Electronic Journal of Mathematical Analysis and Applications
Vol. 9(1) Jan. 2021, pp. 91-111
ISSN: 2090-729X(online)
http://math-frac.org/Journals/EJMAA/

SOME DEGREE BASED TOPOLOGICAL INDICES OF A GENERALISED F SUMS OF GRAPHS

LIJU ALEX , G. INDULAL

Abstract. The degree based structural descriptors such as Zagreb indices, Forgotten index are some important tools for the topological structural analysis of graphs. Various physio-chemical properties like boiling points, binding energy, $\pi-$ electron energy of alkanes and other hydrocarbons were extensively studied using these indices [7]. In [2] a new sum of two graphs is defined with the help of subdivision of edges and cartesian product and obtained the Weiner index of the resulting graph. In [4] the first and second zagreb indices and in [10] the Forgotten index of this new sums of graphs are obtained. In this paper we generalise the F sums of graphs using an arbitrary graph H and compute the First, Second Zagreb indices and the Forgotten index of the generalised F sums thus making the results in [4, 10] as particular cases when H is replaced by K_1. The results obtained are illustrated with some examples also.

1. Introduction

The study of structural descriptors associated with molecular graph is a formost area of importance in analysing chemical compounds. A topological index is a unique tool which associates the structural composition of graphs to a unique real number. From the introduction of the first topological index by H. Wiener [12] in 1947 called path number (which was later renamed as Wiener index) a plethora of research had been carried out in this area. Throughout the years, a lot of new topological indices which are invariant under graph isomorphism has been explored [6] based on distance between vertices and degrees of vertices. Though the distance based indices were introduced first, the degree based indices to name a few Randić index, Forgotten index, first Zagreb index, second Zagreb index, Generalised Zagreb index were found to be a lot more convenient as well as applicable in studying molecular graphs . In 1972 I. Gutman, N. Trinajstić [7] defined the first and second Zagreb indices. They introduced these indices as a an easier approximation in the computation of $\pi-$ electron energy of hydrocarbons, they also defined one another index with similar physical application. Although the first and second zagreb indices were studied extensively and found many applications in molecular graph theory,

2010 *Mathematics Subject Classification.*
Key words and phrases. first Zagreb index ($M_1(G)$), second Zagreb index ($M_2(G)$), Forgotten index ($F(G)$), F sum.
Submitted April 25, 2020.

the third index, doesn't followed up properly for a long time until recently in 2015 B. Furtula, I. Gutman in [3] rediscovered this third index and named it as Forgotten index and studied various properties of this index. The first Zagreb index, $M_1(G)$ is defined as the sum of squares of degrees of each vertices and the second zagreb index, $M_2(G)$ is the sum of the product of the degrees of the end vertices of every edge, and the Forgotten index $F(G)$ is the sum of the cubes of degrees of each vertices.

$$M_1(G) = \sum_{u \in V(G)} d(u)^2$$
$$M_2(G) = \sum_{uv \in E(G)} d(u)d(v)$$
$$F(G) = \sum_{u \in V(G)} d(u)^3 = \sum_{uv \in E(G)} d(u)^2 + d(v)^2$$

where $d(u)$ denote the degree of the vertex u. Some of the mathematical properties as well as physical application of these indices can be found in [6, 8, 9, 11]. Computing the structural descriptors of graph operations like cartesian product, strong product, tensor product, lexicographic product has been a subject of various studies. In [2] M.Eliasi, B. Taeri, computed the Wiener index of four new sums associated with the cartesian product. They considered the associated subdivision graphs in defining this sum which is referred as F-sum . We extend this definition of F sums by defining four subdivision graphs by introducing a new graph H corresponding to each edge and define an analogous generalised F sum. We also compute the first second Zagreb indices and Forgotten index of these four new generalised F sums. For the basics of graph theoretic terminologies see [1].

2. The graphs $S_H(G_1), R_H(G_1), Q_H(G_1), T_H(G_1)$ and the generalised $F-$sums

Let G_1, G_2 be two graphs with vertex set $V_1(G)$,$V_2(G)$ edge set $E_1(G)$ and $E_2(G)$ respectively. Let H be any graph, we define four subdivision related graphs associated with H as $S_H(G_1), R_H(G_1), Q_H(G_1), T_H(G_1)$ as follows.

(1) $S_H(G_1)$ is the graph obtained from G_1 by replacing each edge e_i of G_1 with a copy of H and making every vertex in the ith copy of H adjacent to the end vertices of e_i for each $e_i \in E(G_1)$. That is, $S_H(G_1)$ is a graph with vertex set $V(S_H(G_1)) = V(G_1) \bigcup V_e(H)$ where $V_e(H) = \cup_{i=1}^{|E(G_1)|} V_i(H)$, $V_i(H) = V(H)\ \forall i$ and the edge set $) = \{(v,h),(u,h) : e = vu \in E(G_1), h \in V_e(H)\} \cup E_e(H)$ where $E_e(H) = \cup_{i=1}^{|E(G_1)|} E_i(H)$, $E_i(H) = E(H)\ \forall i$

(2) $R_H(G_1)$ is the graph obtained from G_1 by replacing each edge e_i of G_1 with a copy of H and making every vertex in the ith copy of H adjacent to the end vertices of e_i for each $e_i \in E(G_1)$ also keeping every edge in G_1 as well. That is, $R_H(G_1)$ is a graph with vertex set $V(R_H(G_1)) = V(G_1) \bigcup V_e(H)$ and edge set $E(R_H(G_1)) = \{(v,h),(u,h) : e = vu \in E(G_1), h \in V_e(H)\} \cup E_e(H) \cup E(G_1)$, where $V_e(H) = \cup_{i=1}^{|E(G_1)|} V_i(H)$, $V_i(H) = V(H)\ \forall i$, $E_e(H) = \cup_{i=1}^{|E(G_1)|} E_i(H)$, $E_i(H) = E(H)\ \forall i$.

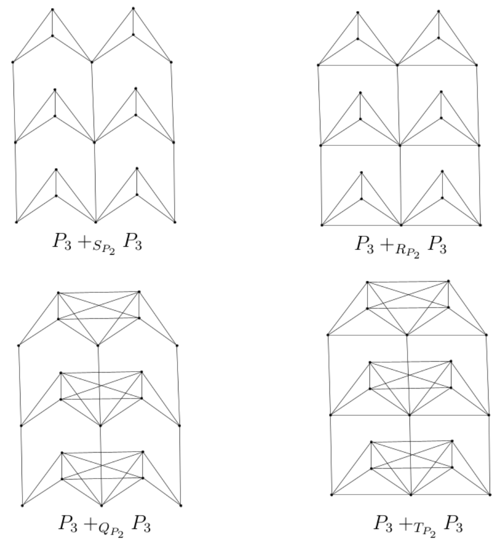

FIGURE 1.

(3) $Q_H(G_1)$ is the graph obtained from G_1 by replacing each edge e_i of G_1 with a copy of H and making every vertex in the ith copy of H adjacent to the end vertices of e_i for each $e_i \in E(G_1)$ along with edges joining all the vertices in the ith copy of H to all the vertices in the jth copy of H whenever e_i adjacent to e_j in G_1. That is, $Q_H(G_1)$ is a graph with vertex set $V(Q_H(G_1)) = V(G_1) \bigcup V_e(H)$ and edge set $E(Q_H(G_1)) = \{(v,h),(u,h) : e = vu \in E(G_1), h \in V_e(H)\} \cup E_e(H) \cup E(H_eVH_s)$ where $V_e(H) = \cup_{i=1}^{|E(G_1)|} V_i(H)$, $V_i(H) = V(H)\ \forall i$, $E(H_eVH_s) = \{(h_e,h_s) : h_e \in V(H_e), h_s \in V(H_s), e,s \textit{ are adjacent in } G_1\}$,$E_e(H) = \cup_{i=1}^{|E(G_1)|} E_i(H)$, $E_i(H) = E(H)\ \forall i$ and H_e, H_s are the copies of H corresponding to the edge $e,s \in E(G_1)$ and e,s are adjacent in G_1.

(4) $T_H(G_1)$ is the graph obtained from G_1 by replacing each edge e_i of G_1 with a copy of H and making every vertex in the ith copy of H adjacent to the end vertices of e_i for each $e_i \in E(G_1)$ along with edges joining all the vertices in the ith copy of H to all the vertices in the jth copy of H whenever e_i adjacent to e_j in G_1 and keeping every edge of G_1 as well . That is, $T_H(G_1)$ is a graph with vertex set $V(T_H(G_1)) = V(G_1) \bigcup V_e(H)$ and edge set $E(T_H(G_1)) = E(G_1) \cup \{(v,h),(u,h) : e = vu \in E(G_1), h \in V(H)\} \cup E_e(H) \cup E(G_1) \cup E(H_eVH_s)$ where $V_e(H) = \cup_{i=1}^{|E(G_1)|} V_i(H)$, $V_i(H) = V(H)\ \forall i$, , $E_e(H) = \cup_{i=1}^{|E(G_1)|} E_i(H)$, $E_i(H) = E(H)\ \forall i$, $E(H_eVH_s) = \{(h_e,h_s) : h_e \in V(H_e), h_s \in V(H_s), e,s \textit{ are adjacent in } G_1\}$ and H_e, H_s are the copies of H corresponding to the edge $e,s \in E(G_1)$ and e,s are adjacent in G_1. $T_H(G_1)$ is called the total graph associated with H.

Corresponding to the four new graphs $S_H(G_1), R_H(G_1), Q_H(G_1), T_H(G_1)$, we define four new sums called generalised F sums associated with the graph H as follows.

Let G_1 and G_2 be two connected graphs and H be any graph. Let F_H be any one of the symbols S_H, R_H, Q_H, T_H. The generalised F sum of G_1 and G_2 is denoted by $G_1 +_{F_H} G_2$, is graph with vertex set $V(G_1 +_{F_H} G_2) = V(F_H(G_1)) \times V(G_2)$ and the edge set $E(G_1 +_{F_H} G_2) = \{(a,b)(c,d) : a = c \in V(G_1)\ and\ bd \in E(G_2)\ or\ ac \in E(F_H(G_1))\ and\ b = d \in V(G_2)\}$. Figure 1 is an example with $G_1, G_2 = P_3$ and $H = P_2$.

