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SOME FIXED POINT THEOREMS IN CONE A_b-METRIC SPACE

K. ANTHONY SINGH, M. R. SINGH AND TH. CHHATRAJIT SINGH

Abstract. In this paper, we introduce the concept of cone A_b-metric space and prove some fixed point theorems in the new space.

1. Introduction

The Banach contraction principle which is the foundation of metric branch of fixed point theory is one of the most widely used fixed point theorems in all analysis. Inspite of its simplicity, it is a very powerful tool in mathematics having useful applications especially in nonlinear analysis. Due to this, the result has been generalized in various directions. As a generalization of metric space, Huang and Zhang [6] introduced the concept of cone metric space by replacing the set of real numbers by a general Banach space E which is partially ordered with respect to a cone $P \subset E$.

On the other hand, Sedghi et al. [22] generalized metric space to S-metric space and Bakhtin generalized it to b-metric space. Nizar and Nabil [17] introduced the concept of S_b-metric space as a generalization of S-metric space. The concepts of S-metric space and S_b-metric space are further extended to A-metric space and A_b-metric space respectively by Mujahid Abbas et al. [12] and Manoj Ughade et al. [10] and used in many other research papers. Dhamodharan and Krishnakumar [4] extended S-metric space to cone S-metric space. K. Anthony Singh and M. R. Singh [7] further extended cone S-metric space to cone S_b-metric space.

The aim of this paper is to generalize the concept of cone S_b-metric space further to cone A_b-metric space and prove some fixed point theorems. The new space can also be looked upon as a generalization of A_b-metric space.

Following definitions, concepts and properties will be needed in the sequel.

Definition 1.1[6] Let E be a Banach space. A subset P of E is called a cone if and only if

1. P is closed, nonempty and $P \neq \{0\}$,
2. $ax + by \in P$ for all $x, y \in P$ and nonnegative real numbers a, b,
3. $P \cap (-P) = \{0\}$.

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Given a cone $P \subset E$, we define a partial ordering $\leq$ in E with respect to P by $x \leq y$ if and only if $y - x \in P$. We will write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in int.P$,where $int.P$ denotes the interior of P. The cone P is called normal if there is a number $K > 0$ such that $0 \leq x \leq y$ implies $\| x \| \leq K \| y \|$ for all $x, y \in E$.The least positive number satisfying the above is called the normal constant of P.

The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}$ is a sequence such that $x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \leq y$ for some $y \in E$,then there is $x \in E$ such that $\| x_n - x \| \to \infty$ as $n \to \infty$. Equivalently, the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone.

Example 1.2[21] Let $K > 1$ be given. Consider the real vector space $E = \left\{ax + b : a, b \in \mathbb{R}; x \in [1 - \frac{1}{K}, 1]\right\}$ with supremum norm and the cone $P = \{ax + b \in E : a \geq 0, b \geq 0\}$ in E . The cone P is regular and so normal.

Definition 1.3[6] Let X be a non-empty set. Suppose the mapping $d : X \times X \to E$ satisfies

1. $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y,\ \forall x, y \in X$,
2. $d(x, y) = d(y, x),\ \forall x, y \in X$,
3. $d(x, y) \leq d(x, z) + d(z, y),\ \forall x, y, z \in X$.

Then (X, d) is called a cone metric space or simply CMS.

Lemma 1.4[23] Every regular cone is normal.

Example 1.5[6] Let $E = \mathbb{R}^2, P = \{(x, y) : x, y \geq 0\}, X = \mathbb{R}$ and $d : X \times X \to E$ such that $d(x, y) = (\mid x - y \mid, \alpha \mid x - y \mid)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition 1.6[17] Let X be a nonempty set and $b \geq 1$ be a given real number. A function $S_b : X \times X \times X \to [0, \infty)$ is said to be S_b-metric if and only if for all $x, y, z, a \in X$ the following conditions are satisfied:

(S_b1) $S_b(x, y, z) = 0$ if and only if $x = y = z$,
(S_b2) $S_b(x, y, z) \leq b[S_b(x, x, a) + S_b(y, y, a) + S_b(z, z, a)]$.

The pair (X, S_b) is called an S_b-metric space.
It is quite obvious that S_b-metric spaces are the generalizations of S-metric spaces since every S-metric is an S_b-metric with $b = 1$

Example 1.7[24] Let (X, S) be an S-metric space, and $S_*(x, y, z) = \{S(x, y, z)\}^p$, where $p > 1$ is a real number. Then, S_* is an S_b-metric on X with $b = 2^{2(p-1)}$.

Example 1.8 Let $X = \mathbb{R}$ and let the function $S : X \times X \times X \to \mathbb{R}$ be defined as $S(x, y, z) = \mid x - z \mid + \mid y - z \mid$. Then S is an S-metric on X. Therefore, the function $S_b(x, y, z) = \{S(x, y, z)\}^2 = \{\mid x - z \mid + \mid y - z \mid\}^2$ is an S_b-metric on X with $b = 2^{2(2-1)} = 4$.

