Electronic Journal of Mathematical Analysis and Applications
Vol. 9(1) Jan. 2021, pp. 124-130.
ISSN: 2090-729X(online)
http://math-frac.org/Journals/EJMAA/

ON THE GAUSS $k-$FIBONACCI POLYNOMIALS

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Abstract. In this paper, we define the new families of Gauss $k-$Fibonacci polynomials. We obtain some exciting properties of the families. We give the relationships between the family of the Gauss $k-$ Fibonacci polynomials and the known Gauss Fibonacci polynomials. We also define the generalized polynomials for these numbers. We also obtain some interesting properties of the polynomials. Furthermore, we find the new generalizations of these families and the polynomials in matrix representation. Then we prove Cassini's Identities for the families and their polynomials.

1. Introduction

Fibonacci numbers have exciting properties and many applications in many branches of mathematics.[5], [8], [12], [13], [14], [15]. In [6], Mikkawy and Sogabe gave a new family of $k-$Fibonacci numbers. In [10], Özkan et.all. defined a new family of $k-$Lucas numbers and give some properties about the family of the numbers. There are some works on polynomials of the families of $k-$Fibonacci numbers and k-Lucas numbers [8], [13], [15]. In [3], Falcon and Plaza gave general k-Fibonacci numbers and showed properties of these numbers were related with elementary matrix algebra. In [2], Bolat and Köse found some important properties about $k-$Fibonacci number. Finally [16], Taş presented $k-$Gauss Fibonacci numbers and give some properties related to these numbers.

In this paper, we define the new families of Gauss $k-$Fibonacci polynomials. We obtain some exciting properties of the families. We give the relationships between the family of the Gauss $k-$ Fibonacci polynomials and the known Fibonacci polynomials. More, we find the new generalizations of these families in matrix representation. Then we prove Cassini's Identities for the families.

2. Material and Methods

Now, we introduce the Fibonacci polynomials $F_n(x)$, the Lucas polynomials $L_n(x)$, the Gauss Fibonacci numbers $GF_n(x)$ and he Gauss Fibonacci polynomials $gf_n(x)$-matrix, $k-$Fibonacci numbers $F_n^{(k)}$, $k-$Gaussian Fibonacci numbers $GF_n^{(k)}$.

1991 *Mathematics Subject Classification.* 11B39, 11B83, 11C20.
Key words and phrases. Gauss Fibonacci polynomials, k-Fibonacci numbers, Gauss k-Fibonacci number.
Submitted April 7, 2020. Revised May 4, 2020.

Definition 1. *The Fibonacci polynomials $F_n(x)$ are defined by*

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x) \tag{1}$$

where $F_1(x) = 1$, $F_2(x) = x$, and $n \geq 3$.[7]

Definition 2. *The Lucas polynomials $L_n(x)$ are defined by*

$$L_n(x) = xL_{n-1}(x) + L_{n-2}(x) \tag{2}$$

where $L_0(x) = 2$, $L_1(x) = 1$, and $n \geq 2$.[7]

The Lucas polynomials satisfy the following properties:

$$L_n(x) = F_{n+1}(x) + F_{n-1}(x) \tag{3}$$

$$xL_n(x) = F_{n+2}(x) + F_{n-2}(x)$$
$$L_n(x) = xF_n(x) + 2F_{n-1}(x)$$

[7].

Definition 3. *The Gaussian Fibonacci numbers $\{GF_n\}_{n=0}^{\infty}$ are defined by the following recurrence relation*

$$GF_{n+1} = GF_n + GF_{n-1}, n \geq 1 \tag{4}$$

with initial conditions $GF_0 = i$ and $GF_1 = 1$.[1]

Binet formulas for GF_n is given by as follows

$$GF_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} + i\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} \tag{5}$$

where $\alpha = \frac{1+\sqrt[2]{5}}{2}$ and $\beta = \frac{1-\sqrt[2]{5}}{2}$.[4]

Definition 4. *The Gaussian Fibonacci polynomials $\{GF_n(x)\}_{n=0}^{\infty}$ are defined following recurrence relation*

$$GF_{n+1}(x) = GF_n(x) + GF_{n-1}(x), n \geq 2 \tag{6}$$

with initial conditions $GF_1(x) = 1$ and $GF_2(x) = x + i$.[11]

