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COMMON COUPLED FIXED POINTS OF GENERALIZED CONTRACTION MAPS IN b-METRIC SPACES

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Abstract. In this paper, we introduce generalized contraction condition for two pairs (F, f) and (G, g) of maps $F, G : X \times X \to X, f, g : X \to X$ where X is a b-metric space and prove the existence and uniqueness of common coupled fixed points of these two pairs under the assumptions that these pairs are w-compatible and satisfying generalized contraction condition by restricting the completeness of X to its subspace. We draw some corollaries from our main results and provide examples in support of our results.

1. Introduction

The main idea of b-metric was initiated from the works of Bourbaki [8] and Bakhtin [4]. The concept of b-metric space or metric type space was introduced by Czerwik [9] as a generalization of metric space. Afterwards, many authors studied fixed point theorems for single-valued and multi-valued mappings in b-metric spaces, for more information we refer [3, 6, 7, 10, 14, 15, 19].

In 2006, Bhaskar and Lakshmikantham [5] introduced the notion of coupled fixed point and established the existence of coupled fixed points for mixed monotone mappings in ordered metric spaces. Later, Lakshmikantham and Ćirić [16] introduced the notion of coupled coincidence points of mappings in two variables. Afterwards, many authors studied coupled fixed point theorems, we refer [11, 16, 17, 20, 21].

Definition 1.1.[9] Let X be a non-empty set. A function $d : X \times X \to [0, \infty)$ is said to be a b-metric if the following conditions are satisfied: for any $x, y, z \in X$

(i) $0 \leq d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) there exists $s \geq 1$ such that $d(x, z) \leq s[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) is called a b-metric space with coefficient s.

Every metric space is a b-metric space with $s = 1$. In general, every b-metric space is not a metric space.

Definition 1.2. [7] Let (X, d) be a b-metric space.

(i) A sequence $\{x_n\}$ in X is called b-convergent if there exists $x \in X$ such that

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$d(x_n, x) \to 0$ as $n \to \infty$. In this case, we write $\lim_{n\to\infty} x_n = x$ and x is unique.

(ii) A sequence $\{x_n\}$ in X is called b-Cauchy if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

(iii) A b-metric space (X, d) is said to be a complete b-metric space if every b-Cauchy sequence in X is b-convergent in X.

(iv) A set $B \subset X$ is said to be b-closed if for any sequence $\{x_n\}$ in B such that $\{x_n\}$ is b-convergent to $z \in X$ then $z \in B$.

In general, a b-metric is not necessarily continuous.

In this paper, we denote $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{N}$ is the set of all natural numbers.

Example 1.3. [13] Let $X = \mathbb{N} \cup \{\infty\}$. We define a mapping $d : X \times X \to [0, \infty)$ as follows:

$$d(m,n) = \begin{cases} 0 & \text{if } m = n, \\ |\frac{1}{m} - \frac{1}{n}| & \text{if one of } m, n \text{ is even and the other is even or } \infty, \\ 5 & \text{if one of } m, n \text{ is odd and the other is odd or } \infty, \\ 2 & \text{otherwise.} \end{cases}$$

Then (X, d) is a b-metric space with coefficient $s = \frac{5}{2}$.

Definition 1.4. [5] Let X be a nonempty set and $F : X \times X \to X$ be a mapping. Then we say that an element $(x, y) \in X \times X$ is a coupled fixed point, if $F(x, y) = x$ and $F(y, x) = y$.

Definition 1.5. [16] Let X be a nonempty set. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings. An element $(x, y) \in X \times X$ ia called

(i) a coupled coincidence point of the mappings F and g if $F(x, y) = gx$ and $F(y, x) = gy$;

(ii) a common coupled fixed point of mappings F and g if $F(x, y) = gx = x$ and $F(y, x) = gy = y$.

In 2010, Abbas, Khan and Radenovic [1] introduced the concept of w-compatible mappings as follows.

Definition 1.6. [1] Let X be a non-empty set. We say that the mappings $F : X \times X \to X$ and $g : X \to X$ are w-compatible if $gF(x, y) = F(gx, gy)$ whenever $gx = F(x, y)$ and $gy = F(x, y)$.

The following lemmas are useful in proving our main results.

Lemma 1.7. [2] Let (X, d) be a b-metric space with coefficient $s \geq 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are b-convergent to x and y respectively, then we have

$$\frac{1}{s^2} d(x, y) \leq \liminf_{n\to\infty} d(x_n, y_n) \leq \limsup_{n\to\infty} d(x_n, y_n) \leq s^2 d(x, y).$$

In particular, if $x = y$, then we have $\lim_{n\to\infty} d(x_n, y_n) = 0$. Moreover for each $z \in X$ we have

$$\frac{1}{s} d(x, z) \leq \liminf_{n\to\infty} d(x_n, z) \leq \limsup_{n\to\infty} d(x_n, z) \leq s d(x, z).$$

Lemma 1.8. [12] Let (X, d) be a b-metric space with coefficient $s \geq 1$ and $T : X \to X$ be a selfmap. Suppose that $\{x_n\}$ is a sequence in X induced by $x_{n+1} = Tx_n$ such that $d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$, where $\lambda \in [0, 1)$ is a constant. Then $\{x_n\}$ is a b-Cauchy sequence in X.

In 1994, Matthews [18] introduced the notion of a partial metric in which the concept of self distance need not be equal to zero.

Definition 1.9. [18] Let X be a nonempty set. A mapping $p : X \times X \to \mathbb{R}^+$ is said to be a partial metric, if it satisfies the following conditions:

For any $x, y, z \in X$

$(P1)$ $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,

$(P2)$ $p(x, x) \leq p(x, y),\ p(y, y) \leq p(x, y)$,

$(P3)$ $p(x,y) = p(y,x)$,
$(P4)$ $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.
The pair (X,p) is called a partial metric space.

Recently, Gu and Shatanawi [11] proved the following theorem in the setting of partial metric spaces.