Definition 1.9[7] Suppose that E is a real Banach space, P is a cone in E with $int.P \neq \phi$ and $\leq$ is partial ordering in E with respect to P. Let X be a non-empty set, and let the function $S : X \times X \times X \to E$ satisfy the following conditions:

1. $S(u, v, z) \geq 0$,
2. $S(u, v, z) = 0$ if and only if $u = v = z$,
3. $S(u, v, z) \leq b[S(u, u, a) + S(v, v, a) + S(z, z, a)],\ \forall u, v, z, a \in X$, where $b \geq 1$ is a constant.

Then, the function S is called a cone S_b-metric on X and the pair (X, S) is called a cone S_b-metric space.
We note that cone S_b-metric spaces are generalizations of cone S-metric spaces since every cone S-metric is a cone S_b-metric with $b = 1$.
Example 1.10[7] Let $E = \mathbb{R}^2$, the Euclidean plane, and $P = \{(x, y) \in E : x, y \geq 0\}$, a normal cone in E. Let $X = \mathbb{R}$ and $S : X \times X \times X \to E$ be such that $S(x, y, z) = (\alpha S_*(x, y, z), \beta S_*(x, y, z))$, where $\alpha, \beta > 0$ are constants and S_* is an S_b-metric on X. Then S is a cone S_b-metric on X.
Definition 1.11[10] Let X be a non-empty set and let $b \geq 1$ be a given real number. A function $A : X^n \to [0, \infty)$ is called an A_b-metric on X if $\forall x_i, a \in X, i = 1, 2, 3, \dots, n$, the following conditions are satisfied:

1. $A(x_1, x_2, \dots, x_{n-1}, x_n) \geq 0$,
2. $A(x_1, x_2, \dots, x_{n-1}, x_n) = 0$ if and only if $x_1 = x_2 = \cdots = x_{n-1} = x_n$,
3. $A(x_1, x_2, \dots, x_{n-1}, x_n) \leq b[A(x_1, x_1, \dots, (x_1)_{n-1}, a)+A(x_2, x_2, \dots, (x_2)_{n-1}, a)$ $+ \cdots + A(x_{n-1}, x_{n-1}, \dots, (x_{n-1})_{n-1}, a) + A(x_n, x_n, \dots, (x_n)_{n-1}, a)]$

The pair (X, A) is called an A_b-metric space.
Example 1.12[10] Let $X = [1, \infty)$. Define $A_b : X^n \to [0, \infty)$ by

$$A_b(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n} \sum_{i<j} \mid x_i - x_j \mid^2$$

for all $x_i \in X, i = 1, 2, 3, \dots, n$.
Then (X, A_b) is an A_b-metric space with $b = 2 > 1$.

2. Cone A_b-metric space

Here, we introduce a new space called cone A_b-metric space.
Definition 2.1 Suppose that E is a real Banach space, P is a cone in E with $int.P \neq \phi$ and $\leq$ is partial ordering in E with respect to P. Let X be a non-empty set, and let the function $A : X^n \to E$ satisfy the following conditions

1. $A(x_1, x_2, \dots, x_{n-1}, x_n) \geq 0$,
2. $A(x_1, x_2, \dots, x_{n-1}, x_n) = 0$ if and only if $x_1 = x_2 = \cdots = x_{n-1} = x_n$,
3. $A(x_1, x_2, \dots, x_{n-1}, x_n) \leq b[A(x_1, x_1, \dots, (x_1)_{n-1}, a)+A(x_2, x_2, \dots, (x_2)_{n-1}, a)$ $+ \cdots + A(x_{n-1}, x_{n-1}, \dots, (x_{n-1})_{n-1}, a) + A(x_n, x_n, \dots, (x_n)_{n-1}, a)]$, $\forall x_i, a \in X, i = 1, 2, 3 \dots, n$,

where $b \geq 1$ is a constant.
Then, the function A is called a cone A_b-metric on X and the pair (X, A) is called a cone A_b-metric space.
We note that cone A_b-metric spaces are generalizations of cone S_b-metric spaces since every cone S_b-metric is a cone A_b-metric with $n = 3$.
Example 2.2 Let $E = \mathbb{R}^2$, the Euclidean plane, and $P = \{(x, y) \in E : x, y \geq 0\}$, a normal cone in E. Let $X = \mathbb{R}$ and $A : X^n \to E$ be such that $A(x_1, x_2, \dots, x_n) = A_*(x_1, x_2, \dots, x_n)(\alpha, \beta)$, where $\alpha, \beta > 0$ are constants and A_* is an A_b-metric on X. We show that A is a cone A_b-metric on X.

(1) Since $A_*(x_1, x_2, \dots, x_n) \geq 0,\ \forall x_1, x_2, \dots, x_n \in X$, we have $A(x_1, x_2, \dots, x_n) = A_*(x_1, x_2, \dots, x_n)(\alpha, \beta) \geq (0, 0) = 0$ i.e. $A(x_1, x_2, \dots, x_n) \geq 0,\ \forall x_1, x_2, \dots, x_n \in X$.
(2) $A(x_1, x_2, \dots, x_n) = 0 \Leftrightarrow A_*(x_1, x_2, \dots, x_n)(\alpha, \beta) = 0$ $\Leftrightarrow A_*(x_1, x_2, \dots, x_n) = 0 \Leftrightarrow x_1 = x_2 = \cdots = x_n$.