For $n > 2$, we have

$$GF_n(x) = F_n(x) + iF_{n-1}(x). \tag{7}$$

Binet formulas for $GF_n(x)$ is given by as follows

$$GF_n(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)} + i\frac{\alpha^{n-1}(x) - \beta^{n-1}(x)}{\alpha(x) - \beta(x)} \tag{8}$$

where $\alpha = \frac{x+\sqrt[2]{x^2+4}}{2}$ and $\beta = \frac{x-\sqrt[2]{x^2+4}}{2}$.[11]

The Gauss Fibonacci polynomial matrix $gf_n(x)$ is defined by

$$gf_n(x) = \begin{bmatrix} GF_{n+1}(x) & GF_n(x) \\ GF_n(x) & GF_{n-1}(x) \end{bmatrix}, n \geq 0.$$

[11]

For $n > 1$, we have

$$GF_{n+1}(x)GF_{n-1}(x) - GF_n^2(x) = (-1)^n(2 - ix). \tag{9}$$

[11]

Definition 5. *There are unique numbers m and r such that $n = mk + r$ where $m, k(\neq 0)$ natural numbers and $0 \leq r < k$. The generalized $k-$Fibonacci numbers $F_n^{(k)}$ are defined by*

$$F_n^{(k)} = \frac{1}{(\sqrt[2]{5})^k}(\alpha^{m+1} - \beta^{m+1})^{k-r}(\alpha^m - \beta^m)^r$$

where $\alpha = \frac{1+\sqrt[2]{5}}{2}$ *and* $\beta = \frac{1-\sqrt[2]{5}}{2}$.[6]

Definition 6. *There are unique numbers m and r such that $n = mk + r$ where $m, k(\neq 0)$ natural numbers and $0 \leq r < k$. The generalized $k-$ Gaussian Fibonacci numbers $GF_n^{(k)}$ are defined by*

$$\begin{aligned} GF_n^{(k)} &= \left[\left(\frac{\sqrt[2]{5}}{5} + \left(\frac{5-\sqrt[2]{5}}{10}\right)i\right)\alpha^m + \left(-\frac{\sqrt[2]{5}}{5} + \left(\frac{5+\sqrt[2]{5}}{10}\right)i\right)\beta^m\right]^{k-r} \\ &\quad \left[\left(\frac{\sqrt[2]{5}}{5} + \left(\frac{5-\sqrt[2]{5}}{10}\right)i\right)\alpha^{m+1} + \left(-\frac{\sqrt[2]{5}}{5} + \left(\frac{5+\sqrt[2]{5}}{10}\right)i\right)\beta^{m+1}\right]^r \end{aligned}$$

where $\alpha = \frac{1+\sqrt[2]{5}}{2}$ *and* $\beta = \frac{1-\sqrt[2]{5}}{2}$.[16]

3. Main Results

Definition 7. *There are unique numbers m and r such that $n = mk + r$ where $m, k(\neq 0)$ natural numbers and $0 \leq r < k$. The generalized Gauss $k-$Fibonacci Polynomials $GF_n^{(k)}(x)$ are defined by*

$$\begin{aligned} GF_n^{(k)}(x) &= \left[\frac{\alpha^m(x) - \beta^m(x)}{\alpha(x) - \beta(x)} + ix\frac{\alpha^{m-1}(x) - \beta^{m-1}(x)}{\alpha(x) - \beta(x)}\right]^{k-r} \\ &\quad \left[\left(\frac{\alpha^{m+1}(x) - \beta^{m+1}(x)}{\alpha(x) - \beta(x)} + ix\frac{\alpha^m(x) - \beta^m(x)}{\alpha(x) - \beta(x)}\right)\right]^r \end{aligned}$$

where $\alpha = \frac{x+\sqrt[2]{x^2+4}}{2}$ *and* $\beta = \frac{x-\sqrt[2]{x^2+4}}{2}$.

Also, we can find the generalized Gauss $k-$Fibonacci polynomials by matrix methods. Indeed, it is clear that

$$GF_n^{k-1}(x)gf_n(x) = \begin{bmatrix} GF_{kn+k+1}^{(k)}(x) & GF_{kn+k}^{(k)}(x) \\ GF_{kn+k}^{(k)}(x) & GF_{kn+k-1}^{(k)}(x) \end{bmatrix}$$

where $gf_n(x) = \begin{bmatrix} GF_{n+1}(x) & GF_n(x) \\ GF_n(x) & GF_{n-1}(x) \end{bmatrix}, n \geq 0.$

Let's give some values for the Gauss k-Fibonacci polynomials in the table below.