3. Zagreb index of the generalised F sum

In this section we derive the expression for the first and second Zagreb indices of the generalised F sum in terms of the Zagreb indices of the factor graphs.

Theorem 1. *Let G_1 and G_2 be two connected graphs and H be any graph. Then*

a. $M_1(G_1+_{S_H}G_2) = |V(G_2)|M_1(S_H(G_1))+|V(G_1)|M_1(G_2)+8||E(G_1)||E(G_2)||V(H)|$
b. $M_2(G_1+_{S_H}G_2) = |V(G_2)|M_2(S_H(G_1))+|E(G_2)||V(H)|^2M_1(G_1)+2|E(G_1)||V(H)|M_1(G_2)+ |V(G_1)|M_2(G_2) + 8|E(G_2)|\left(||E(S_H(G_1))| - |V(H)||E(G_1)|\right)$

Proof. From the definition of first zagreb index, we have

$$\begin{aligned}
M_1(G_1 +_{S_H} G_2) &= \sum_{(u,v)\in V(G_1+_{S_H}G_2)} d^2_{(G_1+_{S_H}G_2)}(u,v) \\
&= \sum_{(u,v)(x,y)\in E(G_1+_{S_H}G_2)} \Big(d_{(G_1+_{S_H}G_2)}(u,v) + d_{(G_1+_{S_H}G_2)}(x,y)\Big) \\
&= \sum_{u\in V(G_1)} \sum_{vy\in E(G_2)} \Big(d_{(G_1+_{S_H}G_2)}(u,v) + d_{(G_1+_{S_H}G_2)}(u,y)\Big) \\
&+ \sum_{v\in V(G_2)} \sum_{ux\in E(S_H(G_1))} \Big(d_{(G_1+_{S_H}G_2)}(u,v) + d_{(G_1+_{S_H}G_2)}(x,v)\Big)
\end{aligned}$$

Now we find the values of the each parts in the sum one by one. First we consider the sum in which $u \in V(G_1)$ and $vy \in E(G_2)$.

$$\begin{aligned}
&\sum_{u\in V(G_1)} \sum_{vy\in E(G_2)} \Big(d_{(G_1+_{S_H}G_2)}(u,v) + d_{(G_1+_{S_H}G_2)}(u,y)\Big) \\
&= \sum_{u\in V(G_1)} \sum_{vy\in E(G_2)} \big[d_{S_H(G_1)}(u) + d_{G_2}(v) + \big(d_{S_H(G_1)}(u) + d_{G_2}(y)\big)\big] \\
&= \sum_{u\in V(G_1)} \sum_{vy\in E(G_2)} \big[2d_{S_H(G_1)}(u) + d_{G_2}(v) + d_{G_2}(y)\big] \\
&= \sum_{u\in V(G_1)} [2|E(G_2)||V(H)|d_{G_1}(u) + M_1(G_2)] = 4|E(G_1)||E(G_2)||V(H)| + |V(G_1)|M_1(G_2)
\end{aligned}$$

Now for each edge $ux \in E(S_H(G_1))$, $v \in V(G_2)$. We divide the sum into two parts, $ux \in E(S_H(G_1)), u \in V(G_1), x \in V_e(H)$ and $ux \in E(S_H(G_1)), u, x \in V_e(H)$.

$$\sum_{v\in V(G_2)} \sum_{ux\in E(S_H(G_1))} \Big(d_{(G_1+_{S_H}G_2)}(u,v) + d_{(G_1+_{S_H}G_2)}(x,v)\Big)$$

$$= \sum_{v\in V(G_2)} \sum_{ux\in E(S_H(G_1))u\in V(G_1)x\in V_e(H)} \big[\big(d_{S_H(G_1)}(u) + d_{G_2}(v)\big) + d_{S_H(G_1)}(x)\big]$$

$$+ \sum_{v\in V(G_2)} \sum_{ux\in E(S_H(G_1))u,x\in V_e(H)} \big[\big(d_{S_H(G_1)}(u)\big) + d_{S_H(G_1)}(x)\big]$$

$$= \sum_{v\in V(G_2)} (2|E(G_1)||V(H)|d_{G_2}(v) + M_1(S_H(G_1)) = 4|E(G_1)||V(H)||E(G_2)| + |V(G_2)|M_1(S_H(G_1)$$

From the expressions we obtain

$$M_1(G_1 +_{S_H} G_2) = |V(G_2)|M_1(S_H(G_1) + |V(G_1)|M_1(G_2) + 8||E(G_1)||E(G_2)|V(H)|$$

Next Consider

$$\begin{aligned} M_2(G_1 +_{S_H} G_2) &= \sum_{(u,v)(x,y)\in E(G_1+_{S_H}G_2)} \Big(d_{G_1+_{S_H}G_2}(u,v)d_{G_1+_{S_H}G_2}(x,y)\Big) \\ &= \sum_{u\in V(G_1)} \sum_{vy\in E(G_2)} \Big(d_{G_1+_{S_H}G_2}(u,v)d_{G_1+_{S_H}G_2}(u,y)\Big) \\ &+ \sum_{v\in V(G_2)} \sum_{ux\in E(S_H(G_1))} \Big(d_{G_1+_{S_H}G_2}(u,v)d_{G_1+_{S_H}G_2}(x,v)\Big) \end{aligned}$$

First part of the sum is

$$\sum_{u\in V(G_1)} \sum_{vy\in E(G_2)} \Big(d_{(G_1+_{S_H}G_2)}(u,v)d_{(G_1+_{S_H}G_2)}(u,y)\Big)$$

$$= \sum_{u\in V(G_1)} \sum_{vy\in E(G_2)} \big[d_{S_H(G_1)}(u) + d_{G_2}(v)\big]\big[d_{S_H(G_1)}(u) + d_{G_2}(y)\big]$$

$$= \sum_{u\in V(G_1)} \sum_{vy\in E(G_2)} \big[|V(H)|^2 d_{G_1}(u)^2 + |V(H)|d_{G_1}(u)\left(d_{G_2}(v) + d_{G_2}(y)\right) + d_{G_2}(v)d_{G_2}(y)\big]$$

$$= |E(G_2)||V(H)|^2 M_1(G_1) + 2|E(G_1)||V(H)|M_1(G_2) + |V(G_1)|M_2(G_2)$$

The next part is,

$$\sum_{v\in V(G_2)} \sum_{ux\in E(S_H(G_1))} \Big(d_{(G_1+_{S_H}G_2)}(u,v)d_{(G_1+_{S_H}G_2)}(x,v)\Big)$$

$$= \sum_{v\in V(G_2)} \sum_{ux\in E(S_H(G_1)),u\in V(G_1),x\in V_e(H)} \big[\big(d_{S_H(G_1)}(u) + d_{G_2}(v)\big)d_{S_H(G_1)}(x)\big]$$

$$+ \sum_{v\in V(G_2)} \sum_{ux\in E(S_H(G_1)),u,x\in V_e(H)} \big[d_{S_H(G_1)}(u)d_{S_H(G_1)}(x)\big]$$

$$= \sum_{v\in V(G_2)} \left(4\left(|E(S_H(G_1))| - |V(H)|E(G_1)\right)d_{G_2}(v) + M_2(S_H(G_1))\right)$$

$$= 8\left(|E(S_H(G_1))| - |V(H)|E(G_1)\right)|E(G_2)| + |V(G_2)|M_2(S_H(G_1))$$

Thus we obtain,

$$\begin{aligned}M_2(G_1 +_{S_H} G_2) &= |V(G_2)|M_2(S_H(G_1)) + |E(G_2)||V(H)|^2 M_1(G_1) + 2|E(G_1)||V(H)|M_1(G_2) + |V(G_1)|M_2(G_2)\\ &+ 8\left(|E(S_H(G_1))| - |V(H)||E(G_1)|\right)|E(G_2)|\end{aligned}$$

□

Theorem 2. *Let G_1 and G_2 be two connected graphs and H be any graph. Then*

a. $M_1(G_1 +_{R_H} G_2) = |V(G_2)|M_1(R_H(G_1)) + |V(G_1)|M_1(G_2) + 8|E(G_1)||E(G_2)|\left(|V(H)| + 1\right)$
b. $M_2(G_1 +_{R_H} G_2) = |V(G_2)|M_2(R_H(G_1)) + |V(G_1)|M_2(G_2)$
$+|E(G_2)|\Big((|V(H)| + 1)^2 + 2(|V(H)| + 1)\Big) M_1(G_1) + |E(G_1)|\left(2(|V(H)| + 1) + 1\right) M_1(G_2)$
$+ 8|E(G_2)|\left(|E(R_H(G_1))| - (|V(H)| + 1)|E(G_1)|\right)$

Proof. Using the definition,

$$\begin{aligned}M_1(G_1 +_{R_H} G_2) &= \sum_{(u,v)\in V(G_1+_{R_H}G_2)} d^2_{(G_1+_{R_H}G_2)}(u,v)\\
&= \sum_{(u,v)(x,y)\in E(G_1+_{R_H}G_2)} \Big(d_{(G_1+_{R_H}G_2)}(u,v) + d_{(G_1+_{R_H}G_2)}(x,y)\Big)\\
&= \sum_{u\in V(G_1)}\sum_{vy\in E(G_2)} \Big(d_{(G_1+_{R_H}G_2)}(u,v) + d_{(G_1+_{R_H}G_2)}(u,y)\Big)\\
&+ \sum_{v\in V(G_2)}\sum_{ux\in E(R_H(G_1)} \Big(d_{(G_1+_{R_H}G_2)}(u,v) + d_{(G_1+_{R_H}G_2)}(x,v)\Big)\end{aligned}$$

