$. Then the above inequality (12) takes the form

$$
\left.
\begin{aligned}
y'(t) &\leq \sigma^*(t), \ \ t \in J \setminus \{t_1, \ldots, t_p\}, \\
y(t_j^+) - y(t_j^-) &\leq \mathcal{I}_j^*(u(t_j)), \ j = 1, \ldots, p, \\
y(0) &\leq y(T) e^{-H(T)},
\end{aligned}
\right\}
\tag{15}
$$

where $\sigma^*(t) = e^{H(t)} \sigma(t)$ and $\mathcal{I}_j^*(u(t_j)) = e^{H(t_j)} \mathcal{I}_j(u(t_j))$.

If $t \in (t_j, t_{i+1}]$, $j = 1, \ldots, p$, we have that

$$
y(t) = y(t_j^+) + \int_{t_j}^t \sigma^*(s) \, ds.
$$

On the other hand,

$$
y(t_j^-) = y(t_{j-1}^+) + \int_{t_{j-1}}^{t_j} y'(s) \, ds
$$

for all $j = 1, \ldots, p$. Therefore, from the theory of integral calculus, it follows that

$$
\begin{aligned}
y(t_1^-) - y(0) &= \int_0^{t_1} y'(s) \, ds \\
y(t_2^-) - y(t_1^+) &= \int_{t_1}^{t_2} y'(s) \, ds \\
&\vdots \\
y(t) - y(t_p^+) &= \int_{t_p}^t y'(s) \, ds.
\end{aligned}
$$

Summing up the above equations,

$$
y(t) - \sum_{0 < t_j < t} \mathcal{I}_j^*(u(t_j)) \leq y(0) + \int_0^t y'(s) \, ds,
$$

or

$$
y(t) \leq y(0) + \sum_{0 < t_j < t} \mathcal{I}_j^*(u(t_j)) + \int_0^t \sigma^*(s) \, ds
\tag{16}
$$

for $t \in J$. Substituting $t = T$ in the above inequality yields

$$
y(0) \leq \frac{1}{e^{H(T)} - 1} \sum_{j=2}^p \mathcal{I}_j^*(u(t_j)) + \frac{1}{e^{H(T)} - 1} \int_0^T \sigma^*(s) \, ds.
$$

Substituting this value of $y(0)$ in (16), we obtain

$$u(t) \leq \frac{e^{-H(t)}}{e^{H(T)} - 1} \sum_{j=1}^{p} \mathcal{I}_j^*(u(t_j)) + \frac{e^{-H(t)}}{e^{H(T)} - 1} \int_0^T \sigma^*(s) \, ds$$

$$+ e^{-H(t)} \sum_{0 < t_j < t} \mathcal{I}_j^*(u(t_j)) + e^{-H(t)} \int_0^t \sigma^*(s) \, ds$$

$$\leq \frac{e^{-H(t)}}{e^{H(T)} - 1} \sum_{j=1}^{p} e^{H(t_j)} \mathcal{I}_j(u(t_j)) + e^{-H(t)} \sum_{0 < t_j < t} e^{H(t_j)} \mathcal{I}_j(u(t_j))$$

$$+ \frac{e^{-H(t)}}{e^{H(T)} - 1} \int_0^T e^{H(s)} \sigma(s) \, ds + e^{-H(t)} \int_0^t e^{H(s)} \sigma(s) \, ds$$

$$= \sum_{j=1}^{p} G(t, t_j) I_j(u(t_j)) + \int_0^T G(t, s) \sigma(s) \, ds$$

for all $t \in J$ and the proof of the lemma is complete. $\qquad\square$

Similarly, we have the following useful result concerning the impulsive differential inequality with reverse sign.

Lemma 3.5. *Given $\sigma \in L^1(J, \mathbb{R})$, if there is a function $v \in PC^1(J, \mathbb{R})$ satisfying the impulsive differential inequality*

$$\left.\begin{array}{r} v'(t) + h(t)v(t) \geq \sigma(t), \ \ t \in J \setminus \{t_1, \ldots, t_p\}, \\ v(t_j^+) - v(t_j^-) \geq \mathcal{I}_j(v(t_j)), \ j = 1, \ldots, p, \\ v(0) \geq v(T), \end{array}\right\} \qquad (17)$$

then it satisfies the impulsive integral inequality

$$v(t) \geq \sum_{j=1}^{p} G(t, t_j) \mathcal{I}_j(v(t_j)) + \int_0^T G(t, s) \sigma(s) \, ds, \ t \in J, \qquad (18)$$

where the Green's function G is defined by the expression (11) on $J \times J$.