Theorem 1.10. [11] Let (X,p) be a partial metric space. Let $F,G : X\times X \to X, f,g : X \to X$ be four mappings. Suppose that there exist k_1,k_2,k_3,k_4 and k_5 in $[0,1)$ with $k_1+k_2+k_3+2k_4+2k_5 < 1$ such that

$$\begin{aligned} p(F(x,y),G(u,v)) + p(F(y,x),G(v,u)) \leq\ & k_1[p(fx,gu)+p(fy,gv)] \\ & + k_2[p(fx,F(x,y))+p(fy,F(y,x))] \\ & + k_3[p(gu,G(u,v))+p(gv,G(v,u))] \\ & + k_4[p(fx,G(u,v))+p(fy,G(v,u))] \\ & + k_5[p(gu,F(x,y))+p(gv,F(y,x))] \end{aligned}$$

for all $x,y,u,v \in X$. Also, suppose the following hypotheses:

- (i) $F(X\times X) \subset g(X)$ and $G(X\times X) \subset f(X)$,
- (ii) either $f(X)$ or $g(X)$ is a complete subspace of X,
- (iii) (F,f) and (G,g) are w-compatible.

Then F,G,f and g have a unique common coupled fixed point in $X\times X$. Moreover, the common coupled fixed point of F,G,f and g has the form (u,v).

Motivated by the works of Gu and Shatanawi [11] (Theorem 1.10) in Section 2, we introduce generalized contraction condition for two pairs (F,f) and (G,g) of maps $F,G : X\times X \to X, f,g : X \to X$ where X is a b-metric space and prove the existence and uniqueness of common coupled fixed points of these two pairs under the assumptions that these pairs are w-compatible and satisfying generalized contraction condition by restricting the completeness of X to its subspace. We draw some corollaries from our main results and provide examples in support of our results in Section 3.

2. Main results

The following we introduce generalized contraction condition for two pairs (F,f) and (G,g) of maps $F,G : X\times X \to X, f,g : X\to X$ in b-metric spaces.

Definition 2.1. Let X be a b-metric space with coefficient $s \geq 1$ and $F,G : X\times X \to X, f,g : X \to X$ be four mappings. Suppose that there exists $k \in [0,1)$ such that

$$s^4[d(F(x,y),G(u,v)) + d(F(y,x),G(v,u))] \leq kM(x,y,u,v) \tag{1}$$

for all $x,y,u,v \in X$, where
$M(x,y,u,v) = \max\{d(fx,gu)+d(fy,gv), d(fx,F(x,y))+d(fy,F(y,x)),$
$d(gu,G(u,v))+d(gv,G(v,u)), \frac{d(fx,G(u,v))+d(fy,G(v,u))}{2s}, \frac{d(gu,F(x,y))+d(gv,F(y,x))}{2s^2}\}$.
In this case, we say that the maps F,G,f,g satisfy generalized contraction condition on X.

Proposition 2.2. Let (X,d) be a b-metric space with coefficient $s\geq 1$ and $F,G : X\times X \to X, f,g : X\to X$ be four mappings satisfy the generalized contraction condition. Suppose that

- (i) If $F(X\times X) \subseteq g(X)$ and the pair (G,g) is w-compatible, and if (u,v) is a common coupled fixed point of F and f then (u,v) is a common coupled fixed point of F,G,f and g and it is unique.

(*ii*) If $G(X \times X) \subseteq f(X)$ and the pair (F, f) is w-compatible, and if (u, v) is a common coupled fixed point of G and g then (u, v) is a common coupled fixed point of F, G, f and g and it is unique.

Then F, G, f and g have a unique common coupled fixed point in $X \times X$.

Proof. First, we assume that (i) holds. Let (u, v) be a common coupled fixed point of F and f.
Then $F(u, v) = fu = u$ and $F(v, u) = fv = v$.
Since $F(X \times X) \subseteq g(X)$, there exist $a, b \in X$ such that $u = F(u, v) = ga$ and $v = F(v, u) = gb$.
We now consider

$$\begin{cases} s^4[d(u, G(a, b)) + d(v, G(b, a))] = s^4[d(F(u, v), G(a, b)) + d(F(v, u), G(b, a))] \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad \leq k\ M(u, v, a, b) \end{cases} \tag{2}$$

$$\begin{aligned} M(u, v, a, b) = \max\{&d(fu, ga) + d(fv, gb), d(fu, F(u, v)) + d(fv, F(v, u)), \\ &d(ga, G(a, b)) + d(gb, G(b, a)), \tfrac{d(fu, G(a,b)) + d(fv, G(b,a))}{2s}, \tfrac{d(ga, F(u,v)) + d(gb, F(v,u))}{2s^2}\} \\ = &\ d(u, G(a, b)) + d(v, G(b, a)) \end{aligned}$$

From the inequality (2), we have

$$\begin{aligned} s^4[d(u, G(a, b)) + d(v, G(b, a))] &\leq k\ [d(u, G(a, b)) + d(v, G(b, a))] \\ &< d(u, G(a, b)) + d(v, G(b, a)), \end{aligned}$$

a contradiction.
Therefore $u = G(a, b) = ga$ and $v = G(b, a) = gb$.
Since the pair (G, g) is w-compatible, we have
$gu = g(G(a, b)) = G(ga, gb) = G(u, v)$ and $gv = g(G(b, a)) = G(gb, ga) = G(v, u)$.
We now prove that $gu = u$ and $gv = v$.
Suppose that $gu \neq u$ and $gv \neq v$.
Now we consider

$$s^4[d(u, gu) + d(v, gv)] = s^4[d(F(u, v), G(u, v)) + d(F(v, u), G(v, u))] \leq k\ M(u, v, u, v) \tag{3}$$

where

$$\begin{aligned} M(u, v, u, v) = \max\{&d(fu, gu) + d(fv, gv), d(fu, F(u, v)) + d(fv, F(v, u)), \\ &d(gu, G(u, v)) + d(gv, G(v, u)), \tfrac{d(fu, G(u,v)) + d(fv, G(v,u))}{2s}, \tfrac{d(gu, F(u,v)) + d(gv, F(v,u))}{2s^4}\} \\ = &\ d(u, gu) + d(v, gv). \end{aligned}$$