Now we find the values of the each part in the sum separately. First we consider the sum in which $u \in V(G_1)$ and $vy \in E(G_2)$, We also use the relation $d_{R_H(G_1)}(u) = (|V(H)| + 1)\, d_{G_1}(u)$

$$\begin{aligned}&\sum_{u\in V(G_1)}\sum_{vy\in E(G_2)} \Big(d_{(G_1+_{R_H}G_2)}(u,v) + d_{(G_1+_{R_H}G_2)}(u,y)\Big)\\
&= \sum_{u\in V(G_1)}\sum_{vy\in E(G_2)} \Big[\big(d_{R_H(G_1)}(u) + d_{G_2}(v)\big) + \big(d_{R_H(G_1)}(u) + d_{G_2}(y)\big)\Big]\\
&= \sum_{u\in V(G_1)}\sum_{vy\in E(G_2)} \Big[2d_{R_H(G_1)}(u) + d_{G_2}(v) + d_{G_2}(y)\Big]\\
&= \sum_{u\in V(G_1)}\sum_{vy\in E(G_2)} \left[2\left(|V(H)| + 1\right) d_{G_1}(u) + d_{G_2}(v) + d_{G_2}(y)\right]\\
&= \sum_{u\in V(G_1)} \left[2|E(G_2)|\left(|V(H)| + 1\right) d_{G_1}(u) + M_1(G_2)\right]\\
&= 4|E(G_1)||E(G_2)|\left(|V(H)| + 1\right) + |V(G_1)|M_1(G_2)\end{aligned}$$

Now for each edge $ux \in E(R_H(G_1))$, $v \in V(G_2)$.

$$\sum_{v\in V(G_2)}\sum_{ux\in E(S_H(G_1))} \Big(d_{(G_1+_{S_H}G_2)}(u,v) + d_{(G_1+_{S_H}G_2)}(x,v)\Big) \qquad (1)$$

Let $D = \Big(d_{(G_1+_{F_H}G_2)}(u,v) + d_{(G_1+_{F_H}G_2)}(x,v)\Big)$ where $F_H = S_H$ *or* R_H *or* Q_H *or* T_H.
We divide the following sum into three parts as

$$\sum_{v\in V(G_2)}\sum_{ux\in E(R_H(G_1))} D = \sum_{v\in V(G_2)}\sum_{ux\in E(R_H(G_1)),u,x\in V(G_1)} D \tag{2}$$

$$+\sum_{v\in V(G_2)}\sum_{ux\in E(R_H(G_1)),u\in V(G_1),x\in V_e(H)} D \tag{3}$$

$$+\sum_{v\in V(G_2)}\sum_{ux\in E(R_H(G_1)),u,x\in V_e(H)} D \tag{4}$$

Now we calculate the value of these three expressions separately

$$\sum_{v\in V(G_2)}\sum_{ux\in E(R_H(G_1)),u,x\in V(G_1)} D = \sum_{v\in V(G_2)}\sum_{ux\in E(R_H(G_1)),u,x\in V(G_1)} \Big(d_{(G_1+_{R_H}G_2)}(u,v) + d_{(G_1+_{R_H}G_2)}(x,v)\Big)$$

$$= \sum_{v\in V(G_2)}\sum_{ux\in E(R_H(G_1))u,x\in V(G_1)} \Big[\big(d_{R_H(G_1)}(u) + d_{G_2}(v)\big) + \big(d_{R_H(G_1)}(x) + d_{G_2}(v)\big)\Big]$$

$$= \sum_{v\in V(G_2)}\sum_{ux\in E(R_H(G_1))u,x\in V(G_1)} \Big[d_{R_H(G_1)}(u) + d_{R_H(G_1)}(x) + 2d_{G_2}(v)\Big]$$

$$= \sum_{v\in V(G_2)}\sum_{ux\in E(R_H(G_1))u,x\in V(G_1)} \Big[d_{R_H(G_1)}(u) + d_{R_H(G_1)}(x)\Big] + 4|E(G_1)||E(G_2)|$$

Also, Consider the case where $ux \in E(R_H(G_1)), u \in V(G_1), x \in V_e(H)$

$$\sum_{v\in V(G_2)}\sum_{ux\in E(R_H(G_1)),u\in V(G_1),x\in V_e(H)} D$$

$$= \sum_{v\in V(G_2)}\sum_{ux\in E(R_H(G_1)),u\in V(G_1),x\in V_e(H)} \Big(d_{(G_1+_{R_H}G_2)}(u,v) + d_{(G_1+_{R_H}G_2)}(x,v)\Big)$$

$$= \sum_{v\in V(G_2)}\sum_{ux\in E(R_H(G_1)),u\in V(G_1),x\in V_e(H)} \big(d_{R_H(G_1)}(u) + d_{G_2}(v) + d_{R_H(G_1)}(x)\big)$$

$$= \sum_{v\in V(G_2)}\sum_{ux\in E(R_H(G_1)),u\in V(G_1),x\in V_e(H)} \big(d_{R_H(G_1)}(u) + d_{R_H(G_1)}(x)\big) + 4|E(G_1)||E(G_2)||V(H)|$$

Now consider the sum

$$\sum_{v\in V(G_2)}\sum_{ux\in E(R_H(G_1)),u,x\in V(H)} D$$

$$= \sum_{v\in V(G_2)}\sum_{ux\in E(R_H(G_1)),u,x\in V(H)} \Big(d_{(G_1+_{R_H}G_2)}(u,v) + d_{(G_1+_{R_H}G_2)}(x,v)\Big)$$

$$= \sum_{v\in V(G_2)}\sum_{ux\in E(R_H(G_1)),u,x\in V(H)} d_{R_H(G_1)}(u) + d_{R_H(G_1)}(x)$$

From the expressions, we obtain

$$M_1(G_1 +_{R_H} G_2) = |V(G_2)|M_1(R_H(G_1)) + |V(G_1)|M_1(G_2) + 8|E(G_1)||E(G_2)|\,(|V(H)| + 1)$$

Similarly,

$$\begin{aligned} M_2(G_1 +_{R_H} G_2) &= \sum_{(u,v)(x,y)\in E(G_1+_{R_H}G_2)} \Big(d_{G_1+_{R_H}G_2}(u,v) d_{G_1+_{R_H}G_2}(x,y) \Big) \\ &= \sum_{u\in V(G_1)} \sum_{vy\in E(G_2)} \Big(d_{G_1+_{R_H}G_2}(u,v) d_{G_1+_{R_H}G_2}(u,y) \Big) \\ &+ \sum_{v\in V(G_2)} \sum_{ux\in E(R_H(G_1)} \Big(d_{(G_1+_{R_H}G_2}(u,v) d_{(G_1+_{R_H}G_2}(x,v) \Big) \end{aligned}$$

Now we separately find the sums,

$$\begin{aligned} & \sum_{u\in V(G_1)} \sum_{vy\in E(G_2)} \Big(d_{G_1+_{R_H}G_2}(u,v) d_{G_1+_{R_H}G_2}(u,y) \Big) \\ &= \sum_{u\in V(G_1)} \sum_{vy\in E(G_2)} \left[\left(d_{R_H(G_1)}(u) + d_{G_2}(v) \right) \left(d_{R_H(G_1)}(u) + d_{G_2}(y) \right) \right] \\ &= \sum_{u\in V(G_1)} \sum_{vy\in E(G_2)} \left[d_{R_H(G_1)}(u)^2 + d_{R_H(G_1)}(u) \left(d_{G_2}(v) + d_{G_2}(y) \right) + d_{G_2}(v) d_{G_2}(y) \right] \\ &= \sum_{u\in V(G_1)} \sum_{vy\in E(G_2)} \Big[(|V(H)|+1)^2 d_{G_1}(u)^2 + (|V(H)|+1) d_{G_1}(u) \left(d_{G_2}(v) + d_{G_2}(y) \right) + d_{G_2}(v) d_{G_2}(y) \Big] \\ &= |E(G_2)| (|V(H)|+1)^2 M_1(G_1) + 2|E(G_1)| (|V(H)|+1) M_1(G_2) + |V(G_1)| M_2(G_2) \end{aligned}$$

Let $T = \Big(d_{(G_1+_{F_H}G_2)}(u,v) d_{(G_1+_{F_H}G_2)}(x,v) \Big)$, Where $F_H = S_H$ *or* R_H *or* Q_H *or* T_H.
We divide the following sum into three parts as

$$\sum_{v\in V(G_2)} \sum_{ux\in E(R_H(G_1))} T = \sum_{v\in V(G_2)} \sum_{ux\in E(R_H(G_1)),u,x\in V(G_1)} T \tag{5}$$

$$+ \sum_{v\in V(G_2)} \sum_{ux\in E(R_H(G_1)),u\in V(G_1),x\in V_e(H)} T \tag{6}$$

$$+ \sum_{v\in V(G_2)} \sum_{ux\in E(R_H(G_1)),u,x\in V_e(H)} T \tag{7}$$

Now we calculate the value of these three expressions separately

$$\begin{aligned} & \sum_{v\in V(G_2)} \sum_{ux\in E(R_H(G_1)),u,x\in V(G_1)} T \\ &= \sum_{v\in V(G_2)} \sum_{ux\in E(R_H(G_1)),u,x\in V(G_1)} \left[\left(d_{R_H(G_1)}(u) + d_{G_2}(v) \right) \left(d_{R_H(G_1)}(x) + d_{G_2}(v) \right) \right] \\ &= \sum_{v\in V(G_2)} \sum_{ux\in E(R_H(G_1)),u,x\in V(G_1)} \left[d_{R_H(G_1)}(u) d_{R_H(G_1)}(x) + d_{G_2}(v) \left(d_{R_H(G_1)}(u) + d_{R_H(G_1)}(x) \right) + d_{G_2}(v)^2 \right] \\ &= \sum_{v\in V(G_2)} \sum_{ux\in E(R_H(G_1)),u,x\in V(G_1)} d_{R_H(G_1)}(u) d_{R_H(G_1)}(x) + 2|E(G_2)| (|V(H)|+1) M_1(G_1) + |E(G_2)| M_1(G_2) \end{aligned}$$