(3) $\alpha A_*(x_1, x_2, \ldots, x_n) \leq b[\alpha A_*(x_1, x_1, \ldots, x_1, a) + \alpha A_*(x_2, x_2, \ldots, x_2, a) + \cdots + \alpha A_*(x_n, x_n, \ldots, x_n, a)]$.
and,
$\beta A_*(x_1, x_2, \ldots, x_n) \leq b[\beta A_*(x_1, x_1, \ldots, x_1, a) + \beta A_*(x_2, x_2, \ldots, x_2, a) + \cdots + \beta A_*(x_n, x_n, \ldots, x_n, a)]$.
$\therefore A(x_1, x_2, \ldots, x_n) = A_*(x_1, x_2, \ldots, x_n)(\alpha, \beta)$
$\leq b[A_*(x_1, x_1, \ldots, x_1, a)(\alpha, \beta) + A_*(x_2, x_2, \ldots, x_2, a)(\alpha, \beta)$
$+ \cdots + A_*(x_n, x_n, \ldots, x_n, a)(\alpha, \beta)]$
$= b[A(x_1, x_1, \ldots, x_1, a) + A(x_2, x_2, \ldots, x_2, a) + \cdots + A(x_n, x_n, \ldots, x_n, a)]$.

Thus, A is a cone A_b-metric on X.
In particular, we have, the function

$$A_*(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} \sum_{i<j} \mid x_i - x_j \mid^2, \; x_i \in X, \; i = 1, 2, 3, \ldots, n,$$

is an A_b-metric on X with $b = 2$.
Therefore, the function

$$A(x_1, x_2, \ldots, x_n) = \left(\sum_{i=1}^{n} \sum_{i<j} \mid x_i - x_j \mid^2, \frac{1}{4} \sum_{i=1}^{n} \sum_{i<j} \mid x_i - x_j \mid^2 \right)$$

is a cone A_b-metric on X with $b = 2$.

Definition 2.3 Let (X, A) be a cone A_b-metric space.

1. A sequence $\{x_n\}$ in X is said to converge to x if for each $c \in E$, $0 \ll c$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $A(x_n, x_n, \ldots, x_n, x) \ll c$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.
2. A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $c \in E$, $0 \ll c$ there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$, $A(x_n, x_n, \ldots, x_n, x_m) \ll c$.
3. The cone A_b-metric space (X, A) is called complete if every Cauchy sequence is convergent.

Lemma 2.4 Let (X, A) be a cone A_b-metric space. Then, for all $x, y, a \in X$,

(i) $A(x, x, \ldots, x, y) \leq bA(y, y, \ldots, y, x)$
(ii) $A(x, x, \ldots, x, y) \leq (n-1)bA(x, x, \ldots, x, a) + bA(y, y, \ldots, y, a)$.

Lemma 2.5 Let (X, A) be a cone A_b-metric space, P be a normal cone with normal constant K. Then a sequence $\{x_n\}$ in X converges to x if and only if $A(x_n, x_n, \ldots, x_n, x) \to 0$ as $n \to \infty$.

Proof. Let $\{x_n\}$ converge to x. And for a given $\varepsilon > 0$, let us choose $c \in E$, $0 \ll c$ such that $K \parallel c \parallel < \varepsilon$. Then, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $A(x_n, x_n, \ldots, x_n, x) \ll c$. Thus for all $n \geq n_0$,we have, $\parallel A(x_n, x_n, \ldots, x_n, x) \parallel \leq K \parallel c \parallel < \varepsilon$. This means that $A(x_n, x_n, \ldots, x_n, x) \to 0$ as $n \to \infty$.

Conversely, let $A(x_n, x_n, \ldots, x_n, x) \to 0$ as $n \to \infty$. And, let $c \in E$ with $0 \ll c$. Then, there exists $\varepsilon > 0$, such that $\parallel x \parallel < \varepsilon$ implies $c - x \in int.P$. For this ε, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\parallel A(x_n, x_n, \ldots, x_n, x) \parallel < \varepsilon$. Therefore, we have $c - A(x_n, x_n, \ldots, x_n, x) \in int.P$. Thus, $A(x_n, x_n, \ldots, x_n, x) \ll c$ for all $n \geq n_0$. Therefore, the sequence $\{x_n\}$ converges to x.