From (3), we have
$s^4[d(u, gu) + d(v, gv)] \leq k[d(u, gu) + d(v, gv)]$ implies that
$(s^4 - k)[d(u, gu) + d(v, gv)] \leq 0$, which is a contradiction.
Therefore $gu = u$ and $gv = v$ and hence $G(u, v) = gu = u$ and $G(v, u) = gv = v$.
Thus (u, v) is a common coupled fixed point of F, G, f and g.
Let (u', v') be another common coupled fixed point of F, G, f and g with $(u, v) \neq (u', v')$.
We now consider

$$\begin{aligned} s^4[d(u, u') + d(v, v')] &= s^4[d(F(u, v), G(u', v')) + d(F(v, u), G(v', u'))] \\ &\leq kM(u, v, u', v') \\ &= k \max\{d(fu, gu') + d(fv, gv'), d(fu, F(u, v)) + d(fv, F(v, u)), \\ &\qquad d(gu', G(u', v')) + d(gv', G(v', u')), \tfrac{d(fu, G(u',v')) + d(fv, G(v',u'))}{2s}, \\ &\qquad \tfrac{d(gu', F(u,v)) + d(gv', F(v,u))}{2s^2}\} \\ &= k[d(u, u') + d(v, v')] < d(u, u') + d(v, v'), \end{aligned}$$

a contradiction.
Therefore $u = u'$ and $v = v'$.
Hence (u, v) is a unique coupled fixed point of F, G, f and g. □

Lemma 2.3. Let (X, d) be a b-metric space with coefficient $s \geq 1$, $F, G : X \times X \to X, f, g : X \to X$ be four mappings satisfy generalized contraction condition and $F(X \times X) \subseteq g(X)$ and $G(X \times X) \subseteq f(X)$. For $x_0 \in X$ and $y_0 \in X$, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $F(x_{2n}, y_{2n}) = gx_{2n+1} = z_{2n}$(say), $F(y_{2n}, x_{2n}) = gy_{2n+1} = w_{2n}$(say), $G(x_{2n+1}, y_{2n+1}) = fx_{2n+2} = z_{2n+1}$(say) and $G(y_{2n+1}, x_{2n+1}) = fy_{2n+2} = w_{2n+1}$(say) for all $n = 0, 1, 2, \ldots$. Then the sequences $\{z_n\}$ and $\{w_n\}$ are b-Cauchy in X.

Proof. Let $x_0 \in X$ and $y_0 \in X$. Then there exist $x_1 \in X$ and $y_1 \in X$ such that $F(x_0, y_0) = gx_1 = z_0$(say) and $F(y_0, x_0) = gy_1 = w_0$(say). In the same way, for $x_1 \in X$ and $y_1 \in X$, there exist $x_2 \in X$ and $y_2 \in X$ such that $G(x_1, y_1) = fx_2 = z_1$(say) and $G(y_1, x_1) = fy_2 = w_1$(say). On continuing this way, we get,
$F(x_{2n}, y_{2n}) = gx_{2n+1} = z_{2n}$, $F(y_{2n}, x_{2n}) = gy_{2n+1} = w_{2n}$,
$G(x_{2n+1}, y_{2n+1}) = fx_{2n+2} = z_{2n+1}$ and $G(y_{2n+1}, x_{2n+1}) = fy_{2n+2} = w_{2n+1}$, for all $n \geq 0$.
We have the following two cases.
Case (i). $h \in [0, \frac{1}{s})$ $(s \geq 1)$.
If n is odd, then $n = 2m + 1, m \in \mathbb{N}$.
We now consider

$$\begin{cases} d(z_n, z_{n+1}) + d(w_n, w_{n+1}) \leq s^4[d(z_{2m+1}, z_{2m+2}) + d(w_{2m+1}, w_{2m+2})] \\ \qquad = s^4[d(F(x_{2m+2}, y_{2m+2}), G(x_{2m+1}, y_{2m+1})) \\ \qquad\qquad +d(F(y_{2m+2}, x_{2m+2}), G(y_{2m+1}, x_{2m+1}))] \\ \qquad \leq kM(x_{2m+2}, y_{2m+2}, x_{2m+1}, y_{2m+1}) \end{cases} \tag{4}$$

where

$$\begin{aligned} M(x_{2m+2}, y_{2m+2}, x_{2m+1}, y_{2m+1}) &= \max\{d(fx_{2m+2}, gx_{2m+1}) + d(fy_{2m+2}, gy_{2m+1}), \\ &d(fx_{2m+2}, F(x_{2m+2}, y_{2m+2})) + d(fy_{2m+2}, F(y_{2m+2}, x_{2m+2})), \\ &d(gx_{2m+1}, G(x_{2m+1}, y_{2m+1})) + d(gy_{2m+1}, G(y_{2m+1}, x_{2m+1})), \\ &\tfrac{d(fx_{2m+2}, G(x_{2m+1}, y_{2m+1})) + d(fy_{2m+2}, G(y_{2m+1}, x_{2m+1}))}{2s}, \\ &\tfrac{d(gx_{2m+1}, F(x_{2m+2}, y_{2m+2})) + d(gy_{2m+1}, F(y_{2m+2}, x_{2m+2}))}{2s^2}\} \\ &= \max\{d(z_{2m+1}, z_{2m}) + d(w_{2m+1}, w_{2m}), \\ &d(z_{2m+1}, z_{2m+2}) + d(w_{2m+1}, w_{2m+2}), \\ &d(z_{2m}, z_{2m+1}) + d(w_{2m}, w_{2m+1}), \\ &\tfrac{d(z_{2m+1}, z_{2m+1}) + d(w_{2m+1}, w_{2m+1})}{2s}, \tfrac{d(z_{2m}, z_{2m+2}) + d(w_{2m}, w_{2m+2})}{2s^2}\} \\ &\leq \max\{d(z_{2m+1}, z_{2m}) + d(w_{2m+1}, w_{2m}), \\ &\quad d(z_{2m+1}, z_{2m+2}) + d(w_{2m+1}, w_{2m+2})\}. \end{aligned}$$