Now,

$$\sum_{v\in V(G_2)}\sum_{ux\in E(R_H(G_1)),u\in V(G_1),x\in V_e(H)} T$$
$$=\sum_{v\in V(G_2)}\sum_{ux\in E(R_H(G_1)),u\in V(G_1),x\in V_e(H)} \left(d_{R_H(G_1)}(u)+d_{G_2}(v)\right)d_{R_H(G_1)}(x)$$
$$=\sum_{v\in V(G_2)}\sum_{ux\in E(R_H(G_1)),u\in V(G_1),x\in V_e(H)} d_{R_H(G_1)}(u)d_{R_H(G_1)}(x)+8|E(G_2)|\left(|E(R_H(G_1))|-(|V(H)|+1)|E(G_1)|\right)$$

Now consider the sum

$$\sum_{v\in V(G_2)}\sum_{ux\in E(R_H(G_1)),u,x\in V_e(H)} T=\sum_{v\in V(G_2)}\sum_{ux\in E(R_H(G_1)),u,x\in V_e(H)} d_{R_H(G_1)}(u)d_{R_H(G_1)}(x)$$

Now collecting all the previous terms, the value is

$$\begin{aligned}M_2(G_1+_{R_H}G_2)=&|V(G_2)|M_2(R_H(G_1))+|V(G_1)|M_2(G_2)+|E(G_2)|\left((|V(H)|+1)^2+2(|V(H)|+1)\right)M_1(G_1)\\&+|E(G_1)|\left(2(|V(H)|+1)+1\right)M_1(G_2)+8|E(G_2)|\left(|E(R_H(G_1))|-(|V(H)|+1)|E(G_1)|\right)\end{aligned}$$

□

Theorem 3. *Let G_1 and G_2 be two connected graphs and H be any graph. Then*

a. $M_1(G_1+_{Q_H}G_2)=|V(G_2)|M_1(Q_H(G_1))+|V(G_1)|M_1(G_2)+8|E(G_1)||E(G_2)||V(H)|$

b. $M_2(G_1+_{Q_H}G_2)=|V(G_2)|M_2(Q_H(G_1))+|V(G_1)|M_2(G_2)+|E(G_2)||V(H)|^2M_1(G_1)+2|E(G_1)||V(H)|M_1(G_2)+8|E(G_2)|\left(|E(Q_H(G_1))|-|E(G_1)||V(H)|\right)$

Proof. Using the definition,

$$\begin{aligned}M_1(G_1+_{Q_H}G_2)&=\sum_{(u,v)\in V(G_1+_{S_H}G_2)} d^2_{(G_1+_{Q_H}G_2)}(u,v)\\&=\sum_{(u,v)(x,y)\in E(G_1+_{Q_H}G_2)}\left(d_{(G_1+_{Q_H}G_2)}(u,v)+d_{(G_1+_{Q_H}G_2)}(x,y)\right)\\&=\sum_{u\in V(G_1)}\sum_{vy\in E(G_2)}\left(d_{(G_1+_{Q_H}G_2)}(u,v)+d_{(G_1+_{Q_H}G_2)}(u,y)\right)\\&+\sum_{v\in V(G_2)}\sum_{ux\in E(Q_H(G_1))}\left(d_{(G_1+_{Q_H}G_2)}(u,v)+d_{(G_1+_{Q_H}G_2)}(x,v)\right)\end{aligned}$$

Now we find the values of the each part in the sum separately. First we consider the sum in which $u\in V(G_1)$ and $vy\in E(G_2)$.

$$\begin{aligned}&\sum_{u\in V(G_1)}\sum_{vy\in E(G_2)}\left(d_{(G_1+_{Q_H}G_2)}(u,v)+d_{(G_1+_{Q_H}G_2)}(u,y)\right)\\&=\sum_{u\in V(G_1)}\sum_{vy\in E(G_2)}\left[\left(d_{Q_H(G_1)}(u)+d_{G_2}(v)\right)+\left(d_{Q_H(G_1)}(u)+d_{G_2}(y)\right)\right]\\&=\sum_{u\in V(G_1)}\sum_{vy\in E(G_2)}\left[2d_{Q_H(G_1)}(u)+d_{G_2}(v)+d_{G_2}(y)\right]\\&=\sum_{u\in V(G_1)}\left[2|E(G_2)||V(H)|d_{G_1}(u)+M_1(G_2)\right]\\&=4|E(G_1)||E(G_2)||V(H)|+|V(G_1)|M_1(G_2)\end{aligned}$$

For each edge $ux \in E(Q_H(G_1))$ and the vertex $v \in V(G_2)$

$$\sum_{v\in V(G_2)} \sum_{ux\in E(Q_H(G_1))} D = \sum_{v\in V(G_2)} \sum_{ux\in E(Q_H(G_1)),u\in V(G_1),x\in V_e(H)} D + \sum_{v\in V(G_2)} \sum_{ux\in E(Q_H(G_1)),u,x\in V(H)} D$$

Now we separately find both the expressions. Consider the facts $d_{Q_H(G_1)}(u) = |V(H)|d_{G_1}(u)$ for each $u \in V(G_1)$

$$\begin{aligned}
&\sum_{v\in V(G_2)} \sum_{ux\in E(Q_H(G_1)),u\in V(G_1),x\in V_e(H)} D \\
&= \sum_{v\in V(G_2)} \sum_{ux\in E(Q_H(G_1)),u\in V(G_1),x\in V_e(H)} \Big(d_{(G_1+_{Q_H}G_2)}(u,v) + d_{(G_1+_{Q_H}G_2)}(x,v)\Big) \\
&= \sum_{v\in V(G_2)} \sum_{ux\in E(Q_H(G_1)),u\in V(G_1),x\in V_e(H)} \big(d_{Q_H(G_1)}(u) + d_{G_2}(v)\big) + d_{Q_H(G_1)}(x) \\
&= \sum_{v\in V(G_2)} \sum_{ux\in E(Q_H(G_1)),u\in V(G_1),x\in V_e(H)} |V(H)|d_{G_1}(u) + d_{G_2}(v) + d_{Q_H(G_1)}(x) \\
&= \sum_{v\in V(G_2)} \left(|V(H)|\right)^2 M_1(G_1) + 2|E(G_1)||V(H)|d_{G_2}(v) + \sum_{v\in V(G_2)} \sum_{ux\in E(Q_H(G_1)),u\in V(G_1),x\in V_e(H)} d_{Q_H(G_1)}(x) \\
&= \left(|V(H)|\right)^2 M_1(G_1)|V(G_2)| + 4|E(G_1)||E(G_2)||V(H)| + \sum_{v\in V(G_2)} \sum_{ux\in E(Q_H(G_1)),u\in V(G_1),x\in V_e(H)} d_{Q_H(G_1)}(x)
\end{aligned}$$

Now, we find the value of the last expression

$$\begin{aligned}
&\sum_{v\in V(G_2)} \sum_{ux\in E(Q_H(G_1)),u\in V(G_1),x\in V_e(H)} d_{Q_H(G_1)}(x) \\
&= \sum_{v\in V(G_2)} 2\left(\sum_{u\in V(Q_H(G_1))} d_{Q_H(G_1)}(u) - \sum_{u\in V(Q_H(G_1))\cap V(G_1)} d_{Q_H(G_1)}(u)\right) \\
&= \sum_{v\in V(G_2)} 2\left(2|E(Q_H(G_1))| - \sum_{u\in V(G_1)} |V(H)|d_{G_1}(u)\right) \\
&= 4|V(G_2)|\left(|E(Q_H(G_1))| - |E(G_1)||V(H)|\right)
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{v\in V(G_2)} \sum_{ux\in E(Q_H(G_1)),u\in V(G_1),x\in V_e(H)} D \\
&\quad = \left(|V(H)|\right)^2 M_1(G_1)|V(G_2)| + 4|E(G_1)||E(G_2)||V(H)| + 4|V(G_2)|\left(|E(Q_H(G_1))| - |E(G_1)||V(H)|\right)
\end{aligned}$$

Then again,

$$\sum_{v\in V(G_2)}\sum_{ux\in E(Q_H(G_1)),u,x\in V(H)} D = \sum_{v\in V(G_2)}\sum_{ux\in E(Q_H(G_1)),u,x\in V(H)} \Big(d_{(G_1+_{Q_H}G_2)}(u,v)+d_{(G_1+_{Q_H}G_2)}(x,v)\Big)$$

$$= \sum_{v\in V(G_2)}\sum_{ux\in E(Q_H(G_1)),u,x\in V(H)} \big(d_{Q_H(G_1)}(u)+d_{Q_H(G_1)}(x)\big)$$

$$= \sum_{v\in V(G_2)}\left(\sum_{ux\in E(Q_H(G_1)),u,x\in V(H)} \big(d_{Q_H(G_1)}(u)+d_{Q_H(G_1)}(x)\big)\right)$$

$$= \sum_{v\in V(G_2)}\left(\sum_{ux\in E(Q_H(G_1))} \big(d_{Q_H(G_1)}(u)+d_{Q_H(G_1)}(x)\big) - \sum_{ux\in E(Q_H(G_1)),u\in V(G_1),x\in V(H)} \big(d_{Q_H(G_1)}(u)+d_{Q_H(G_1)}(x)\big)\right.$$

$$= \sum_{v\in V(G_2)}\left(M_1(Q_H(G_1)) - |V(H)|^2\sum_{u\in V(G_1)} d_{G_1}(u)^2 - 2\sum_{x\in V(H)} d_{Q_H(G_1)}(x)\right)$$

$$= \sum_{v\in V(G_2)} M_1(Q_H(G_1)) - |V(H)|^2 M_1(G_1) - 4\left(|E(Q_H(G_1))| - |E(G_1)||V(H)|\right)$$

$$= |V(G_2)|\left(M_1(Q_H(G_1)) - |V(H)|^2 M_1(G_1) - 4|E(Q_H(G_1))| + 4|E(G_1)||V(H)|\right)$$