Theorem 3.1. *Suppose that hypotheses (H_1) through (H_4) and (H_6) hold. Furthermore, if $M_G \sum_{j=1}^{p} L_{I_j} < 1$, then the IDE (1) has a impulsive solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by*

$$x_0(t) = u(t), \quad x_{n+1}(t) = \sum_{j=1}^{p} G(t, t_j) \mathcal{I}_j(x_n(t_j)) + \int_0^T G(t, s) f(s, x_n(s)) \, ds \quad (19)$$

for all $t \in J$, converges monotonically to x^*.

Proof. Set $E = PC(J, \mathbb{R})$ and place the IDE (1) in the function space E. Then, by Lemma 3.2, every compact chain C in E possesses the compatibility property with respect to the norm $\| \cdot \|_{PC}$ and the order relation $\preceq$ so that every compact chain C is Janhavi a set in E.

Now, by Lemma 3.3, the IDE (1) is equivalent to the nonlinear impulsive integral equation

$$x(t) = \sum_{j=1}^{p} G(t, t_j) \mathcal{I}_j(x(t_j)) + \int_0^T G(t, s) f(s, x(s)) \, ds \qquad (20)$$

for all $t \in J$.

Define two operators $\mathcal{A}$ and $\mathcal{B}$ on E by

$$\mathcal{A}x(t) = \sum_{j=1}^{p} G(t, t_j) \mathcal{I}_j(x(t_j)), \ t \in J, \tag{21}$$

and

$$\mathcal{B}x(t) = \int_0^T G(t, s) f(s, x(s)) \, ds, \ t \in J. \tag{22}$$

From the piecewise continuity of the Green's function G on $J \times J$, it follows that $\mathcal{A}$ and $\mathcal{B}$ define the operators $\mathcal{A}, \mathcal{B} : E \to E$ and the impulsive integral equation (20) is transformed into the operator equation as

$$\mathcal{A}x(t) + \mathcal{B}x(t) = x(t), \ \ t \in J. \tag{23}$$

Now, the problem of finding the impulsive solution of the IDE (1) is just reduced to finding impulsive solution of the operator equation (23) on J. We show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.2 in a series of following steps.

Step I: *$\mathcal{A}$ and $\mathcal{B}$ are nondecreasing on E.*

Let $x, y \in E$ be such that $x \succeq y$. Then, by hypothesis (H_1), we get

$$\mathcal{A}x(t) = \sum_{j=1}^{p} Gk(t, t_j) \mathcal{I}_j(x(t_j)) \geq \sum_{j=1}^{p} G(t, t_j) \mathcal{I}_j(y(t_j)) = \mathcal{A}y(t),$$

for all $t \in J$. By definition of the order relation in E, we obtain $\mathcal{A}x \succeq \mathcal{A}y$ and a fortiori, $\mathcal{A}$ is a nondecreasing operator on E. Similarly, using hypothesis (H_3),

$$\mathcal{B}x(t) = \int_0^T G(t, s) f(s, x(s)) \, ds \geq \int_0^T G(t, s) f(s, x(s)) \, ds = \mathcal{B}y(t),$$

for all $t \in J$. Therefore, the operator $\mathcal{B}$ is also nondecreasing on E into itself.

Step II: *$\mathcal{A}$ is partially bounded and partially contraction on E.*

Let $x \in E$ be arbitrary. Then by (H_1) we have

$$|\mathcal{A}x(t)| \leq \left| \sum_{j=1}^{p} G(t, t_j) \mathcal{I}_j(x(t_j)) \right| \leq \sum_{j=1}^{p} |G(t, t_j)| \, |\mathcal{I}_j(y(t_j))| \leq \sum_{j=1}^{p} M_G \, M_{\mathcal{I}_j}$$

for all $t \in J$. Taking the supremum over t, we obtain $\|\mathcal{A}x\| \leq \sum_{j=1}^{p} M_G \, M_{\mathcal{I}_j}$ for all $x \in E$, so $\mathcal{A}$ is a bounded operator on E. This further implies that $\mathcal{A}$ is partially bounded on E.