If $M(x_{2m+2}, y_{2m+2}, x_{2m+1}, y_{2m+1}) = d(z_{2m+1}, z_{2m+2}) + d(w_{2m+1}, w_{2m+2})$ then from (4), we get that
$s^4[d(z_{2m+1}, z_{2m+2}) + d(w_{2m+1}, w_{2m+2})] \leq k[d(z_{2m+1}, z_{2m+2}) + d(w_{2m+1}, w_{2m+2})]$
implies that
$d(z_{2m+1}, z_{2m+2}) + d(w_{2m+1}, w_{2m+2}) \leq 0$,
a contradiction.
Therefore, $M(x_{2m+2}, y_{2m+2}, x_{2m+1}, y_{2m+1}) = d(z_{2m}, z_{2m+1}) + d(w_{2m}, w_{2m+1})$.
Hence from (4), we have
$s^4[d(z_{2m+1}, z_{2m+2}) + d(w_{2m+1}, w_{2m+2})] \leq k[d(z_{2m}, z_{2m+1}) + d(w_{2m}, w_{2m+1})]$ implies

that

$$d(z_{2m+1}, z_{2m+2}) + d(w_{2m+1}, w_{2m+2}) \leq h[d(z_{2m}, z_{2m+1}) + d(w_{2m}, w_{2m+1})] \quad (5)$$

where $h = \frac{k}{s^4} < 1$.
On the similar lines, if n is even, it follows that

$$d(z_{2m+1}, z_{2m+2}) + d(w_{2m+1}, w_{2m+2}) \leq h[d(z_{2m}, z_{2m+1}) + d(w_{2m}, w_{2m+1})] \quad (6)$$

From (5) and (6), it follows that

$$\begin{cases} d(z_n, z_{n+1}) + d(w_n, w_{n+1}) \leq h[d(z_{n-1}, z_n) + d(w_{n-1}, w_n)] \\ d(z_n, z_{n+1}) + d(w_n, w_{n+1}) \leq h[d(z_{n-1}, z_n) + d(w_{n-1}, w_n)] \\ \qquad \leq h^2[d(z_{n-2}, z_{n-1}) + d(w_{n-2}, w_{n-1})] \\ \qquad \vdots \\ \qquad \leq h^n[d(z_0, z_1) + d(w_0, w_1)]. \end{cases} \quad (7)$$

For each $n, m \in \mathbb{N}$ with $n > m$ and using (4), we obtain that

$$\begin{aligned} d(z_m, z_n) + d(w_m, w_n) &\leq s[d(z_m, z_{m+1}) + d(z_{m+1}, z_n) + d(w_m, w_{m+1}) + d(w_{m+1}, w_n)] \\ &\leq s[d(z_m, z_{m+1}) + d(w_m, w_{m+1})] \\ &\quad + s^2[d(z_{m+1}, z_{m+2}) + d(z_{m+2}, z_n) + d(w_{m+1}, w_{m+2}) \\ &\quad + d(w_{m+2}, w_n)] \\ &\leq s[d(z_m, z_{m+1}) + d(w_m, w_{m+1})] + s^2[d(z_{m+1}, z_{m+2}) + d(w_{m+1}, w_{m+2})] \\ &\quad + s^3[d(z_{m+2}, z_{m+3}) + d(w_{m+2}, w_{m+3})] + \ldots + \\ &\quad s^{n-m-1}[d(z_{n-1}, z_n) + d(w_{n-1}, w_n)] \\ &\leq sh^m[d(z_0, z_1) + d(w_0, w_1)] + s^2h^{m+1}[d(z_0, z_1) + d(w_0, w_1)] \\ &\quad + s^3h^{m+2}[d(z_0, z_1) + d(w_0, w_1)] + \ldots + + s^{n-m-1}h^{n-1}[d(z_0, z_1) \\ &\quad + d(w_0, w_1)] \\ &= sh^m[1 + sh + (sh)^2 + \ldots + (sh)^{n-m-1}][d(z_0, z_1) + d(w_0, w_1)] \\ &\leq sh^m[1 + sh + (sh)^2 + \ldots][d(z_0, z_1) + d(w_0, w_1)] \\ &= \frac{sh^m}{1 - sh}[d(z_0, z_1) + d(w_0, w_1)] \to 0 \text{ as } m \to \infty \end{aligned}$$

which implies that $\lim_{m,n\to\infty} d(z_m, z_n) = 0$ and $\lim_{m,n\to\infty} d(w_m, w_n) = 0$.
Therefore $\{z_n\}$ and $\{w_n\}$ are b-Cauchy sequences in (X, d).
Case (ii). $h \in [\frac{1}{s}, 1)$.
In this case, we have $h^n \to 0$ as $n \to \infty$, so there exists $n_0 \in \mathbb{N}$ such that $h^{n_0} < \frac{1}{s}$.
Thus by Case (i), we have $\{z_{n_0}, z_{n_0+1}, z_{n_0+2}, \ldots, z_{n_0+n}, \ldots\}$ and $\{w_{n_0}, w_{n_0+1}, w_{n_0+2}, \ldots, w_{n_0+n}, \ldots\}$ are b-Cauchy sequences.
Therefore $z_n = \{z_0, z_1, z_2, \ldots, z_{n_0-1}\} \cup \{z_{n_0}, z_{n_0+1}, z_{n_0+2}, \ldots, z_{n_0+n}, \ldots\}$ and $w_n = \{w_0, w_1, w_2, \ldots, w_{n_0-1}\} \cup \{w_{n_0}, w_{n_0+1}, w_{n_0+2}, \ldots, w_{n_0+n}, \ldots\}$ are b-Cauchy sequences in X. □

Theorem 2.4. Let (X, d) be a b-metric space with coefficient $s \geq 1$ and $F, G : X \times X \to X, f, g : X \to X$ be four mappings satisfying generalized contraction condition. Assume that

(i) $F(X \times X) \subseteq g(X)$ and $G(X \times X) \subseteq f(X)$,
(ii) either $f(X)$ or $g(X)$ is a complete subspace of X,
(iii) (F, f) and (G, g) are w-compatible.

Then F, G, f and g have a unique common coupled fixed point in $X \times X$.