Thus we obtain

$$M_1(G_1+_{Q_H}G_2) = |V(G_2)|M_1(Q_H(G_1)) + |V(G_1)|M_1(G_2) + 8|E(G_1)||E(G_2)||V(H)|$$

Similarly,

$$\begin{aligned} M_2(G_1+_{Q_H}G_2) &= \sum_{(u,v)(x,y)\in E(G_1+_{Q_H}G_2)} \Big(d_{(G_1+_{Q_H}G_2)}(u,v)d_{(G_1+_{Q_H}G_2)}(x,y)\Big) \\ &= \sum_{u\in V(G_1)}\sum_{vy\in E(G_2)} \Big(d_{(G_1+_{Q_H}G_2)}(u,v)d_{(G_1+_{Q_H}G_2)}(u,y)\Big) \\ &+ \sum_{v\in V(G_2)}\sum_{ux\in E(Q_H(G_1))} \Big(d_{(G_1+_{Q_H}G_2)}(u,v)d_{(G_1+_{Q_H}G_2)}(x,v)\Big) \end{aligned}$$

Now we find the values of each part in the sum.

$$\sum_{u\in V(G_1)}\sum_{vy\in E(G_2)} \Big(d_{(G_1+_{Q_H}G_2)}(u,v)d_{(G_1+_{Q_H}G_2)}(u,y)\Big)$$

$$= \sum_{u\in V(G_1)}\sum_{vy\in E(G_2)} \left[\big(d_{Q_H(G_1)}(u)+d_{G_2}(v)\big)\big(d_{Q_H(G_1)}(u)+d_{G_2}(y)\big)\right]$$

$$= \sum_{u\in V(G_1)}\sum_{vy\in E(G_2)} \left[d_{Q_H(G_1)}(u)^2 + d_{Q_H(G_1)}(u)\left(d_{G_2}(v)+d_{G_2}(y)\right) + d_{G_2}(v)d_{G_2}(y)\right]$$

$$= |E(G_2)||V(H)|^2 M_1(G_1) + 2|E(G_1)||V(H)|M_1(G_2) + |V(G_1)|M_2(G_2)$$

Now,

$$\sum_{v\in V(G_2)}\sum_{ux\in E(R_H(G_1))} T = \sum_{v\in V(G_2)}\sum_{ux\in E(Q_H(G_1)),u\in V(G_1),x\in V_e(H)} T + \sum_{v\in V(G_2)}\sum_{ux\in E(Q_H(G_1)),u,x\in V_e(H)} T$$

Now we separately find each expressions

$$\sum_{v\in V(G_2)}\sum_{ux\in E(Q_H(G_1)),u\in V(G_1),x\in V_e(H)} T$$
$$= \sum_{v\in V(G_2)}\sum_{ux\in E(Q_H(G_1)),u\in V(G_1),x\in V_e(H)} \left(d_{Q_H(G_1)}(u)+d_{G_2}(v)\right)d_{Q_H(G_1)}(x)$$
$$= \sum_{v\in V(G_2)}\sum_{ux\in E(Q_H(G_1)),u\in V(G_1),x\in V_e(H)} d_{Q_H(G_1)}(u)d_{Q_H(G_1)}(x)+d_{G_2}(v)d_{Q_H(G_1)}(x)$$
$$= 8|E(G_2)|\left(|E(Q_H(G_1))|-|E(G_1)||V(H)|\right)+\sum_{v\in V(G_2)}\sum_{ux\in E(Q_H(G_1)),u\in V(G_1),x\in V_e(H)} d_{Q_H(G_1)}(u)d_{Q_H(G_1)}(x)$$

Then again,

$$\sum_{v\in V(G_2)}\sum_{ux\in E(Q_H(G_1)),u,x\in V_e(H)} T = \sum_{v\in V(G_2)}\left(\sum_{ux\in E(Q_H(G_1)),u,x\in V_e(H)}\left(d_{Q_H(G_1)}(u)d_{Q_H(G_1)}(x)\right)\right)$$

Thus we obtain,

$$M_2(G_1+_{Q_H}G_2)=|V(G_2)|M_2(Q_H(G_1))+|V(G_1)|M_2(G_2)+|E(G_2)||V(H)|^2M_1(G_1)+2|E(G_1)||V(H)|M_1(G_2)$$
$$+8|E(G_2)|\left(|E(Q_H(G_1))|-|E(G_1)||V(H)|\right)$$

□

The following theorem is a consequence of Theorem 2 and 3.

Theorem 4. *Let G_1 and G_2 be two connected graphs and H be any graph. Then*

a. $M_1(G_1+_{T_H}G_2)=|V(G_2)|M_1(T_H(G_1))+|V(G_1)|M_1(G_2)+8|E(G_1)||E(G_2)|\left(|V(H)|+1\right)$
b. $M_2(G_1+_{T_H}G_2)=|V(G_2)|M_2(T_H(G_1))+|V(G_1)|M_2(G_2)$
$+|E(G_2)|\left(\left(|V(H)|+1\right)^2+2\left(|V(H)|+1\right)\right)M_1(G_1)$
$+|E(G_1)|\left(2(|V(H)|+1)+1\right)M_1(G_2)+8|E(G_2)|\left(|E(T_H(G_1))|-(|V(H)|+1)|E(G_1)|\right)$

Illustration 1. *When $G_1=P_n$, $G_2=P_m$ $H=P_r$, $n,m,r>3$, using the theorem, we easily obtain the following results*

(1) $M_1(P_n+_{S_{P_r}}P_m)=4r^2mn-6r^2m+24rmn-10mn-24mr-8nr+14m-6n+8r$
(2) $M_2(P_n+_{S_{P_r}}P_m)=20r^2mn-30r^2m+4r^2n+32rmn+6r^2-28mn-28mr-28nr+32m+28r-8$
(3) $M_1(P_n+_{R_{P_r}}P_m)=4r^2mn-6r^2m+32rmn+2mn-36mr-8nr-14n+8r+8$
(4) $M_2(P_n+_{R_{P_r}}P_m)=24r^2mn-38r^2m-4r^2n+72rmn+6r^2-8mn-92mr-44nr+6m-30n+52r+28$
(5) $M_1(P_n+_{Q_{P_r}}P_m)=4r^3mn+20r^2mn-10r^3m-38r^2m+16rmn-10mn-8mr-8nr+14mn-6n+8r$
(6) $M_2(P_n+_{Q_{P_r}}P_m)=4r^4mn+28r^3mn+48r^2mn-12r^4m-68r^3m-80r^2m-12r^2n-4rmn+22r^2+52rm-28nr-24mn+24m+20r-8$
(7) $M_1(P_n+_{T_{P_r}}P_m)=4r^3mn+20r^2mn-10r^3m-38r^2m+24mnr-20mr+2mn-8nr-14n+8r+8$
(8) $M_2(P_n+_{T_{P_r}}P_m)=4r^4mn+28r^3mn+60r^2mn-12r^4m-68r^3m-106r^2m-12r^2n+36rmn+22r^2-32mr-32nr-4mn-2m-30n+52r+28$

4. Forgotten index of the generalised cartesian F sum

In this section we derive the expression for the Forgotten index or F index of the generalised F sums of graphs in terms of the F index of the factor graphs.

Theorem 5. *Let G_1 and G_2 be two connected graphs and H be any graph. Then*

$$\begin{aligned}F(G_1 +_{S_H} G_2) &= |V(G_2)|F(S_H(G_1)) + |V(G_1)|F(G_2) + 6|E(G_2)||V(H)|^2 M_1(G_1)\\ &+ 6|E(G_1)||V(H)|M_1(G_2)\end{aligned}$$

Proof. From the definition of F index, we have

$$\begin{aligned}F(G_1 +_{S_H} G_2) &= \sum_{(u,v)\in V(G_1+_{S_H}G_2)} d^3_{(G_1+_{S_H}G_2)}(u,v)\\ &= \sum_{(u,v)(x,y)\in E(G_1+_{S_H}G_2)} \Big(d_{(G_1+_{S_H}G_2)}(u,v)^2 + d_{(G_1+_{S_H}G_2)}(x,y)^2\Big)\\ &= \sum_{u\in V(G_1)}\sum_{vy\in E(G_2)} \Big(d_{(G_1+_{S_H}G_2)}(u,v)^2 + d_{(G_1+_{S_H}G_2)}(u,y)^2\Big)\\ &+ \sum_{v\in V(G_2)}\sum_{ux\in E(S_H(G_1))} \Big(d_{(G_1+_{S_H}G_2)}(u,v)^2 + d_{(G_1+_{S_H}G_2)}(x,v)^2\Big)\end{aligned}$$

Now we find the values of the each part in the sum separately. First we consider the sum in which $u \in V(G_1)$ and $vy \in E(G_2)$.

$$\begin{aligned}&\sum_{u\in V(G_1)}\sum_{vy\in E(G_2)} \Big(d_{(G_1+_{S_H}G_2)}(u,v)^2 + d_{(G_1+_{S_H}G_2)}(u,y)^2\Big)\\ &= \sum_{u\in V(G_1)}\sum_{vy\in E(G_2)} \Big[\big(d_{S_H(G_1)}(u) + d_{G_2}(v)\big)^2 + \big(d_{S_H(G_1)}(u) + d_{G_2}(y)\big)^2\Big]\\ &= \sum_{u\in V(G_1)}\sum_{vy\in E(G_2)} \big[2d_{S_H(G_1)}(u)^2 + d_{G_2}(v)^2 + d_{G_2}(y)^2 + 2d_{S_H(G_1)}(u)\left(d_{G_2}(v) + d_{G_2}(y)\right)\big]\\ &= \sum_{u\in V(G_1)} \big[2|E(G_2)||V(H)|^2 d_{G_1}(u)^2 + F(G_2) + 2|V(H)|d_{G_1}(u)M_1(G_2)\big]\\ &= 2|E(G_2)||V(H)|^2 M_1(G_1) + |V(G_1)|F(G_2) + 4|E(G_1)||V(H)|M_1(G_2)\end{aligned}$$

Now for each edge $ux \in E(S_H(G_1))$, $v \in V(G_2)$.