Next, let $x, y \in E$ be such that $x \succeq y$. Then by (H_2), we have

$$|\mathcal{A}x(t) - \mathcal{A}y(t)| \leq \left| \sum_{j=1}^{p} G(t, t_j) \mathcal{I}_j(x(t_j)) - \sum_{j=1}^{p} G(t, t_j) \mathcal{I}_j(y(t_j)) \right|$$

$$\leq \left| \sum_{j=1}^{p} G(t, t_j) \big[\mathcal{I}_j(x(t_j)) - \mathcal{I}_j(x(t_j)) \big] \right|$$

$$\leq \sum_{j=1}^{p} G(t, t_j) \, L_{\mathcal{I}_j} \big[x(t_j) - x(t_j) \big]$$

$$\leq L_{\mathcal{A}} \, \|x - y\|_{PC} \,,$$

for all $t \in J$, where $L_{\mathcal{A}} = \sum_{j=1}^{p} M_G L_{\mathcal{I}_j} < 1$. Taking the supremum over t, we obtain

$$\|\mathcal{A}x - \mathcal{A}y\|_{PC} \leq L_{\mathcal{A}} \|x - y\|_{PC}$$

for all $x, y \in E$ with $x \succeq y$. Hence $\mathcal{A}$ is a partial contraction on E which also implies that $\mathcal{A}$ is partial continuous on E.

Step III: $\mathcal{B}$ *is partially continuous on* E.

Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in a chain C such that $x_n \to x$, for all $n \in \mathbb{N}$. Then

$$\lim_{n \to \infty} \mathcal{B}x_n(t) = \lim_{n \to \infty} \int_0^T G(t, s) f(s, x_n(s)) \, ds$$

$$= \int_0^T G(t, s) \left[\lim_{n \to \infty} f(s, x_n(s)) \right] \, ds$$

$$= \int_0^T G(t, s) f(s, x(s)) \, ds = \mathcal{B}x(t),$$

for all $t \in J$. This shows that $\mathcal{B}x_n$ converges to $\mathcal{B}x$ pointwise on J.

Now, we show that $\{\mathcal{B}x_n\}_{n \in \mathbb{N}}$ is a quasi-equicontinuous sequence of functions in E. Let $\tau_1, \tau_2 \in (t_{j-1}, t_j] \cap J$. Then, we have

$$\left| \mathcal{B}x_n(\tau_1) - \mathcal{B}x_n(\tau_2) \right|$$

$$= \left| \int_0^T G(\tau_1, s) f(s, x_n(s)) \, ds - \int_0^T G(\tau_2, s) f(s, x_n(s)) \, ds \right|$$

$$\leq \left| \int_0^{\tau_1} f(s, x_n(s)) e^{-(H(\tau_1) - H(s))} \, ds - \int_0^{\tau_2} f(s, x_n(s)) e^{-(H(\tau_2) - H(s))} \, ds \right|$$

$$+ \frac{e^{-H(T)}}{1 - e^{-H(T)}} \int_0^T \left| f(s, x_n(s)) \left[e^{-(H(\tau_1) - H(s))} - e^{-(H(\tau_2) - H(s))} \right] \right| \, ds$$

$$\leq M_f \left| \int_0^{\tau_1} e^{-(H(\tau_1) - H(s))} \, ds - \int_0^{\tau_2} e^{-(H(\tau_2) - H(s))} \, ds \right|$$

$$+ \frac{M_f e^{-H(T)}}{1 - e^{-H(T)}} \int_0^T \left| \left[e^{-(H(\tau_1) - H(s))} - e^{-(H(\tau_2) - H(s))} \right] \right| \, ds$$

$$\leq M_f \left| \int_0^{t_j} \left| e^{-(H(\tau_1) - H(s))} - e^{-(H(\tau_2) - H(s))} \right| \, ds \right|$$

$$+ M_f \left| \int_{\tau_2}^{\tau_1} \left| e^{-(H(\tau_2) - H(s))} \right| \, ds \right|$$

$$+ \frac{M_f e^{-H(T)}}{1 - e^{-H(T)}} \int_0^T \left| \left[e^{-(H(\tau_1) - H(s))} - e^{-(H(\tau_2) - H(s))} \right] \right| \, ds$$

$$\to 0 \quad \text{as} \quad \tau_1 \to \tau_2,$$

uniformly for all $n \in \mathbb{N}$. This shows that the $\{\mathcal{B}x_n\}$ is a quasi-equicontinuous sequence of functions and so convergence $\mathcal{B}x_n \to \mathcal{B}x$ is uniform and hence $\mathcal{B}$ is partially continuous on E (see Lemma 2.4 of Bainov and Simeonov [1] and Li *et al* [22] for the details).

Step IV: $\mathcal{B}$ *is partially compact operator on* E.