Proof. From (i), there exist sequences $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{w_n\}$ in X such that
$F(x_{2n}, y_{2n}) = gx_{2n+1} = z_{2n}$, for all $n \geq 0$
$F(y_{2n}, x_{2n}) = gy_{2n+1} = w_{2n}$, for all $n \geq 0$

$G(x_{2n+1}, y_{2n+1}) = fx_{2n+2} = z_{2n+1}$, for all $n \geq 0$
$G(y_{2n+1}, x_{2n+1}) = fy_{2n+2} = w_{2n+1}$, for all $n \geq 0$.
Assume that $z_n = z_{n+1}$ and $w_n = w_{n+1}$ for some $n = \{0, 1, 2, \dots\}$.
Case (i): n even.
We write $n = 2m, m \in \mathbb{N}$.
Now we consider

$$\begin{aligned} d(z_{n+1}, z_{n+2}) + d(w_{n+1}, w_{n+2}) &\leq s^4[d(z_{2m+1}, z_{2m+2}) + d(w_{2m+1}, w_{2m+2})] \\ &= s^4[d(F(x_{2m+2}, y_{2m+2}), G(x_{2m+1}, y_{2m+1})) \\ &\quad + d(F(y_{2m+2}, x_{2m+2}), G(y_{2m+1}, x_{2m+1}))] \\ &\leq kM(x_{2m+2}, y_{2m+2}, x_{2m+1}, y_{2m+1}) \end{aligned} \tag{8}$$

where

$$\begin{aligned} M(x_{2m+2}, y_{2m+2}, x_{2m+1}, y_{2m+1}) &= \max\{d(fx_{2m+2}, gx_{2m+1}) + d(fy_{2m+2}, gy_{2m+1}), \\ & d(fx_{2m+2}, F(x_{2m+2}, y_{2m+2})) + d(fy_{2m+2}, F(y_{2m+2}, x_{2m+2})), \\ & d(gx_{2m+1}, G(x_{2m+1}, y_{2m+1})) + d(gy_{2m+1}, G(y_{2m+1}, x_{2m+1})), \\ & \tfrac{d(fx_{2m+2}, G(x_{2m+1}, y_{2m+1})) + d(fy_{2m+2}, G(y_{2m+1}, x_{2m+1}))}{2s}, \\ & \tfrac{d(gx_{2m+1}, F(x_{2m+2}, y_{2m+2})) + d(gy_{2m+1}, F(y_{2m+2}, x_{2m+2}))}{2s^2}\} \\ &= \max\{d(z_{2m+1}, z_{2m}) + d(w_{2m+1}, w_{2m}), \\ & d(z_{2m+1}, z_{2m+2}) + d(w_{2m+1}, w_{2m+2}), \\ & d(z_{2m}, z_{2m+1}) + d(w_{2m}, w_{2m+1}), \\ & \tfrac{d(z_{2m+1}, z_{2m+1}) + d(w_{2m+1}, w_{2m+1})}{2s}, \tfrac{d(z_{2m}, z_{2m+2}) + d(w_{2m}, w_{2m+2})}{2s^2}\} \\ &\leq \max\{0, d(z_{2m+1}, z_{2m+2}) + d(w_{2m+1}, w_{2m+2}), 0, 0, \\ & \tfrac{d(z_{2m+1}, z_{2m+2}) + d(w_{2m+1}, w_{2m+2})}{2s}\} \\ &= d(z_{2m+1}, z_{2m+2}) + d(w_{2m+1}, w_{2m+2}). \end{aligned}$$

From (8), we have
$s^4 d(z_{2m+1}, z_{2m+2}) + d(w_{2m+1}, w_{2m+2}) \leq k[d(z_{2m+1}, z_{2m+2}) + d(w_{2m+1}, w_{2m+2})]$ implies that
$d(z_{2m+1}, z_{2m+2}) + d(w_{2m+1}, w_{2m+2}) \leq 0$ which implies that $z_{2m+1} = z_{2m+2}$ and $w_{2m+1} = w_{2m+2}$.
Hence $z_{2m} = z_{2m+1} = z_{2m+2}$ and $w_{2m} = w_{2m+1} = w_{2m+2}$.
In general, $z_{2m} = z_{2m+k}$ and $w_{2m} = w_{2m+k}$ for $k = 0, 1, 2, \dots$.
Case (ii): n odd.
We write $n = 2m + 1, m \in \mathbb{N}$.
Now we consider

$$\begin{aligned} d(z_{n+1}, z_{n+2}) + d(w_{n+1}, w_{n+2}) &\leq s^4[d(z_{2m+2}, z_{2m+3}) + d(w_{2m+2}, w_{2m+3})] \\ &= s^4[d(F(x_{2m+2}, y_{2m+2}), G(x_{2m+3}, y_{2m+3})) \\ &\quad + d(F(y_{2m+2}, x_{2m+2}), G(y_{2m+3}, x_{2m+3}))] \\ &\leq kM(x_{2m+2}, y_{2m+2}, x_{2m+3}, y_{2m+3}) \end{aligned} \tag{9}$$

where

$$\begin{aligned} M(x_{2m+2}, y_{2m+2}, x_{2m+3}, y_{2m+3}) &= \max\{d(fx_{2m+2}, gx_{2m+3}) + d(fy_{2m+2}, gy_{2m+3}), \\ & d(fx_{2m+2}, F(x_{2m+2}, y_{2m+2})) + d(fy_{2m+2}, F(y_{2m+2}, x_{2m+2})), \\ & d(gx_{2m+3}, G(x_{2m+3}, y_{2m+3})) + d(gy_{2m+3}, G(y_{2m+3}, x_{2m+3})), \\ & \tfrac{d(fx_{2m+2}, G(x_{2m+3}, y_{2m+3})) + d(fy_{2m+2}, G(y_{2m+3}, x_{2m+3}))}{2s}, \\ & \tfrac{d(gx_{2m+3}, F(x_{2m+2}, y_{2m+2})) + d(gy_{2m+3}, F(y_{2m+2}, x_{2m+2}))}{2s^2}\} \\ &= \max\{d(z_{2m+1}, z_{2m+2}) + d(w_{2m+1}, w_{2m+2}), \\ & d(z_{2m+1}, z_{2m+2}) + d(w_{2m+1}, w_{2m+2}), \end{aligned}$$

$$\begin{aligned}
&d(z_{2m+2}, z_{2m+3}) + d(w_{2m+2}, w_{2m+3}),\\
&\tfrac{d(z_{2m+1},z_{2m+3})+d(w_{2m+1},w_{2m+3})}{2s}, \tfrac{d(z_{2m+2},z_{2m+2})+d(w_{2m+2},w_{2m+2})}{2s^2}\}\\
&\leq \max\{0, 0, d(z_{2m+2}, z_{2m+3})+d(w_{2m+2}, w_{2m+3}),\\
&\tfrac{d(z_{2m+2},z_{2m+3})+d(w_{2m+2},w_{2m+3})}{2}, 0\}\\
&= d(z_{2m+2}, z_{2m+3}) + d(w_{2m+2}, w_{2m+3}).
\end{aligned}$$