Lemma 2.6 Let (X, A) be a cone A_b-metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to w_1 and $\{x_n\}$ converges to w_2, then $w_1 = w_2$. That is, the limit of a convergent sequence is

unique.
Proof. We have

$$\begin{aligned} A(w_1,w_1,\dots,w_1,w_2) &\leq (n-1)bA(w_1,w_1,\dots,w_1,x_n)+bA(w_2,w_2,\dots,w_2,x_n)\\ &\leq (n-1)b^2A(x_n,x_n,\dots,x_n,w_1)+b^2A(x_n,x_n,\dots,x_n,w_2)\\ \Rightarrow\parallel A(w_1,w_1,\dots,w_1,w_2)\parallel &\leq K[(n-1)b^2\parallel A(x_n,x_n,\dots,x_n,w_1)\parallel\\ &\quad +b^2\parallel A(x_n,x_n,\dots,x_n,w_2)\parallel]\\ &\quad \to 0 \ as\ n\to\infty \end{aligned}$$

$$\begin{aligned} &\Rightarrow A(w_1,w_1,\dots,w_1,w_2)=0\\ &\qquad\Rightarrow w_1=w_2. \end{aligned}$$

Lemma 2.7 Let (X,A) be a cone A_b-metric space, P be a normal cone with normal constant K. Then a sequence $\{x_n\}$ in X is a Cauchy sequence if and only if $A(x_n,x_n,\dots,x_n,x_m)\to 0$ as $n,m\to\infty$.
Proof. Let $\{x_n\}$ be a Cauchy sequence. And for a given $\varepsilon>0$, let us choose $c\in E$, $0\ll c$ such that $K\parallel c\parallel<\varepsilon$. Then, there exists $n_0\in\mathbb{N}$ such that for all $n,m\geq n_0$, $A(x_n,x_n,\dots,x_n,x_m)\ll c$. Thus for all $n,m\geq n_0$,we have, $\parallel A(x_n,x_n,\dots,x_n,x_m)\parallel\leq K\parallel c\parallel<\varepsilon$. This means that $A(x_n,x_n,\dots,x_n,x_m)\to 0$ as $n,m\to\infty$.

Conversely, let $A(x_n,x_n,\dots,x_n,x_m)\to 0$ as $n,m\to\infty$. And, let $c\in E$ with $0\ll c$. Then, there exists $\varepsilon>0$, such that $\parallel x\parallel<\varepsilon$ implies $c-x\in int.P$. For this ε, there exists $n_0\in\mathbb{N}$ such that for all $n,m\geq n_0$, $\parallel A(x_n,x_n,\dots,x_n,x_m)\parallel<\varepsilon$. Therefore, we have $c-A(x_n,x_n,\dots,x_n,x_m)\in int.P$. Thus, $A(x_n,x_n,\dots,x_n,x_m)\ll c$ for all $n,m\geq n_0$. Therefore, the sequence $\{x_n\}$ is a Cauchy sequence.
Lemma 2.8 Let (X,A) be a cone A_b-metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be sequence in X. If $\{x_n\}$ converges to w, then $\{x_n\}$ is a Cauchy sequence. That is, every convergent sequence is Cauchy.
Proof. We have

$$\begin{aligned} A(x_n,x_n,\dots,x_n,x_m) &\leq (n-1)bA(x_n,x_n,\dots,x_n,w)\\ &\quad +bA(x_m,x_m,\dots,x_m,w)\\ \Rightarrow\parallel A(x_n,x_n,\dots,x_n,x_m)\parallel &\leq K[(n-1)b\parallel A(x_n,x_n,\dots,x_n,w)\parallel\\ &\quad +b\parallel A(x_m,x_m,\dots,x_m,w)\parallel]\\ &\quad \to 0\ as\ n,m\to\infty \end{aligned}$$

$$\Rightarrow A(x_n,x_n,\dots,x_n,x_m)\to 0\ as\ n,m\to\infty.$$

Therefore $\{x_n\}$ is a Cauchy sequence.

3. Main Results

We now state and prove our main results.
Theorem 3.1 Let (X,A) be a complete cone A_b-metric space, P be a normal cone with normal constant K. Suppose the mapping $T:X\to X$ satisfies the following condition

$$A(Tx,\dots,Tx,Ty)\leq hA(x,\dots,x,y) \tag{1}$$

for all $x,y\in X$,where $h\in\left[0,\frac{1}{b^2}\right)$ is a constant. Then, T has a unique fixed point $w\in X$ and we have $\lim_{n\to\infty}T^nx=w$, for all $x\in X$.

Proof. Let $x_0 \in X$ and a sequence $\{x_n\}$ be defined by $T^n x_0 = x_n$. Suppose that $x_n \neq x_{n+1}$ for all n. Using condition (1), we obtain

$$A(x_n, \ldots, x_n, x_{n+1}) \leq hA(x_{n-1}, \ldots, x_{n-1}, x_n) \leq \cdots \leq h^n A(x_0, \ldots, x_0, x_1) \quad (2)$$

Using condition 3(Def. 2.1) and (2), we have for all $n, m \in \mathbb{N}$ with $m > n$,