Let C be an arbitrary chain in E. We show that $\mathcal{B}(C)$ is uniformly bounded and quasi-equicontinuous set in E. First we show that $\mathcal{B}(C)$ is uniformly bounded. Let $y \in \mathcal{B}(C)$ be any element. Then there is an element $x \in C$ such that $y = \mathcal{B}x$. By hypothesis (H_3),

$$|y(t)| = |\mathcal{B}x(t)|$$
$$= \left| \int_0^t G(t,s)f(s,x(s))\, ds \right|$$
$$\leq \int_0^T |G(t,s)|\, |f(s,x(s))|\, ds$$
$$\leq M_G M_f T = r,$$

for all $t \in J$. Taking the supremum over t we obtain $\|y\|_{PC} \leq \|\mathcal{B}x\|_{PC} \leq r$, for all $y \in \mathcal{B}(C)$. Hence $\mathcal{B}(C)$ is uniformly bounded subset of functions E. Next we show that $\mathcal{B}(C)$ is a quasi-equicontinuous set in E. Let Let $\tau_1, \tau_2 \in (t_{j-1}, t_j] \cap J$. Then proceeding as in the Step II, it can be shown that $\mathcal{B}(C)$ is an quasi-equicontinuous subset of functions in E. So $\mathcal{B}(C)$ is a uniformly bounded and quasi-equicontinuous family of functions in E and hence it is compact in view of Arzelá-Ascoli theorem (see Bainov [1] and Li *et al* [22] for the details). Consequently $\mathcal{B} : E \to E$ is a partially compact operator of E into itself.

Step V: *u is a lower impulsive solution of the operator equation $x = \mathcal{A}x + \mathcal{B}x$.*

By hypothesis (H_4), the IDE (1) has a lower impulsive solution u defined on J. Then, we have

$$\left.\begin{aligned} u'(t) + h(t)u(t) &\leq f(t, u(t)), \ \ t \in J \setminus \{t_1, \ldots, t_p\}, \\ u(t_j^+) - u(t_j^-) &\leq \mathcal{I}_j(u(t_j)), \ j = 1, \ldots, p, \\ u(0) &\leq u(T). \end{aligned}\right\} \tag{24}$$

Now, by a direct application of the impulsive differential inequality established in Lemma 3.4 yields that

$$u(t) \leq \sum_{j=1}^p G(t, t_j)\mathcal{I}_j(u(t_j)) + \int_0^T G(t,s)f(s, u(s))\, ds \tag{25}$$

for $t \in J$. Furthermore, from definitions of the operators $\mathcal{A}$ and $\mathcal{B}$ it follows that $u(t) \leq \mathcal{A}u(t) + \mathcal{B}u(t)$ for all $t \in J$. Hence $u \preceq \mathcal{A}u + \mathcal{B}u$. Thus the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.2 and so the operator equation $\mathcal{A}x + \mathcal{B}x = x$ has a impulsive solution. Consequently the integral equation and a fortiori, the IDE (1) has a impulsive solution x^* defined on J. Furthermore, the sequence $\{x_n\}_{n=0}^\infty$ of successive approximations defined by (19) converges monotonically to x^*. This completes the proof. $\qquad\square$

Next, we prove the uniqueness theorem for the IDE on the interval J.

Theorem 3.2. *Suppose that hypotheses (H_1)-(H_3) and (H_5)-(H_6) hold. Furthermore, if $M_G \left(\sum_{j=1}^p L_{\mathcal{I}_j} + L_f T \right) < 1$, then the IDE (1) has a unique impulsive solution solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by (19) converges monotonically to x^*.*

Proof. Set $E = PC(J, \mathbb{R})$. Then, every pair of elements in $PC(J, \mathbb{R})$ has a lower bound as well as an upper bound so it is a lattice with respect to the order relation $\preceq$ in E.

Now, by Lemma 3.3, the IDE (1) is equivalent to the nonlinear impulsive integral equation (20). Define two operators $\mathcal{A}$ and $\mathcal{B}$ on E by (21) and (22). Now, consider the mapping $\mathcal{T} : E \to E$ defined by

$$\mathcal{T}x(t) = \mathcal{A}x(t) + \mathcal{B}x(t), \ t \in J. \tag{26}$$

Then the impulsive integral equation (10) is reduced to the operator equation as

$$\mathcal{T}x(t) = x(t), \ t \in J. \tag{27}$$

Now, proceeding with the arguments as in the proof of Theorem 3.1 it can shown that the operator $\mathcal{A}$ is a partial Lipschitzian with Lipschitz constant $L_{\mathcal{A}} = \sum_{j=1}^{p} M_G L_{\mathcal{I}_j}$. Similarly, we show that $\mathcal{B}$ is also a Lipschitzian on E into itself. Let $x, y \in E$ be such that $x \succeq y$. Then, by hypothesis (H$_5$), one has