From (9), we have
$s^4 d(z_{2m+2}, z_{2m+3})+d(w_{2m+2}, w_{2m+3}) \leq k[d(z_{2m+2}, z_{2m+3})+d(w_{2m+2}, w_{2m+3})]$ implies that
$d(z_{2m+2}, z_{2m+3}) + d(w_{2m+2}, w_{2m+3}) \leq 0$ which implies that $z_{2m+2} = z_{2m+3}$ and $w_{2m+2} = w_{2m+3}$.
Hence $z_{2m+1} = z_{2m+2} = z_{2m+3}$ and $w_{2m+1} = w_{2m+2} = w_{2m+3}$.
In general, $z_{2m+1} = z_{2m+k}$ and $w_{2m+1} = w_{2m+k}$ for $k = 0, 1, 2, \ldots$.
From Case (i) and Case (ii), we have $z_{n+k} = z_n$ and $w_{n+k} = w_n$ for $k = 0, 1, 2, \ldots$.
Therefore, $\{z_{n+k}\}$ and $\{w_{n+k}\}$ are constant sequences and hence $\{z_{n+k}\}$ and $\{w_{n+k}\}$ are Cauchy sequences.
Now we assume that $z_n \neq z_{n+1}$ and $w_n \neq w_{n+1}$ for all $n \in \mathbb{N}$.
If n is odd, then $n = 2m + 1, m \in \mathbb{N}$.
We now consider

$$\begin{aligned}
d(z_n, z_{n+1}) + d(w_n, w_{n+1}) &\leq s^4[d(z_{2m+1}, z_{2m+2}) + d(w_{2m+1}, w_{2m+2})]\\
&= s^4[d(F(x_{2m+2}, y_{2m+2}), G(x_{2m+1}, y_{2m+1}))\\
&\quad + d(F(y_{2m+2}, x_{2m+2}), G(y_{2m+1}, x_{2m+1}))]\\
&\leq kM(x_{2m+2}, y_{2m+2}, x_{2m+1}, y_{2m+1})
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
M(x_{2m+2}, y_{2m+2}, x_{2m+1}, y_{2m+1}) &= \max\{d(fx_{2m+2}, gx_{2m+1}) + d(fy_{2m+2}, gy_{2m+1}),\\
&d(fx_{2m+2}, F(x_{2m+2}, y_{2m+2}))+d(fy_{2m+2}, F(y_{2m+2}, x_{2m+2})),\\
&d(gx_{2m+1}, G(x_{2m+1}, y_{2m+1}))+d(gy_{2m+1}, G(y_{2m+1}, x_{2m+1})),\\
&\tfrac{d(fx_{2m+2},G(x_{2m+1},y_{2m+1}))+d(fy_{2m+2},G(y_{2m+1},x_{2m+1}))}{2s},\\
&\tfrac{d(gx_{2m+1},F(x_{2m+2},y_{2m+2}))+d(gy_{2m+1},F(y_{2m+2},x_{2m+2}))}{2s^2}\}\\
&= \max\{d(z_{2m+1}, z_{2m}) + d(w_{2m+1}, w_{2m}),\\
&d(z_{2m+1}, z_{2m+2}) + d(w_{2m+1}, w_{2m+2}),\\
&d(z_{2m}, z_{2m+1}) + d(w_{2m}, w_{2m+1}),\\
&\tfrac{d(z_{2m+1},z_{2m+1})+d(w_{2m+1},w_{2m+1})}{2s}, \tfrac{d(z_{2m},z_{2m+2})+d(w_{2m},w_{2m+2})}{2s^2}\}\\
&\leq \max\{d(z_{2m+1}, z_{2m}) + d(w_{2m+1}, w_{2m}),\\
&d(z_{2m+1}, z_{2m+2})+d(w_{2m+1}, w_{2m+2})\}.
\end{aligned}$$

If $M(x_{2m+2}, y_{2m+2}, x_{2m+1}, y_{2m+1}) = d(z_{2m+1}, z_{2m+2})+d(w_{2m+1}, w_{2m+2})$ then from (10), we get that
$s^4[d(z_{2m+1}, z_{2m+2}) + d(w_{2m+1}, w_{2m+2})] \leq k[d(z_{2m+1}, z_{2m+2}) + d(w_{2m+1}, w_{2m+2})]$ implies that
$d(z_{2m+1}, z_{2m+2}) + d(w_{2m+1}, w_{2m+2}) \leq 0$,
a contradiction.
Therefore, $M(x_{2m+2}, y_{2m+2}, x_{2m+1}, y_{2m+1}) = d(z_{2m}, z_{2m+1}) + d(w_{2m}, w_{2m+1})$.
Hence from (10), we have
$s^4[d(z_{2m+1}, z_{2m+2})+d(w_{2m+1}, w_{2m+2})] \leq k[d(z_{2m}, z_{2m+1})+d(w_{2m}, w_{2m+1})]$ implies that

$$d(z_{2m+1}, z_{2m+2}) + d(w_{2m+1}, w_{2m+2}) \leq h[d(z_{2m}, z_{2m+1}) + d(w_{2m}, w_{2m+1})] \tag{11}$$

where $h = \frac{k}{s^4} < 1$.
On the similar lines, if n is even, it follows that

$$d(z_{2m+1}, z_{2m+2}) + d(w_{2m+1}, w_{2m+2}) \leq h[d(z_{2m}, z_{2m+1}) + d(w_{2m}, w_{2m+1})] \quad (12)$$