$$\begin{aligned}
A(x_n, \ldots, x_n, x_m) &\leq b[(n-1)A(x_n, \ldots, x_n, x_{n+1}) + A(x_m, \ldots, x_m, x_{n+1})] \\
&\leq (n-1)bA(x_n, \ldots, x_n, x_{n+1}) + b^2 A(x_{n+1}, \ldots, x_{n+1}, x_m) \\
&\leq (n-1)bA(x_n, \ldots, x_n, x_{n+1}) \\
&\quad +(n-1)b^3 A(x_{n+1}, \ldots, x_{n+1}, x_{n+2}) \\
&\quad +b^4 A(x_{n+2}, \ldots, x_{n+2}, x_m) \\
&\leq (n-1)bA(x_n, \ldots, x_n, x_{n+1}) \\
&\quad +(n-1)b^3 A(x_{n+1}, \ldots, x_{n+1}, x_{n+2}) \\
&\quad +(n-1)b^5 A(x_{n+2}, \ldots, x_{n+2}, x_{n+3}) + \ldots \\
&\quad +b^{2(m-n-1)} A(x_{m-1}, \ldots, x_{m-1}, x_m) \\
&< (n-1)b\{A(x_n, \ldots, x_n, x_{n+1}) + b^2 A(x_{n+1}, \ldots, x_{n+1}, x_{n+2}) \\
&\quad +b^4 A(x_{n+2}, \ldots, x_{n+2}, x_{n+3}) + \ldots \\
&\quad +b^{2(m-n-1)} A(x_{m-1}, \ldots, x_{m-1}, x_m)\} \\
&\leq (n-1)b\{h^n + b^2 h^{n+1} + b^4 h^{n+2} + \ldots \\
&\quad +b^{2(m-n-1)} h^{m-1}\} A(x_0, \ldots, x_0, x_1) \\
&= (n-1)bh^n\{1 + b^2 h + (b^2 h)^2 + \ldots \\
&\quad +(b^2 h)^{m-n-1}\} A(x_0, \ldots, x_0, x_1) \\
&\leq \frac{(n-1)bh^n}{1 - b^2 h} A(x_0, \ldots, x_0, x_1) \\
\Rightarrow \| A(x_n, \ldots, x_n, x_m) \| &\leq \frac{(n-1)bh^n K}{1 - b^2 h} \| A(x_0, \ldots, x_0, x_1) \|
\end{aligned}$$

Since $h \in \left[0, \frac{1}{b^2}\right)$ and $b \geq 1$, we have $0 \leq h < 1$. Therefore, taking limit for $n \to \infty$ (and consequently $n, m \to \infty$), we have $\| A(x_n, \ldots, x_n, x_m) \| \to 0$.
Thus, $A(x_n, \ldots, x_n, x_m) \to 0$ as $n, m \to \infty$.
Therefore, the sequence $\{x_n\}$ is Cauchy. Using the completeness hypothesis, there exists $w \in X$ such that $\lim_{n\to\infty} x_n = w$ i.e. $\lim_{n\to\infty} T^n x_0 = w$.
We show that w is a fixed point of T.
Using condition 3(Def. 2.1) and (1), we have

$$\begin{aligned}
A(Tw, \ldots, Tw, w) &\leq (n-1)bA(Tw, \ldots, Tw, Tx_n) + bA(w, \ldots, w, Tx_n) \\
&\leq (n-1)bhA(w, \ldots, w, x_n) + bA(w, \ldots, w, x_{n+1}) \\
\Rightarrow \| A(Tw, \ldots, Tw, w) \| &\leq K\{(n-1)bh \| A(w, \ldots, w, x_n) \| \\
&\quad +b \| A(w, \ldots, w, x_{n+1}) \|\} \to 0 \; as \; n \to \infty \\
\Rightarrow \| A(Tw, \ldots, Tw, w) \| &= 0 \\
\Rightarrow A(Tw, \ldots, Tw, w) &= 0 \\
\Rightarrow Tw &= w.
\end{aligned}$$

To prove T has unique fixed point.
Let w, w_1 be two fixed points of T. Taking $x = w$ and $y = w_1$ in condition

(1), we have $A(w,\ldots,w,w_1) = A(Tw,\ldots,Tw,Tw_1) \leq hA(w,\ldots,w,w_1)$, where $h \in \left[0, \frac{1}{b^2}\right), b \geq 1$. And this implies that $A(w,\ldots,w,w_1) = 0$. Thus, we have $w = w_1$.

Theorem 3.2 Let (X, A) be a complete cone A_b-metric space and P be a normal cone with normal constant K. Suppose the mapping $T : X \to X$ satisfies the following condition

$$A(Tx,\ldots,Tx,Ty) \leq h[A(x,\ldots,x,Tx) + A(y,\ldots,y,Ty)] \tag{3}$$

for all $x, y \in X$,where $0 \leq h < min\left\{\frac{1}{2}, \frac{1}{(n-1)b^2}\right\}$ is a constant. Then T has a unique fixed point $w \in X$ and we have $\lim_{n\to\infty} T^n x = w$, for each $x \in X$.