$$\begin{aligned}
\left| \mathcal{B}x(t) - \mathcal{B}y(t) \right| &= \left| \int_0^T G(t,s) f(s, x(s)) \, ds - \int_0^T G(t,s) f(s, y(s)) \, ds \right| \\
&\leq \int_0^T |G(t,s)| \left| f(s, x(s)) - f(s, y(s)) \right| ds \\
&\leq L_f \int_0^T M_G \left| x(t) - y(t) \right| ds \\
&\leq M_G L_f T \, \|x - y\|_{PC}
\end{aligned}$$

for all $t \in J$ and $x, y \in E$. Taking the supremum over t in the above inequality, we obtain

$$\|\mathcal{B}x - \mathcal{B}y\|_{PC} \leq L_{\mathcal{B}} \, \|x - y\|_{PC}$$

for all $x, y \in E$, $x \succeq y$, where $L_{\mathcal{B}} = M_G L_f T$. This shows that $\mathcal{B}$ is again a partial Lipschitzian operator on E into itself with a Lipschitz constant $L_{\mathcal{B}}$. Next, by definition of the operator $\mathcal{T}$, one has

$$\|\mathcal{T}x - \mathcal{T}y\|_{PC} \leq \|\mathcal{A}x - \mathcal{A}y\|_{PC} + \|\mathcal{B}x - \mathcal{B}y\|_{PC} \leq (L_{\mathcal{A}} + L_{\mathcal{B}}) \, \|x - y\|_{PC}$$

for all $x, y \in E$, $x \succeq y$, where $L_{\mathcal{A}} + L_{\mathcal{B}} = M_G \left(\sum_{j=1}^{p} L_{\mathcal{I}_j} + L_f T \right) < 1$. Hence $\mathcal{T}$ is a partial contraction operator on E into itself. Since the hypothesis (H$_6$) holds, it is proved as in the step V of the proof of Theorem 3.1 that the operator equation (25) has a lower solution u in E. Then, by an application of Theorem 2.1, we obtain that the operator equation (27) and consequently the IDE (1) has a unique impulsive solution x^* and the sequence $\{x_n\}$ of successive approximations defined by (19) converges monotonically to x^*. This competes the proof. $\qquad\square$

Remark 3.1. The conclusion of Theorems 3.1 and 3.2 also remains true if we replace the hypothesis (H$_6$) with the following one.

(H$_7$) The IDE (1) has an upper impulsive solution $v \in PC^1(J, \mathbb{R})$.

The proofs of the existence and attractivity theorems for the IDE (1) under this new hypothesis are obtained using the arguments similar to Theorems 3.1 and 3.2 with appropriate modifications. In this case we invoke the use of Lemma 3.5 in the proofs.

Example 3.1. Given the interval $J = [0,1]$ of the real line $\mathbb{R}$ and given the points $t_1 = \frac{1}{5}$, $t_2 = \frac{2}{5}$, $t_3 = \frac{3}{5}$ and $t_4 = \frac{4}{5}$ in $[0,1]$, consider the periodic boundary value problem (in short PBVP) for the first order impulsive differential equations (in short IDE)

$$\left.\begin{aligned} x'(t) + x(t) &= \tanh x(t), \ \ t \in [0,1] \setminus \{t_1, t_2, t_3, t_4\}, \\ x(t_j^+) - x(t_j^-) &= \mathcal{I}_j(x(t_j)), \ j = 1, \ldots, 4, \\ x(0) &= x(1), \end{aligned}\right\} \tag{28}$$

for $t_j \in \{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\}$; where $x(t_j^-)$ and $x(t_j^-)$ are respectively, the right and left limit of x at $t = t_j$ such that $x(t_j) = x(t_j^-)$ and $\mathcal{I}_j(x(t_j))$ are the impulsive effects at the points $t = t_j$, $j = 1, \ldots, 4$ given by

$$\mathcal{I}_j(x) = \begin{cases} \dfrac{1}{2^{j+1}} \cdot \dfrac{x}{1+x} + 2, & \text{if } x > 0, \\[2mm] 2, & \text{if } x \leq 0, \end{cases}$$