From (11) and (12), it follows that

$$\begin{aligned} d(z_n, z_{n+1}) + d(w_n, w_{n+1}) &\leq h[d(z_{n-1}, z_n) + d(w_{n-1}, w_n)] \\ &\leq h^2[d(z_{n-2}, z_{n-1}) + d(w_{n-2}, w_{n-1})] \\ &\vdots \\ &\leq h^n[d(z_0, z_1) + d(w_0, w_1)] \to 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore $\lim_{n\to\infty} d(z_n, z_{n+1}) = 0$ and $\lim_{n\to\infty} d(z_n, z_{n+1}) = 0$.
By Lemma 2.3, we have $\{z_n\}$ and $\{w_n\}$ are Cauchy sequences in b-metric space (X, d). Therefore $\{z_{2n+1}\}$ and $\{w_{2n+1}\}$ are Cauchy sequences in the subspace $(f(X), d)$.
Suppose that $f(X)$ is complete. Since $\{z_{2n+1}\} \subseteq f(X)$ and $\{w_{2n+1}\} \subseteq f(X)$, it follows that the sequences $\{z_{2n+1}\}$ and $\{w_{2n+1}\}$ are convergent in $(f(X), d)$.
Hence, there exist $u, v \in f(X)$ such that $\lim_{n\to\infty} d(z_{2n+1}, u) = 0$ and $\lim_{n\to\infty} d(w_{2n+1}, v) = 0$.
Since $u, v \in f(X)$, there exist $s, t \in X$ such that $u = fs$ and $v = ft$.
Since $\{z_n\}$ and $\{w_n\}$ are b-Cauchy sequences in X and $\{z_{2n+1}\} \to u$ and $\{w_{2n+1}\} \to v$ as $n \to \infty$, so that $\{z_{2n}\} \to u$ and $\{w_{2n}\} \to v$ as $n \to \infty$. Therefore $\lim_{n\to\infty} d(z_{2n}, u) = 0$ and $\lim_{n\to\infty} d(w_{2n}, v) = 0$.
By Lemma 1.7, we have

$$\frac{1}{s} d(F(s,t), u) \leq \liminf_{n\to\infty} d(F(s,t), z_{2n+1}) \leq \limsup_{n\to\infty} d(F(s,t), z_{2n+1}) \leq s\ d(F(s,t), u)$$

and

$$\frac{1}{s} d(F(t,s), v) \leq \liminf_{n\to\infty} d(F(t,s), w_{2n+1}) \leq \limsup_{n\to\infty} d(F(t,s), w_{2n+1}) \leq s\ d(F(t,s), v).$$

We now prove that $F(s,t) = u = fs$ and $F(t,s) = v = ft$.
Suppose that $F(s,t) \neq u \neq fs$ and $F(t,s) \neq v \neq ft$.
Now we consider

$$\begin{aligned} d(F(s,t), z_{2n+1}) + d(F(t,s), w_{2n+1}) &= d(F(s,t), G(x_{2n+1}, y_{2n+1})) \\ &\quad + d(F(t,s), G(y_{2n+1}, x_{2n+1})) \\ &\leq s^4[d(F(s,t), G(x_{2n+1}, y_{2n+1})) \\ &\quad + d(F(t,s), G(y_{2n+1}, x_{2n+1}))] \\ &\leq k\ M(s, t, x_{2n+1}, y_{2n+1}) \end{aligned} \quad (13)$$

where

$$\begin{aligned} M(s, t, x_{2n+1}, y_{2n+1}) &= \max\{d(fs, gx_{2n+1}) + d(ft, gy_{2n+1}), d(fs, F(s,t)) + d(ft, F(t,s)), \\ &\quad d(gx_{2n+1}, G(x_{2n+1}, y_{2n+1})) + d(gy_{2n+1}, G(y_{2n+1}, x_{2n+1})), \\ &\quad \tfrac{d(fs, G(x_{2n+1}, y_{2n+1})) + d(ft, G(y_{2n+1}, x_{2n+1}))}{2s}, \tfrac{d(gx_{2n+1}, F(s,t)) + d(gy_{2n+1}, F(t,s))}{2s^2}\} \\ &= \max\{d(u, z_{2n}) + d(v, w_{2n}), d(u, F(s,t)) + d(v, F(t,s)), \\ &\quad d(z_{2n}, z_{2n+1}) + d(w_{2n}, w_{2n+1}), \tfrac{d(u, z_{2n+1}) + d(v, w_{2n+1})}{2s}, \\ &\quad \tfrac{d(z_{2n}, F(s,t)) + d(w_{2n}, F(t,s))}{2s^2}\} \\ &\leq \max\{d(u, z_{2n}) + d(v, w_{2n}), d(u, F(s,t)) + d(v, F(t,s)), \\ &\quad d(z_{2n}, z_{2n+1}) + d(w_{2n}, w_{2n+1}), \tfrac{d(u, z_{2n+1}) + d(v, w_{2n+1})}{2s}, \\ &\quad \tfrac{d(z_{2n}, z_{2n+1}) + d(z_{2n+1}, F(s,t)) + d(w_{2n}, w_{2n+1}) + d(w_{2n+1}, F(t,s))}{2s}\} \end{aligned}$$

On letting limit superior as $n \to \infty$ on $M(s, t, x_{2n+1}, y_{2n+1})$, we get

$\limsup_{n\to\infty} M(s,t,x_{2n+1},y_{2n+1}) \leq d(u,F(s,t)) + d(v,F(t,s))$.
On taking limit superior as $n \to \infty$ in (13), we get

$$\begin{aligned} s^4\tfrac{1}{s}[d(u,F(s,t)) + d(v,F(t,s))] &\leq s^4 \limsup_{n\to\infty}[d(F(s,t),G(x_{2n+1},y_{2n+1})) \\ &\qquad + d(F(t,s),G(y_{2n+1},x_{2n+1}))] \\ &\leq k \limsup_{n\to\infty} M(s,t,x_{2n+1},y_{2n+1}) \\ &\leq k[d(u,F(s,t)) + d(v,F(t,s))] \\ &< d(u,F(s,t)) + d(v,F(t,s)) \end{aligned}$$

which implies that $(s^3-1)[d(u,F(s,t)) + d(v,F(t,s))] < 0$,
which is a contracdiction.
Therefore $d(u,F(s,t)) + d(v,F(t,s)) = 0$ implies that $F(s,t) = u = fs$ and $F(t,s) = v = ft$.
Hence (s,t) is a coincidence point of F and f. Since the pair (F,f) is w-compatible, we have
$fu = f(F(s,t)) = F(fs,ft) = F(u,v)$ and $fv = f(F(t,s)) = F(ft,fs) = F(v,u)$.
We now prove that $fu = u$ and $fv = v$. Suppose that $fu \neq u$ and $fv \neq v$.
We now consider

$$\begin{aligned} s^4[d(fu,u) + d(fv,v)] &\leq s^5[d(fu,z_{2n+1}) + d(fv,w_{2n+1})] + s^5[d(z_{2n+1},u) + d(w_{2n+1},v)] \\ &= s(s^4[d(F(u,v),G(x_{2n+1},y_{2n+1})) + d(F(v,u),G(y_{2n+1},x_{2n+1}))]) \\ &\qquad + s^5[d(z_{2n+1},u) + d(w_{2n+1},v)] \\ &\leq s\ k\ M(u,v,x_{2n+1},y_{2n+1}) + s^5[d(z_{2n+1},u) + d(w_{2n+1},v)] \end{aligned} \tag{14}$$