Proof. Let $x_0 \in X$ and a sequence $\{x_n\}$ be defined by $T^n x_0 = x_n$. Suppose that $x_n \neq x_{n+1}$ for all n. Using condition 3(Def. 2.1) and (3), we get

$$\begin{array}{rcl}
A(x_n,\ldots,x_n,x_{n+1}) & = & A(Tx_{n-1},\ldots,Tx_{n-1},Tx_n) \\
& \leq & h[A(x_{n-1},\ldots,x_{n-1},Tx_{n-1}) + A(x_n,\ldots,x_n,Tx_n)] \\
& = & h[A(x_{n-1},\ldots,x_{n-1},x_n) + A(x_n,\ldots,x_n,x_{n+1})] \\
\Rightarrow A(x_n,\ldots,x_n,x_{n+1}) & \leq & \dfrac{h}{1-h}A(x_{n-1},\ldots,x_{n-1},x_n) \\
\Rightarrow A(x_n,\ldots,x_n,x_{n+1}) & \leq & kA(x_{n-1},\ldots,x_{n-1},x_n),\ where\ k = \dfrac{h}{1-h} < 1,\ as\ h < \dfrac{1}{2} \\
\Rightarrow A(x_n,\ldots,x_n,x_{n+1}) & \leq & kA(x_{n-1},\ldots,x_{n-1},x_n) \\
& \leq & k^2A(x_{n-2},\ldots,x_{n-2},x_{n-1}) \\
& & \ldots \\
& \leq & k^nA(x_0,\ldots,x_0,x_1) \\
\Rightarrow A(x_n,\ldots,x_n,x_{n+1}) & \leq & k^nA(x_0,\ldots,x_0,x_1)
\end{array}$$

Now for $m > n$, we have

$$\begin{array}{rcl}
A(x_n,\ldots,x_n,x_m) & \leq & h[A(x_{n-1},\ldots,x_{n-1},x_n) \\
& & +A(x_{m-1},\ldots,x_{m-1},x_m)] \\
& \leq & h[k^{n-1}A(x_0,\ldots,x_0,x_1) \\
& & +k^{m-1}A(x_0,\ldots,x_0,x_1)] \\
& = & h\left(k^{n-1} + k^{m-1}\right)A(x_0,\ldots,x_0,x_1) \\
\Rightarrow \| A(x_n,\ldots,x_n,x_m) \| & \leq & Kh\left(k^{n-1} + k^{m-1}\right) \| A(x_0,\ldots,x_0,x_1) \| \\
& & \to 0\ as\ m,n \to \infty
\end{array}$$

$$\Rightarrow A(x_n,\ldots,x_n,x_m) \to 0\ as\ m,n \to \infty.$$

Therefore the sequence $\{x_n\}$ is Cauchy. By the completeness of X, there exists $w \in X$ such that $\lim_{n\to\infty} x_n = w$ i.e. $\lim_{n\to\infty} T^n x_0 = w$.

Also, we have

$$\begin{aligned}
A(Tw,\ldots,Tw,w) &\leq (n-1)bA(Tw,\ldots,Tw,Tx_n)+bA(w,\ldots,w,Tx_n)] \\
&\leq (n-1)b\left[h\{A(w,\ldots,w,Tw)+A(x_n,\ldots,x_n,Tx_n)\}\right] \\
&\quad +b^2A(Tx_n,\ldots,Tx_n,w) \\
&\leq (n-1)b^2hA(Tw,\ldots,Tw,w) \\
&\quad +(n-1)bhA(x_n,\ldots,x_n,x_{n+1})+b^2A(x_{n+1},\ldots,x_{n+1},w) \\
\Rightarrow A(Tw,\ldots,Tw,w) &\leq \frac{1}{1-(n-1)b^2h}[(n-1)bhA(x_n,\ldots,x_n,x_{n+1}) \\
&\quad +b^2A(x_{n+1},\ldots,x_{n+1},w)] \\
\Rightarrow\| A(Tw,\ldots,Tw,w)\| &\leq \frac{K}{1-(n-1)b^2h}[(n-1)bh\parallel A(x_n,\ldots,x_n,x_{n+1})\parallel \\
&\quad +b^2\parallel A(x_{n+1},\ldots,x_{n+1},w)\parallel] \\
&\quad \to 0 \ as\ n\to\infty \\
\Rightarrow\| A(Tw,\ldots,Tw,w)\| &= 0 \\
\Rightarrow Tw &= w.
\end{aligned}$$

Therefore, w is a fixed point of T.
To show that the fixed point of T is unique.
Let there be another point w_1 in X such that $Tw_1=w_1$. Then,

$$\begin{aligned}
A(w,\ldots,w,w_1) &= A(Tw,\ldots,Tw,Tw_1) \\
&\leq h\left[A(w,\ldots,w,Tw)+A(w_1,\ldots,w_1,Tw_1)\right] \\
&= h\left[A(w,\ldots,w,w)+A(w_1,\ldots,w_1,w_1)\right] \\
&= 0 \\
\Rightarrow A(w,\ldots,w,w_1)=0 & \\
\Rightarrow w=w_1 &
\end{aligned}$$

Hence, the fixed point of T is unique.
Theorem 3.3 Let (X,A) be a complete cone A_b-metric space and P be a normal cone with normal constant K. Suppose the mapping $T:X\to X$ satisfies the following condition

$$A(Tx,\ldots,Tx,Ty)\leq h\left[A(x,\ldots,x,Ty)+A(y,\ldots,y,Tx)\right] \tag{4}$$

for all $x,y\in X$, where $h\in\left[0,\frac{1}{b^2\{(n-1)b+1\}}\right)$ is a constant. Then T has a unique fixed point $w\in X$ and we have $\lim_{n\to\infty}T^nx=w$, for each $x\in X$.
Proof. Let $x_0\in X$ and a sequence $\{x_n\}$ be defined by $T^nx_0=x_n$. Suppose that