$$\begin{aligned}
&\sum_{v\in V(G_2)}\sum_{ux\in E(S_H(G_1))}\Big(d_{(G_1+_{S_H}G_2)}(u,v)^2+d_{(G_1+_{S_H}G_2)}(x,v)^2\Big)\\
&=\sum_{v\in V(G_2)}\sum_{ux\in E(S_H(G_1))u\in V(G_1),x\in V_e(H)}\Big[\big(d_{S_H(G_1)}(u)+d_{G_2}(v)\big)^2+d_{S_H(G_1)}(x)^2\Big]\\
&+\sum_{v\in V(G_2)}\sum_{ux\in E(S_H(G_1))u,x\in V_e(H)}\big[d_{S_H(G_1)}(u)^2+d_{S_H(G_1)}(x)^2\big]\\
&=\sum_{v\in V(G_2)}\sum_{ux\in E(S_H(G_1))}\big[d_{S_H(G_1)}(u)^2+d_{G_2}(v)^2+d_{S_H(G_1)}(x)^2+2d_{S_H(G_1)}(u)(d_{G_2}(v)\big]\\
&+\sum_{v\in V(G_2)}\sum_{ux\in E(S_H(G_1))u,x\in V_e(H)}\big[d_{S_H(G_1)}(u)^2+d_{S_H(G_1)}(x)^2\big]\\
&=\sum_{v\in V(G_2)}\Big[2|E(G_1)|^{|}V(H)|d_{G_2}(v)^2+F(S_H(G_1))+2|V(H)|^2|M_1(G_1)d_{G_2}(v)\Big]\\
&=2|E(G_1)||V(H)|M_1(G_2)+|V(G_2)|F(S_H(G_1))+4|V(H)|^2|E(G_2)|M_1(G_1)
\end{aligned}$$

From the expressions, we obtain

$$F(G_1+_{S_H}G_2)=|V(G_2)|F(S_H(G_1))+|V(G_1)|F(G_2)+6|E(G_2)||V(H)|^2M_1(G_1)+6|E(G_1)||V(H)|M_1(G_2)$$

□

Theorem 6. *Let G_1 and G_2 be two connected graphs and H be any graph. Then*

$$\begin{aligned}
F(G_1+_{R_H}G_2)&=|V(G_2)|F(R_H(G_1))+|V(G_1)|F(G_2)+6|E(G_2)|\Big((|V(H)|+1)^2\Big)M_1(G_1)\\
&+6|E(G_1)|\left(|V(H)|+1\right)M_1(G_2)
\end{aligned}$$

Proof. From the definition of F index, we have

$$\begin{aligned}
F(G_1+_{R_H}G_2)&=\sum_{(u,v)\in V(G_1+_{R_H}G_2)}d^3_{(G_1+_{R_H}G_2)}(u,v)\\
&=\sum_{u\in V(G_1)}\sum_{vy\in E(G_2)}\Big(d_{(G_1+_{R_H}G_2)}(u,v)^2+d_{(G_1+_{R_H}G_2)}(u,y)^2\Big)\\
&+\sum_{v\in V(G_2)}\sum_{ux\in E(R_H(G_1)}\Big(d_{(G_1+_{R_H}G_2)}(u,v)^2+d_{(G_1+_{R_H}G_2)}(x,v)^2\Big)
\end{aligned}$$

Now we find the values of the each part in the sum. First we consider the sum in which $u \in V(G_1)$ and $vy \in E(G_2)$, We also use $d_{R_H(G_1)}(u) = (|V(H)|+1)\, d_{G_1}(u)$

$$\sum_{u\in V(G_1)} \sum_{vy\in E(G_2)} \Big(d_{(G_1+_{R_H}G_2)}(u,v)^2 + d_{(G_1+_{R_H}G_2)}(u,y)^2\Big)$$
$$= \sum_{u\in V(G_1)} \sum_{vy\in E(G_2)} \Big[\big(d_{R_H(G_1)}(u) + d_{G_2}(v)\big)^2 + \big(d_{R_H(G_1)}(u) + d_{G_2}(y)\big)^2\Big]$$
$$= \sum_{u\in V(G_1)} \sum_{vy\in E(G_2)} \big[d_{R_H(G_1)}(u)^2 + d_{G_2}(v)^2 + +d_{R_H(G_1)}(u)^2 + d_{G_2}(y)^2 + 2d_{R_H(G_1)}(u)\,(d_{G_2}(v) + d_{G_2}(y))\big]$$
$$= \sum_{u\in V(G_1)} \sum_{vy\in E(G_2)} \Big[2\left((|V(H)|+1)\, d_{G_1}(u)\right)^2 + d_{G_2}(v)^2 + +d_{G_2}(y)^2 + 2\left((|V(H)|+1)\, d_{G_1}(u)\right)(d_{G_2}(v) + d_{G_2}(y))$$
$$= \sum_{u\in V(G_1)} \Big[2|E(G_2)|\,(|V(H)|+1)^2\, d_{G_1}(u)^2 + F(G_2) + 2\,(|V(H)|+1)\, d_{G_1}(u) M_1(G_2)\Big]$$
$$= \Big[2|E(G_2)|\,(|V(H)|+1)^2\, M_1(G_1) + |V(G_1)|F(G_2) + 4|E(G_1)|\,(|V(H)|+1)\, M_1(G_2)\Big]$$

Now for each edge $ux \in E(R_H(G_1))$, $v \in V(G_2)$. Consider the relation $E(R_H(G_1)) = (2|V(H)| + |E(H)|)\,|E(G_1)| + |E(G_1)|$.

$$\sum_{v\in V(G_2)} \sum_{ux\in E(S_H(G_1))} \Big(d_{(G_1+_{S_H}G_2)}(u,v)^2 + d_{(G_1+_{S_H}G_2)}(x,v)^2\Big) \tag{8}$$

Let $D_1 = \Big(d_{(G_1+_{F_H}G_2)}(u,v)^2 + d_{(G_1+_{F_H}G_2)}(x,v)^2\Big)$. Where $F_H = S_H, R_H, Q_H, T_H$. We divide the following sum into three parts as

$$\sum_{v\in V(G_2)} \sum_{ux\in E(R_H(G_1))} D_1 = \sum_{v\in V(G_2)} \sum_{ux\in E(R_H(G_1)),u,x\in V(G_1)} D_1 \tag{9}$$
$$+ \sum_{v\in V(G_2)} \sum_{ux\in E(R_H(G_1)),u\in V(G_1),x\in V_e(H)} D_1 \tag{10}$$
$$+ \sum_{v\in V(G_2)} \sum_{ux\in E(R_H(G_1)),u,x\in V(H)} D_1 \tag{11}$$

Now we calculate the value for three expressions separately

$$\sum_{v\in V(G_2)} \sum_{ux\in E(R_H(G_1)),u,x\in V(G_1)} D_1 = \sum_{v\in V(G_2)} \sum_{ux\in E(R_H(G_1)),u,x\in V(G_1)} \Big(d_{(G_1+_{R_H}G_2)}(u,v)^2 + d_{(G_1+_{R_H}G_2)}(x,v)^2\Big)$$
$$= \sum_{v\in V(G_2)} \sum_{ux\in E(G_1)} \Big[\big(d_{R_H(G_1)}(u) + d_{G_2}(v)\big)^2 + \big(d_{R_H(G_1)}(x) + d_{G_2}(v)\big)^2\Big]$$
$$= \sum_{v\in V(G_2)} \sum_{ux\in E(G_1)} \big[d_{R_H(G_1)}(u)^2 + 2d_{G_2}(v)^2 + d_{R_H(G_1)}(x)^2 + 2d_{G_2}(v)\,\big(d_{R_H(G_1)}(u) + d_{R_H(G_1)}(x)\big)\big]$$
$$= \sum_{v\in V(G_2)} \sum_{ux\in E(G_1)} d_{R_H(G_1)}(u)^2 + d_{R_H(G_1)}(x)^2 + 2d_{G_2}(v)^2 + 2d_{G_2}(v)\,(|V(H)+1|)\,(d_{G_1}(u) + d_{G_1}(x))$$
$$= \sum_{v\in V(G_2)} \sum_{ux\in E(G_1)} \big(d_{R_H(G_1)}(u)^2 + d_{R_H(G_1)}(x)^2\big) + 2|E(G_1)|M_1(G_2) + 4|E(G_2)|\,(|V(H)|+1)\, M_1(G_1)$$

Also, Consider the case where $ux \in E(R_H(G_1)), u \in V(G_1), x \in V_e(H)$

$$\sum_{v\in V(G_2)} \sum_{ux\in E(R_H(G_1)), u\in V(G_1), x\in V_e(H)} D_1$$
$$= \sum_{v\in V(G_2)} \sum_{ux\in E(R_H(G_1)), u\in V(G_1), x\in V_e(H)} \left(d_{(G_1+_{R_H}G_2)}(u,v)^2 + d_{(G_1+_{R_H}G_2)}(x,v)^2\right)$$
$$= \sum_{v\in V(G_2)} \sum_{ux\in E(R_H(G_1)), u\in V(G_1), x\in V_e(H)} \left(d_{R_H(G_1)}(u) + d_{G_2}(v)\right)^2 + d_{R_H(G_1)}(x)^2$$
$$= \sum_{v\in V(G_2)} \sum_{ux\in E(R_H(G_1)), u\in V(G_1), x\in V_e(H)} d_{R_H(G_1)}(u)^2 + d_{R_H(G_1)}(x)^2 + d_{G_2}(v)^2 + 2d_{R_H(G_1)}(u)d_{G_2}(v)$$
$$= 2|E(G_1)||V(H)|M_1(G_2) + 4|E(G_2)|\left(|V(H)|+1\right)|V(H)|M_1(G_1)$$
$$+ \sum_{v\in V(G_2)} \sum_{ux\in E(R_H(G_1)), u\in V(G_1), x\in V_e(H)} d_{R_H(G_1)}(u)^2 + d_{R_H(G_1)}(x)^2$$