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$GF_0^{(k)}(x)$	0	0	0	0	0	0
$GF_1^{(k)}(x)$	1	0	0	0	0	0
$GF_2^{(k)}(x)$	$x+i$	1	0	0	0	0
$GF_3^{(k)}(x)$	x^2+1+ix	$x+i$	1	0	0	0
$GF_4^{(k)}(x)$	x^3+2x+ ix^2+i	x^2-1+ $2ix$	$x+i$	1	0	0
$GF_5^{(k)}(x)$	x^4+3x^2+ $1+ix^3+2ix$	x^3+2ix^2+ i	x^2-1+ $2ix$	$x+i$	1	0
$GF_6^{(k)}(x)$	x^5+4x^3+3x+ ix^4+3ix^2+i	x^4+x^2+1+ $2ix^3+2ix$	x^3-3x+ $3ix^2-i$	x^2-1+ $2ix$	$x+i$	1
$GF_7^{(k)}(x)$	x^6+5x^4+ $6x^2+1+$ ix^5+4ix^3+4ix	x^5+2x^3+ $x+2ix^4$ $+4ix^2+i$	x^4-2x^2-1 $3ix^3+ix$	x^3-3x+ $3ix^2-i$	x^2-1+ $2ix$	$x+i$
$GF_8^{(k)}(x)$	$x^7+6x^5+10x^3+$ $4x+ix^6+5ix^4+$ $6ix^2+1$	$x^6+3x^4+2x^2-$ $1+2ix^5+$ $6ix^3-4ix$	x^5-x^3-x+ $3ix^4+3ix^2$ $+i$	x^4-6x^2+ $1+4ix^3-$ $4ix$	x^3-3x+ $3ix^2-i$	x^2-1+ $2ix$

From (8) and Definition 7, we can give the generalized Gauss $k-$Fibonacci polynomials via the Gauss Fibonacci polynomials.

$$GF_n^{(k)}(x) = (GF_m(x))^{k-r}(GF_{m+1}(x))^r, n = mk + r. \tag{10}$$

If $k = 1$ in last equation, we have that $m = n$ and $r = 0$ so $GF_n^{(1)}(x) = GF_n(x)$.

Throughout this paper, let $k, m \in \{1, 2, 3, \ldots\}$.

Theorem 1. *For k and m, we have*

$$\left[(GF_{m+1}(x))^k - (GF_m(x))^k\right] = \left[GF_{(m+1)k}^{(k)}(x) - GF_{mk}^{(k)}(x)\right].$$

Proof. By using (10), we get

$$\begin{aligned} GF_{(m+1)k}^{(k)}(x) - GF_{mk}^{(k)}(x) &= \left[(GF_m(x))^{k-k}(GF_{m+1}(x))^k\right] - \\ &\quad \left[(GF_m(x))^{k-0}(GF_{m+1}(x))^0\right] \\ &= (GF_{m+1}(x))^k - (GF_m(x))^k. \end{aligned}$$

□

For $k = 2, 3, 4$ and n, we have the interesting following properties between these numbers.

$$\begin{aligned} GF^{(2)}_{2n}(x) &= GF^2_n(x) \\ GF^{(2)}_{2n+1}(x) &= GF_n(x)GF_{n+1}(x) \\ GF^{(3)}_{3n}(x) &= GF^3_n(x) \\ GF^{(3)}_{3n+1}(x) &= GF^2_n(x)GF_{n+1}(x) \\ GF^{(3)}_{3n+2}(x) &= GF_n(x)GF^2_{n+1}(x) \\ GF^{(4)}_{4n}(x) &= GF^4_n(x) \\ GF^{(4)}_{4n+1}(x) &= GF^3_n(x)GF_{n+1}(x) \\ GF^{(4)}_{4n+2}(x) &= GF^2_n(x)GF^2_{n+1}(x) \\ GF^{(4)}_{4n+3}(x) &= GF_n(x)GF^3_{n+1}(x) \end{aligned} \tag{11}$$

From Definition 7, we know that $n = mk + r$ where $m, k(\neq 0)$ natural numbers and $0 \leq r < k$. Let's suppose that $GF^{(k)}_{-n} = 0$ for $k = 1, 2, \ldots$.