for all $t \in [0,1]$. Here $f(t,x) = \tanh x$, so it is continuous and bounded on $[0,1] \times \mathbb{R}$ with bound $M_f = 1$. Again, the map $x \mapsto f(t,x)$ is nondecreasing for each $t \in [0,1]$. Next, the impulsive function $\mathcal{I}_j$ are continuous and bounded on $\mathbb{R}$ with bound $M_{\mathcal{I}_j} = 3$ for each $j = 1, \ldots, 4$. It is easy to verify that the impulsive operators $\mathcal{I}_j$ satisfy the hypothesis (H$_2$) with Lipschitz constants $L_{\mathcal{I}_j} = \dfrac{1}{2^{j+1}}$ for $j = 1, \ldots, 4$. In this case the Green's function $G_1(t,s)$ defined on $[0,1] \times [0,1]$ is given by

$$G_1(t,s) = \begin{cases} \dfrac{e^s}{e-1}, & 0 \leq s \leq t \leq 1, \\[3mm] \dfrac{e^{-t+s}}{e-1}, & 0 \leq t \leq s \leq 1, \end{cases}$$

It is obvious that, $M_G \sum_{j=1}^{4} L_{\mathcal{I}_j} = \dfrac{e}{e-1} \sum_{j=1}^{4} \dfrac{1}{2^{j+1}} < 1$. Finally, the functions

$$u(t) = 2 \sum_{j=1}^{4} G_1(t,t_j) - \int_0^1 G_1(t,s) \, ds$$

and

$$v(t) = 4 \sum_{j=1}^{4} G_1(t,t_j) + \int_0^1 G_1(t,s) \, ds$$

are respectively the lower and upper impulsive solutions of the IDE (28) defined on $[0,1]$. Thus, all the conditions of Theorem 3.1 are satisfied and so the IDE (28) has a impulsive solution ξ^* and the sequence $\{x_n\}$ of successive approximations defined by

$$x_0(t) = 2 \sum_{j=1}^{4} G_1(t,t_j) - \int_0^1 G_1(t,s) \, ds,$$

$$x_{n+1}(t) = \sum_{j=1}^{4} G_1(t,t_j) \mathcal{I}_j(x_n(t_j)) + \int_0^1 G_1(t,s) \tanh x_n(s) \, ds$$

for all $t \in J$, converges monotonically to x^*. Similarly, the sequence $\{y_n\}$ of successive approximations defined by

$$y_0(t) = 4 \sum_{j=1}^{4} G_1(t, t_j) + \int_0^1 G_1(t, s) \, ds,$$

$$y_{n+1}(t) = \sum_{j=1}^{4} G_1(t, t_j) \mathcal{I}_j(y_n(t_j)) + \int_0^1 G_1(t, s) \tanh y_n(s)) \, ds$$

for all $t \in J$, also converges monotonically to the impulsive solution y^* of the IDE (28) in view of Remark 3.1.

Remark 3.2. We note that if the IDE (1) has a lower impulsive solution $u \in PC^1(J, \mathbb{R})$ as well as an upper impulsive solution $v \in PC^1(J, \mathbb{R})$ such that $u \preceq v$, then under the given conditions of Theorem 3.1 it has corresponding impulsive solutions x_* and y^* and these impulsive solutions satisfy the inequality

$$u = x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq x_* \preceq y^* \preceq y_n \preceq \cdots \preceq y_1 \preceq y_0 = v.$$

Hence x_* and y^* are respectively the minimal and maximal impulsive solutions of the IDE (1) in the vector segment $[u, v]$ of the Banach space $E = PC(J, \mathbb{R})$, where the vector segment $[u, v]$ is a set of elements in $PC(J, \mathbb{R})$ defined by

$$[u, v] = \{x \in PC(J, \mathbb{R}) \mid u \preceq x \preceq v\}.$$

This is because of the order cone K defined by (5) is a closed set in $PC(J, \mathbb{R})$. A few details concerning the order relation by the order cones and the Janhavi sets in an ordered Banach space are given in Dhage [11].

Remark 3.3. In this paper we considered a very simple PBVP of nonlinear first order impulsive differential equation for the existence and approximation theorem via Dhage iteration method, however the same method may be extended to other complex and higher order PBVPs of nonlinear impulsive differential equations for obtaining the existence and approximation theorems along with algorithms for the approximate solution. In a forthcoming paper, we discuss the PBVPs of impulsive nonlinear second order ordinary differential equations for existence and approximate solution via the method of successive approximations.

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