Now consider the sum

$$\sum_{v\in V(G_2)} \sum_{ux\in E(R_H(G_1)), u,x\in V_e(H)} D_1 = \sum_{v\in V(G_2)} \sum_{ux\in E(R_H(G_1)), u,x\in V_e(H)} \left(d_{(G_1+_{R_H}G_2)}(u,v)^2 + d_{(G_1+_{R_H}G_2)}(x,v)^2\right)$$
$$= \sum_{v\in V(G_2)} \sum_{ux\in E(R_H(G_1)), u,x\in V(H)} d_{R_H(G_1)}(u)^2 + d_{R_H(G_1)}(x)^2$$

From the expressions, we obtain

$$F(G_1 +_{R_H} G_2) = |V(G_2)|F(R_H(G_1)) + |V(G_1)|F(G_2) + 6|E(G_2)|\left(\left(|V(H)|+1\right)^2\right)M_1(G_1)$$
$$+ 6|E(G_1)|\left(|V(H)|+1\right)M_1(G_2)$$

□

Theorem 7. *Let G_1 and G_2 be two connected graphs and H be any graph. Then*

$$F(G_1 +_{Q_H} G_2) = |V(G_2)|F(Q_H(G_1)) + |V(G_1)|F(G_2) + \left(4|E(G_2)||V(H)|^2 + 2|E(G_2)|\right)M_1(G_1)$$
$$+ \left(4|E(G_1)| + 2|E(G_1)||V(H)|\right)M_1(G_2)$$

Proof. Using the definition of F index and generalised F sum,

$$F(G_1 +_{Q_H} G_2) = \sum_{(u,v)\in V(G_1+_{S_H}G_2)} d^3_{(G_1+_{Q_H}G_2)}(u,v)$$
$$= \sum_{(u,v)(x,y)\in E(G_1+_{Q_H}G_2)} \left(d_{(G_1+_{Q_H}G_2)}(u,v)^2 + d_{(G_1+_{Q_H}G_2)}(x,y)^2\right)$$
$$= \sum_{u\in V(G_1)} \sum_{vy\in E(G_2)} \left(d_{(G_1+_{Q_H}G_2)}(u,v)^2 + d_{(G_1+_{Q_H}G_2)}(u,y)^2\right)$$
$$+ \sum_{v\in V(G_2)} \sum_{ux\in E(Q_H(G_1))} \left(d_{(G_1+_{Q_H}G_2)}(u,v)^2 + d_{(G_1+_{Q_H}G_2)}(x,v)^2\right)$$

Now we find the values of the each part in the sum separately . First we consider the sum in which $u \in V(G_1)$ and $vy \in E(G_2)$.

$$\begin{aligned}
&\sum_{u\in V(G_1)}\sum_{vy\in E(G_2)}\Big(d_{(G_1+_{Q_H}G_2)}(u,v)^2+d_{(G_1+_{Q_H}G_2)}(u,y)^2\Big)\\
&=\sum_{u\in V(G_1)}\sum_{vy\in E(G_2)}\Big[\big(d_{Q_H(G_1)}(u)+d_{G_2}(v)\big)^2+\big(d_{Q_H(G_1)}(u)+d_{G_2}(y)\big)^2\Big]\\
&=\sum_{u\in V(G_1)}\sum_{vy\in E(G_2)}\big[2d_{Q_H(G_1)}(u)^2+d_{G_2}(v)^2+d_{G_2}(y)^2+2d_{Q_H(G_1)}(u)\left(d_{G_2}(v)+d_{G_2}(y)\right)\big]\\
&=\sum_{u\in V(G_1)}\big[2|E(G_2)||V(H)|^2d_{G_1}(u)^2+F(G_2)+2|V(H)|d_{G_1}(u)M_1(G_2)\big]\\
&=\big[2|E(G_2)||V(H)|^2M_1(G_1)+|V(G_1)|F(G_2)+4|E(G_1)||V(H)|M_1(G_2)\big]
\end{aligned}$$

For each edge $ux \in E(Q_H(G_1))$ and the vertex $v \in V(G_2)$

$$\begin{aligned}
\sum_{v\in V(G_2)}\sum_{ux\in E(Q_H(G_1))}D_1 =& \sum_{v\in V(G_2)}\sum_{ux\in E(Q_H(G_1)),u\in V(G_1),x\in V_e(H)}D_1\\
&+\sum_{v\in V(G_2)}\sum_{ux\in E(Q_H(G_1)),u,x\in V_e(H)}D_1
\end{aligned}$$

Now we separately find each expressions,

$$\begin{aligned}
&\sum_{v\in V(G_2)}\sum_{ux\in E(Q_H(G_1)),u\in V(G_1),x\in V_e(H)}D_1\\
&=\sum_{v\in V(G_2)}\sum_{ux\in E(Q_H(G_1)),u\in V(G_1),x\in V_e(H)}\Big(d_{(G_1+_{Q_H}G_2)}(u,v)^2+d_{(G_1+_{Q_H}G_2)}(x,v)^2\Big)\\
&=\sum_{v\in V(G_2)}\sum_{ux\in E(Q_H(G_1)),u\in V(G_1),x\in V_e(H)}\big(d_{Q_H(G_1)}(u)+d_{G_2}(v)\big)^2+d_{Q_H(G_1)}(x)^2\\
&=\sum_{v\in V(G_2)}\sum_{ux\in E(Q_H(G_1)),u\in V(G_1),x\in V_e(H)}d_{Q_H(G_1)}(u)^2+d_{G_2}(v)^2+2d_{Q_H(G_1)}(u)d_{G_2}(v)+d_{Q_H(G_1)}(x)^2\\
&=\sum_{v\in V(G_2)}\sum_{ux\in E(Q_H(G_1)),u\in V(G_1),x\in V_e(H)}d_{Q_H(G_1)}(u)^2+d_{Q_H(G_1)}(x)^2+d_{G_2}(v)^2+2|V(H)|d_{G_1}(u)d_{G_2}(v)\\
&=\sum_{v\in V(G_2)}\sum_{ux\in E(Q_H(G_1)),u\in V(G_1),x\in V_e(H)}d_{Q_H(G_1)}(u)^2+d_{Q_H(G_1)}(x)^2\\
&+2|E(G_1)||V(H)|M_1(G_2)+4|E(G_2)||V(H)|^2M_1(G_1)
\end{aligned}$$

Now, we find the value of the last expression

$$\begin{aligned}
&\sum_{v\in V(G_2)}\sum_{ux\in E(Q_H(G_1)),u,x\in V_e(H)} D_1\\
&=\sum_{v\in V(G_2)}\sum_{ux\in E(Q_H(G_1)),u,x\in V_e(H)}\Big(d_{(G_1+_{Q_H}G_2)}(u,v)^2+d_{(G_1+_{Q_H}G_2)}(x,v)^2\Big)\\
&=\sum_{v\in V(G_2)}\sum_{ux\in E(Q_H(G_1)),u,x\in V_e(H)}\left(d_{Q_H(G_1)}(u)^2+d_{Q_H(G_1)}(x)^2\right)\\
&=\sum_{v\in V(G_2)}\left(\sum_{ux\in E(Q_H(G_1)),u,x\in V(H)}\left(d_{Q_H(G_1)}(u)^2+d_{Q_H(G_1)}(x)^2\right)\right)
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
F(G_1+_{Q_H}G_2)&=|V(G_2)|F(Q_H(G_1))+|V(G_1)|F(G_2)+\left(6|E(G_2)||V(H)|^2\right)M_1(G_1)\\
&\quad+\left(6|E(G_1)||V(H)|\right)M_1(G_2)
\end{aligned}$$

□

Theorem 8. *Let G_1 and G_2 be two connected graphs and H be any graph. Then*

$$\begin{aligned}
F(G_1+_{T_H}G_2)&=|V(G_2)|F(T_H(G_1))+|V(G_1)|F(G_2)+6|E(G_2)|\Big((|V(H)|+1)^2\Big)M_1(G_1)\\
&\quad+6|E(G_1)|\left(|V(H)|+1\right)M_1(G_2)
\end{aligned}$$

Proof. For each vertex $(u,v)\in V(G_1+_{T_H}G_2)$ where $u\in V(G_1)$ and $v\in V(G_2)$, $d_{G_1+_{T_H}G_2}(u,v)=d_{G_1+_{R_H}G_2}(u,v)$ and For each vertex $(u,v)\in V(G_1+_{T_H}G_2)$ where $u\in V(T_H(G_1)/V(G_1)$ and $v\in V(G_2)$, $d_{G_1+_{T_H}G_2}(u,v)=d_{G_1+_{Q_H}G_2}(u,v)$

$$\begin{aligned}
&\sum_{u\in V(G_1)}\sum_{vy\in E(G_2)}\Big(d_{(G_1+_{T_H}G_2)}(u,v)^2+d_{(G_1+_{T_H}G_2)}(u,y)^2\Big)\\
&=\Big[2|E(G_2)|\left(|V(H)|+1\right)^2M_1(G_1)+|V(G_1)|F(G_2)+4|E(G_1)|\left(|V(H)|+1\right)M_1(G_2)\Big]
\end{aligned}$$

For each edge $ux\in E(T_H(G_1))$ and the vertex $v\in V(G_2)$

$$\begin{aligned}
\sum_{v\in V(G_2)}\sum_{ux\in E(T_H(G_1))} D_1 &=\sum_{v\in V(G_2)}\sum_{ux\in E(T_H(G_1)),u,x\in V(G_1)} D_1\\
&\quad+\sum_{v\in V(G_2)}\sum_{ux\in E(T_H(G_1)),u\in V(G_1),x\in V_e(H)} D_1\\
&\quad+\sum_{v\in V(G_2)}\sum_{ux\in E(T_H(G_1)),u,x\in V_e(H)} D_1
\end{aligned}$$

By proceeding like theorem 2 we get,

$$\begin{aligned}
\sum_{v\in V(G_2)}\sum_{ux\in E(T_H(G_1)),u,x\in V(G_1)} D_1 &=\sum_{v\in V(G_2)}\sum_{ux\in E(Q_H(G_1)),u,x\in V(G_1)} d_{T_H(G_1)}(u)^2+d_{T_H(G_1)}(x)^2\\
&\quad+2|E(G_1)|M_1(G_2)+4|E(G_2)|(|V(H)|+1)M_1(G_1)
\end{aligned}$$