$x_n \neq x_{n+1}$ for all n. Using condition 3(Def. 2.1) and (4), we get

$$\begin{aligned}
A(x_n,\ldots,x_n,x_{n+1}) &= A(Tx_{n-1},\ldots,Tx_{n-1},Tx_n)\\
&\leq h\left[A(x_n,\ldots,x_n,Tx_{n-1})+A(x_{n-1},\ldots,x_{n-1},Tx_n)\right]\\
&= h\left[A(x_n,\ldots,x_n,x_n)+A(x_{n-1},\ldots,x_{n-1},x_{n+1})\right]\\
&= hA(x_{n-1},\ldots,x_{n-1},x_{n+1})\\
&\leq (n-1)bhA(x_{n-1},\ldots,x_{n-1},x_n)+bhA(x_{n+1},\ldots,x_{n+1},x_n)\\
&\leq (n-1)bhA(x_{n-1},\ldots,x_{n-1},x_n)+b^2hA(x_n,\ldots,x_n,x_{n+1})\\
\Rightarrow A(x_n,\ldots,x_n,x_{n+1}) &\leq \frac{(n-1)bh}{1-b^2h}A(x_{n-1},\ldots,x_{n-1},x_n)\\
\Rightarrow A(x_n,\ldots,x_n,x_{n+1}) &\leq kA(x_{n-1},\ldots,x_{n-1},x_n),\ \textit{where}\\
& \quad k=\frac{(n-1)bh}{1-b^2h}<1\left(\begin{array}{ll} h<\frac{1}{b^2\{(n-1)b+1\}}\Rightarrow\frac{(n-1)b^3h}{1-b^2h}<1, & \\ \Rightarrow\frac{(n-1)bh}{1-b^2h}<\frac{1}{b^2}<1. & \end{array}\right).\\
\Rightarrow A(x_n,\ldots,x_n,x_{n+1}) &\leq kA(x_{n-1},\ldots,x_{n-1},x_n)\\
&\leq k^2A(x_{n-2},\ldots,x_{n-2},x_{n-1})\\
&\quad \ldots\\
&\leq k^nA(x_0,\ldots,x_0,x_1)\\
\Rightarrow A(x_n,\ldots,x_n,x_{n+1}) &\leq k^nA(x_0,\ldots,x_0,x_1)
\end{aligned}$$

Now, for $m>n$, we have

$$\begin{aligned}
A(x_n,\ldots,x_n,x_m) &\leq b\left[(n-1)A(x_n,\ldots,x_n,x_{n+1})+A(x_m,\ldots,x_m,x_{n+1})\right]\\
&\leq (n-1)bA(x_n,\ldots,x_n,x_{n+1})+b^2A(x_{n+1},\ldots,x_{n+1},x_m)\\
&\leq (n-1)bA(x_n,\ldots,x_n,x_{n+1})+(n-1)b^3A(x_{n+1},\ldots,x_{n+1},x_{n+2})\\
&\quad +b^4A(x_{n+2},\ldots,x_{n+2},x_m)\\
&\leq (n-1)bA(x_n,\ldots,x_n,x_{n+1})+(n-1)b^3A(x_{n+1},\ldots,x_{n+1},x_{n+2})\\
&\quad +(n-1)b^5A(x_{n+2},\ldots,x_{n+2},x_{n+3})+\ldots\\
&\quad +b^{2(m-n-1)}A(x_{m-1},\ldots,x_{m-1},x_m)\\
&< (n-1)b\{A(x_n,\ldots,x_n,x_{n+1})+b^2A(x_{n+1},\ldots,x_{n+1},x_{n+2})\\
&\quad +b^4A(x_{n+2},\ldots,x_{n+2},x_{n+3})+\ldots\\
&\quad +b^{2(m-n-1)}A(x_{m-1},\ldots,x_{m-1},x_m)\}\\
&\leq (n-1)b\{k^n+b^2k^{n+1}+b^4k^{n+2}+\ldots\\
&\quad +b^{2(m-n-1)}k^{m-1}\}A(x_0,\ldots,x_0,x_1)\\
&= (n-1)bk^n\{1+b^2k+(b^2k)^2+\ldots\\
&\quad +(b^2k)^{m-n-1}\}A(x_0,\ldots,x_0,x_1)\\
&\leq \frac{(n-1)bk^n}{1-b^2k}A(x_0,\ldots,x_0,x_1)\\
\Rightarrow\parallel A(x_n,\ldots,x_n,x_m)\parallel &\leq \frac{(n-1)bk^nK}{1-b^2k}\parallel A(x_0,\ldots,x_0,x_1)\parallel
\end{aligned}$$