Theorem 2. *For the $GF^{(2)}_n(x)$, we have*

$$\begin{aligned} GF^{(2)}_{2(n-1)}(x) - GF_n(x)GF_{n-2}(x) &= (-1)^{n+1}(2 - ix) \\ GF^{(2)}_{2n}(x) + GF^{(2)}_{2n+2}(x) &= F_{2n}(x)(x + 2i). \end{aligned}$$

Proof. By using (11) and (9), we get

$$GF^{(2)}_{2(n-1)}(x) = GF^{(2)}_{n-1}(x).$$

$$\begin{aligned} GF^{(2)}_{n-1}(x) - GF_n(x)GF_{n-2}(x) &= -(GF_n(x)GF_{n-2}(x) - GF^{(2)}_{n-1}(x)), \\ &= -(-1)^n(2 - ix), \\ &= (-1)^{n+1}(2 - ix). \end{aligned}$$

By using (3), (7) and (11), we obtain

$$\begin{aligned} GF^{(2)}_{2n}(x) + GF^{(2)}_{2n+2}(x) &= GF^2_n(x) + GF^2_{n+1}(x). \\ GF^2_n(x) + GF^2_{n+1}(x) &= (F_n(x) + iF_{n-1}(x))^2 + (F_{n+1}(x) + iF_n(x))^2 \\ &= F^2_n(x) + 2iF_n(x)F_{n-1}(x) + i^2 + \\ &\quad F^2_{n+1}(x) + 2iF_{n+1}(x)F_n(x) + i^2F^2_n(x) \\ &= F^2_{n+1}(x) - F^2_{n-1}(x) + 2iF_n(x)(F_{n-1}(x) + F_{n+1}(x)) \\ &= (F_{n+1}(x) + F_{n-1}(x))(F_{n+1}(x) + F_{n-1}(x)) + \\ &\quad 2iF_n(x)L_n(x) \\ &= xF_n(x)L_n(x) + 2iF_n(x)L_n(x) \\ &= F_n(x)L_n(x)(x + 2i) \\ &= F_{2n}(x)(x + 2i). \end{aligned}$$

□

Theorem 3. *(Cassini's Identity) Let $GF_n^{(k)}(x)$ be the generalized Gauss k-Fibonacci polynomials. For $n, k \geq 2$, Cassini's Identity $GF_n^{(k)}(x)$ is as follows:*

$$GF_{kn+t}^{(k)}(x)GF_{kn+t-2}^{(k)}(x)-\left(GF_{kn+t-1}^{(k)}(x)\right)^2 = \left\{ \begin{array}{ll} GF_n^{2k-2}(x)(-1)^n(2-ix), & t=1 \\ 0, & t \neq 1 \end{array} \right\}.$$

Proof. By using (9) and (10), we get

$$\begin{aligned} & GF_{kn+t}^{(k)}(x)GF_{kn+t-2}^{(k)}(x) - \left(GF_{kn+t-1}^{(k)}(x)\right)^2 \\ = & \left(GF_{n-1}^{k-1}(x)GF_{n+t}(x)\right)\left(GF_n^{k-1}(x)GF_{n+t-2}(x)\right) - \left(GF_n^{k-1}(x)GF_{n+t-1}(x)\right)^2 \\ = & \left(GF_n^{k-1}(x)\right)^2\left(GF_{n+t}(x)GF_{n+t-2}(x) - \left(GF_{n+t-1}(x)\right)^2\right) \\ = & GF_n^{2k-2}(x)\left(GF_{n+t}(x)GF_{n+t-2}(x) - GF_{n+t-1}(x))^2\right) \end{aligned}$$

For $t = 1$;

$$\begin{aligned} = & \; GF_n^{2k-2}(x)\left(GF_{n+t}(x)GF_{n+t-2}(x) - GF_{n+t-1}(x))^2\right) \\ = & \; GF_n^{2k-2}(x)(-1)^n(2-ix). \end{aligned}$$

For $t \neq 1$, $t = m$,$(m \in N)$;

$$\begin{aligned} = & \; GF_n^{2k-2}(x)\left(GF_{n+m}(x)GF_{n+m-2}(x) - GF_{n+m-1}(x))^2\right) \\ = & \; GF_n^{2k-2}(x)(GF_{2n+2m-2}^{(2)} - GF_{2n+2m-2}^{(2)}) \\ = & \; 0. \end{aligned}$$

□

For n, we obtain an interesting relation between the known Gauss Fibonacci polynomials and the generalized Gauss $k-$Fibonacci polynomials

$$GF_{nk+t}^{(k)}(x) = GF_n^{k-t}(x)GF_{n+1}^t(x);$$

$t = 0, 1, \ldots, k-1$.

Conclusion

In the present paper, we defined the new families of Gauss k- Fibonacci polynomials. We gave some relations among these families and the known Gauss Fibonacci polynomials. Furthermore, we find the new generalizations of these families and the polynomials in matrix representation. Then we prove Cassini's Identities for the families and their polynomials.

Acknowledgments

The authors are grateful to the anonymous the editor and referees for the helpful suggestions and comments.

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