Similarly,

$$\sum_{v\in V(G_2)}\sum_{ux\in E(T_H(G_1)),u\in V(G_1),x\in V_e(H)} D_1 = \sum_{v\in V(G_2)}\sum_{ux\in E(R_H(G_1)),u\in V(G_1),x\in V_e(H)} d_{T_H(G_1)}(u)^2 + d_{T_H(G_1)}(x)^2 + 2|E(G_1)||V(H)|M_1(G_2) + 4|E(G_2)|\left(|V(H)|+1\right)M_1(G_1)$$

$$\sum_{v\in V(G_2)}\sum_{ux\in E(T_H(G_1)),u\in u,x\in V_e(H)} D_1 = d_{T_H(G_1)}(u)^2 + d_{T_H(G_1)}(x)^2$$

Now collecting all of them we get,

$$\sum_{v\in V(G_2)}\sum_{ux\in E(T_H(G_1))} D_1 = |V(G_2)|F(T_H(G_1)) + 4|E(G_2)|(|V(H)|+1)M_1(G_1) + 2|E(G_1)|(|V(H)|+1)M_1(G_2) + 8|E(G_1)||E(G_2)|\left(|V(H)|+1\right)$$

Thus,

$$F(G_1 +_{T_H} G_2) = |V(G_2)|F(T_H(G_1)) + |V(G_1)|F(G_2) + 6|E(G_2)|\left(\left(|V(H)|+1\right)^2\right)M_1(G_1) + 6|E(G_1)|\left(|V(H)|+1\right)M_1(G_2)$$

□

Illustration 2. *Let P_n be a path of n vertices, Let $H = P_r$ $n, m, r \geq 4$*

- $F(P_n +_{S_H} P_m) = 8mnr^3 + 24mnr^2 + 88mnr - 66mn - 14mr^3 - 36mr^2 - 88mr + 74m - 24nr^2 - 36nr - 14n + 36r^2 + 36r$
- $F(P_n +_{R_H} P_m) = 8mnr^3 + 48mnr^2 + 160mnr - 10mn - 14mr^3 - 78mr^2 - 202mr - 24nr^2 - 84nr - 74n + 36r^2 + 108r + 72$
- $F(P_n +_{Q_H} P_m) = 8mnr^4 + 56mnr^3 + 96mnr^2 + 4mnr - 66mn - 22mr^4 - 134mr^3 - 168mr^2 + 80mr + 74m - 24nr^2 - 36nr - 14n + 36r^2 + 36r$
- $F(P_n +_{T_H} P_m) = 8mnr^4 + 56mnr^3 + 120mnr^2 + 76mnr - 10mn - 22mr^4 - 134mr^3 - 210mr^2 - 34mr - 24nr^2 - 84nr - 74n + 36r^2 + 108r + 72$

Illustration 3. *Let C_n be a cycle of n vertices , Let $H = P_r$ $n, m, r \geq 4$*

- $F(C_n +_{S_H} P_m) = 8mnr^3 + 24mnr^2 + 88mnr - 66mn - 24nr^2 - 36nr - 14n$
- $F(C_n +_{R_H} P_m) = 8mnr^3 + 48mnr^2 + 160mnr - 10mn - 24nr^2 - 84nr - 74n$
- $F(C_n +_{Q_H} P_m) = 8mnr^4 + 56mnr^3 + 96mnr^2 + 4mnr - 66mn - 24nr^2 - 36nr - 14n$
- $F(C_n +_{T_H} P_m) = 8mnr^4 + 56mnr^3 + 120mnr^2 + 76mnr - 10mn - 24nr^2 - 84nr - 74n$

Illustration 4. *Let C_n be a cycle of n vertices, Let $H = P_r$, $n, m, r \geq 4$*

- $F(C_n +_{S_H} C_m) = 8mnr^3 + 24mnr^2 + 88mnr - 66mn$
- $F(C_n +_{R_H} C_m) = 8mnr^3 + 48mnr^2 + 160mnr - 10mn$
- $F(C_n +_{Q_H} C_m) = 8mnr^4 + 56mnr^3 + 96mnr^2 + 4mnr - 66mn$
- $F(C_n +_{T_H} C_m) = 8mnr^4 + 56mnr^3 + 120mnr^2 + 76mnr - 10mn$

Illustration 5. *Let P_n be a path of n vertices, Let $H = C_r$, $n, m, r \geq 4$*

- $F(P_n +_{S_H} P_m) = 8mnr^3 + 24mnr^2 + 88mnr + 8mn - 14mr^3 - 36mr^2 - 88mr - 24nr^2 - 36nr - 14n + 36r^2 + 36r$
- $F(P_n +_{R_H} P_m) = 8mnr^3 + 48mnr^2 + 160mnr + 64mn - 14mr^3 - 78mr^2 - 202mr - 74m - 24nr^2 - 84nr - 74n + 36r^2 + 108r + 72$

- $F(P_n +_{Q_H} P_m) = 16mnr^4+104mnr^3+216mnr^2+152mnr+8mn-46mr^4-278mr^3-516mr^2-280mr-24nr^2-36nr-14n+36r^2+36r$
- $F(P_n +_{T_H} P_m) = 16mnr^4+104mnr^3+240mnr^2+224mnr+64mn-48mr^4-278mr^3-558mr^2-394mr-74m-24nr^2-84nr-74n+36r^2+108r+72$

Illustration 6. *Let C_n be a cycle with n vertices, Let $H = C_r$, $n, m, r \geq 4$*

- $F(C_n +_{S_H} P_m) = 8mnr^3+24mnr^2+88mnr+8mn-64mr-24nr^2-36nr-14n$
- $F(C_n +_{R_H} P_m) = 8mnr^3+48mnr^2+160mnr+64mn-64mr-24nr^2-84nr-74n$
- $F(C_n +_{Q_H} P_m) = 8mnr^4+56mnr^3+120mnr^2+88mnr-8mn-8mr^4-48mr^3-96mr^2-96mr-24nr^2-36nr-14n$
- $F(C_n +_{T_H} P_m) = 8mnr^4+56mnr^3+144mnr^2+160mnr+64mn-8mr^4-48mr^3-96mr^2-64mr-24nr^2-84nr-74n$

Illustration 7. *Let C_n be a cycle with n vertices, Let $H = C_r$, $n, m, r \geq 4$*

- $F(C_n +_{S_H} C_m) = 8mnr^3+24mnr^2+88mnr+8mn-64mr$
- $F(C_n +_{R_H} C_m) = 8mnr^3+48mnr^2+160mnr+64mn-64mr$
- $F(C_n +_{Q_H} C_m) = 8mnr^4+56mnr^3+120mnr^2+88mnr-8mn-8mr^4-48mr^3-96mr^2-96mr$
- $F(C_n +_{T_H} C_m) = 8mnr^4+56mnr^3+144mnr^2+160mnr+64mn-8mr^4-48mr^3-96mr^2-64mr$

5. Summary and Conclusion

The F sum was introduced by M. Eliasi, B. Taeri in [2] and the first and second Zagreb indices of F sums were computed by Hanyuan Deng, D. Sarala, S. K. Ayyaswamy, S. Balachandran in [4] and the Forgotten index of F sum was computed by Shehnaz Akhter,Muhammad Imran in [10]. We have extended this F sum to a generalised F sum and found the first,second Zagreb indices and Forgotten index of this F sum. These sums can be defined in terms of strong product, lexicographic product, tensor product and other topological indices can be computed in the case of such generalised F sums which we hope that a prosperous area of further research.

References

[1] R. Balakrishnan, A Text Book of Graph Theory (Springer, New York, 2012).

[2] M. Eliasi, B. Taeri, *Four new sums of graphs and their Wiener indices*, Discrete Appl. Math. 157 (2009) 794803.

[3] B. Furtula, I. Gutman,*A Forgotten topological index*, J. Math. Chem. 53(2015) 1184-1190

[4] Hanyuan Deng, D. Sarala, S. K. Ayyaswamy, S. Balachandran, *The Zagreb indices of four operations on graphs*, Applied Mathematics and Computation, Volume 275(2016)422-431

[5] I. Gutman, K.C. Das, *The first Zagreb index 30 years after*, MATCH Commun. Math. Comput. Chem. 50 (2004) 8392.

[6] I. Gutman, Emina Milovanovic, Igor Milovanovic, *Beyond the Zagreb indices*, AKCE International Journal of Graphs and Combinatorics

[7] I. Gutman, N. Trinajstić, *Graph theory amd molecular orbits.Total π- electron energy of alternant hydrocarbons*, Chem. Phys. Lett. 17(1972) 535-538.

[8] I. Gutman, *An exceptional property of the first Zagreb index*, MATCH Commun. Math. Comput. Chem. 72 (2014) 733740.

[9] S. Nikolic, G. Kovacevic, A. Milicevic, N. Trinajstić, *The Zagreb indices 30 years after*, Croat. Chem. Acta 76 (2003) 113124.

[10] Shehnaz Akhter,Muhammad Imran *Computing the Forgotten topological index of four operations on graphs*, AKCE International Journal of Graphs and Combinatorics Vol 14, Issue 1,April (2017), 70-79

[11] D. Stevanović, *Mathematical Properties of Zagreb Indices*, Akademska misao, Beograd, 2014 (in Serbian).

[12] H. Wiener, *Structural determination of paraffin boiling points*, J. Am. Chem. Soc. 69 (1947) 1720.

Liju Alex

Department of Mathematics, Bishop Chulaparambil Memorial College, Kottayam - 686001, India, Marthoma College, Thiruvalla, Pathanamthitta - 689103, India,

E-mail address: `lijualex0@gmail.com`

G. Indulal

Department of Mathematics, St.Aloysius College, Edathua, Alappuzha - 689573, India

E-mail address: `indulalgopal@gmail.com`