Taking limit for $n\to\infty$ (consequently $n,m\to\infty$), we have

$$\parallel A(x_n,\ldots,x_n,x_m)\parallel\to 0$$

Thus we have, $A(x_n, \ldots, x_n, x_m) \to 0$ as $n, m \to \infty$.
Therefore, the sequence $\{x_n\}$ is Cauchy. By the completeness of X, there exists $w \in X$ such that $\lim_{n\to\infty} x_n = w$ i.e $\lim_{n\to\infty} T^n x_0 = w$.
Also, we have,

$$\begin{aligned}
A(Tw, \ldots, Tw, w) &\leq (n-1)bA(Tw, \ldots, Tw, Tx_n) + bA(w, \ldots, w, Tx_n) \\
&\leq (n-1)b\left[h\{A(w, \ldots, w, Tx_n) + A(x_n, \ldots, x_n, Tw)\}\right] \\
&\quad +bA(w, \ldots, w, x_{n+1}) \\
&= \{(n-1)bh + b\}A(w, \ldots, w, x_{n+1}) \\
&\quad +(n-1)bhA(x_n, \ldots, x_n, Tw) \\
&\leq \{(n-1)bh + b\}A(w, \ldots, w, x_{n+1}) \\
&\quad +(n-1)bh[(n-1)bA(x_n, \ldots, x_n, w) + bA(Tw, \ldots, Tw, w)] \\
\Rightarrow A(Tw, \ldots, Tw, w) &\leq \frac{1}{1-(n-1)b^2h}[\{(n-1)bh + b\}A(w, \ldots, w, x_{n+1}) \\
&\quad +(n-1)^2b^2hA(x_n, \ldots, x_n, w)] \\
\Rightarrow \| A(Tw, \ldots, Tw, w) \| &\leq \frac{K}{1-(n-1)b^2h}[\{(n-1)bh + b\} \| A(w, \ldots, w, x_{n+1}) \| \\
&\quad +(n-1)^2b^2h \| A(x_n, \ldots, x_n, w) \|] \\
&\quad \to 0 \ as \ n \to \infty \\
\Rightarrow A(Tw, \ldots, Tw, w) &= 0 \\
\Rightarrow Tw = w.
\end{aligned}$$

Therefore w is a fixed point of T.
To show that the fixed point of T is unique.
Let there be another fixed point w_1 of T in X so that $Tw_1 = w_1$.
Then,

$$\begin{aligned}
A(w, \ldots, w, w_1) &= A(Tw, \ldots, Tw, Tw_1) \\
&\leq h[A(w, \ldots, w, Tw_1) + A(w_1, \ldots, w_1, Tw)] \\
&= h[A(w, \ldots, w, w_1) + A(w_1, \ldots, w_1, w)] \\
&\leq h[A(w, \ldots, w, w_1) + bA(w, \ldots, w, w_1)] \\
&= h(b+1)A(w, \ldots, w, w_1) \\
\Rightarrow A(w, \ldots, w, w_1) &\leq h(b+1)A(w, \ldots, w, w_1)
\end{aligned}$$

But,

$$\begin{aligned}
h &< \frac{1}{b^2\{(n-1)b+1\}} < \frac{1}{b^2(b+1)} \\
\Rightarrow h(b+1) &< \frac{1}{b^2} < 1, \ since \ b \geq 1
\end{aligned}$$

Therefore, we have, $A(w, \ldots, w, w_1) = 0 \Rightarrow w = w_1$.
Hence, the fixed point of T is unique.
Note: If we take $n = 3$ in the above Theorems 3.1, 3.2 and 3.3, then we get the fixed point theorems of cone S_b-metric space in [7].
Example 3.4 Let $E = \mathbb{R}^2$,the Euclidean plane, and $P = \{(x, y) \in E : x, y \geq 0\}$,a

normal cone in E. Let $X=[-1,1]$ and $A: X^n \to E$ be defined as follows

$$A(x_1,x_2,\ldots,x_n)=(1,1)\sum_{i=1}^{n}\sum_{i<j} \mid x_i-x_j \mid^2,\ \forall x_i\in X, i=1,2,\ldots,n.$$

Then, (X,A) is complete cone A_b-metric space with $b=2$.
If we define $T:X\to X$ by $Tx=\frac{x}{4}$, then T satisfies the following condition for all $x_i\in X, i=1,2,3,\ldots,n$

$$\begin{aligned} A(Tx_1,Tx_2,\ldots,Tx_n) &= A\left(\frac{x_1}{4},\frac{x_2}{4},\ldots,\frac{x_n}{4}\right)\\ &= \frac{1}{16}A(x_1,x_2,\ldots,x_n)\\ &\leq kA(x_1,x_2,\ldots,x_n)\end{aligned}$$

where $k\in\left[\frac{1}{16},\frac{1}{b^2}\right)\subset\left[0,\frac{1}{b^2}\right), b=2$. And $x=0$ is the unique fixed point of T in X as asserted by Theorem 3.1.

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K. Anthony Singh
Department of Mathematics, D.M. College of Science, Imphal-795001, India
E-mail address: `anthonykumam@manipuruniv.ac.in`

M. R. Singh
Department of Mathematics, Manipur University, Canchipur, Imphal-795003, India
E-mail address: `mranjitmu@rediffmail.com`

Th. Chhatrajit Singh
Department of Mathematics, Manipur Technical University, Tayelpat, Imphal-795004, India
E-mail address: `chhatrajit@mtu.ac.in`