where

$$\begin{aligned} M(u,v,x_{2n+1},y_{2n+1}) &= \max\{d(fu,gx_{2n+1}) + d(fv,gy_{2n+1}), d(fu,F(u,v)) + d(fv,F(v,u)), \\ &\qquad d(gx_{2n+1},G(x_{2n+1},y_{2n+1})) + d(gy_{2n+1},G(y_{2n+1},x_{2n+1})), \\ &\qquad \tfrac{d(fu,G(x_{2n+1},y_{2n+1})) + d(fv,G(y_{2n+1},x_{2n+1}))}{2s}, \tfrac{d(gx_{2n+1},F(u,v)) + d(gy_{2n+1},F(v,u))}{2s^2}\} \\ &= \max\{d(fu,z_{2n}) + d(fv,w_{2n}), d(fu,F(u,v)) + d(fv,F(v,u)), \\ &\qquad d(z_{2n},z_{2n+1}) + d(w_{2n},w_{2n+1}), \tfrac{d(fu,z_{2n+1}) + d(fv,w_{2n+1})}{2s}, \tfrac{d(z_{2n},fu) + d(w_{2n},fv)}{2s^2}\} \end{aligned}$$

On taking limit superior as $n \to \infty$, we get
$\limsup_{n\to\infty} M(u,v,x_{2n+1},y_{2n+1}) \leq d(fu,F(u,v)) + d(fv,F(v,u))$.
On letting as $n \to \infty$ in (14), we have
$s^3[d(fu,u) + d(fv,v)] \leq k[d(fu,F(u,v)) + d(fv,F(v,u))] < d(fu,F(u,v)) + d(fv,F(v,u))$,
a contradiction.
Therefore $fu = u$ and $fv = v$.
Thus $F(u,v) = fu = u$ and $F(v,u) = fv = v$.
Hence (u,v) is a common coupled fixed point of F and f.
By Proposition 2.2, we have
(u,v) is a unique common coupled fixed point of F,G,f and g. □

Theorem 2.5. Let (X,d) be a b-metric space with coefficient $s \geq 1$. Let $F,G : X \times X \to X$, $f,g : X \to X$ be four mappings. Suppose that there exist k_1,k_2,k_3,k_4 and k_5 in $[0,1)$ with $k_1 + k_2 + k_3 + 2sk_4 + 2sk_5 < 1$ such that

$$\left\{ \begin{aligned} s^4[d(F(x,y),G(u,v)) + d(F(y,x),G(v,u))] &\leq k_1[d(fx,gu) + d(fy,gv)] \\ &\quad + k_2[d(fx,F(x,y)) + d(fy,F(y,x))] \\ &\quad + k_3[d(gu,G(u,v)) + d(gv,G(v,u))] \\ &\quad + k_4[d(fx,G(u,v)) + d(fy,G(v,u))] \\ &\quad + k_5[d(gu,F(x,y)) + d(gv,F(y,x))] \end{aligned} \right. \tag{15}$$

for all $x, y, u, v \in X$. Also, suppose the following hypotheses:

(i) $F(X \times X) \subset g(X)$ and $G(X \times X) \subset f(X)$,
(ii) either $f(X)$ or $g(X)$ is a complete subspace of X,
(iii) (F, f) and (G, g) are w-compatible.

Then F, G, f and g have a unique common coupled fixed point in $X \times X$.

Proof. We define the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{w_n\}$ same as in Theorem 2.4. Assume that $z_n = z_{n+1}$ and $w_n = w_{n+1}$ for some $n = \{0, 1, 2, \ldots\}$.
Case (i): n even.
We write $n = 2m, m \in \mathbb{N}$.
Now we consider and using (15), we have

$$\begin{aligned}
d(z_{2m+1}, z_{2m+2}) + d(w_{2m+1}, w_{2m+2}) &\leq s^4[d(z_{2m+1}, z_{2m+2}) + d(w_{2m+1}, w_{2m+2})] \\
&= s^4[d(F(x_{2m+2}, y_{2m+2}), G(x_{2m+1}, y_{2m+1})) \\
&\quad + d(F(y_{2m+2}, x_{2m+2}), G(y_{2m+1}, x_{2m+1}))] \\
&\leq k_1 d(fx_{2m+2}, gx_{2m+1}) + d(fy_{2m+2}, gy_{2m+1}) \\
&+ k_2 d(fx_{2m+2}, F(x_{2m+2}, y_{2m+2})) + d(fy_{2m+2}, F(y_{2m+2}, x_{2m+2})) \\
&+ k_3 d(gx_{2m+1}, G(x_{2m+1}, y_{2m+1})) + d(gy_{2m+1}, G(y_{2m+1}, x_{2m+1})) \\
&+ k_4 d(fx_{2m+2}, G(x_{2m+1}, y_{2m+1})) + d(fy_{2m+2}, G(y_{2m+1}, x_{2m+1})) \\
&+ k_5 d(gx_{2m+1}, F(x_{2m+2}, y_{2m+2})) + d(gy_{2m+1}, F(y_{2m+2}, x_{2m+2})) \\
&= k_1 d(z_{2m+1}, z_{2m}) + d(w_{2m+1}, w_{2m}) \\
&\quad + k_2 d(z_{2m+1}, z_{2m+2}) + d(w_{2m+1}, w_{2m+2}) \\
&\quad + k_3 d(z_{2m}, z_{2m+1}) + d(w_{2m}, w_{2m+1}) \\
&\quad + k_4 d(z_{2m+1}, z_{2m+1}) + d(w_{2m+1}, w_{2m+1}) \\
&\quad + k_5 d(z_{2m}, z_{2m+2}) + d(w_{2m}, w_{2m+2}) \\
&\leq k_2 d(z_{2m+1}, z_{2m+2}) + d(w_{2m+1}, w_{2m+2}) \\
&\quad + sk_5 d(z_{2m+1}, z_{2m+2}) + d(w_{2m+1}, w_{2m+2})
\end{aligned}$$

which implies that $(1 - k_2 - sk_5)[d(z_{2m+1}, z_{2m+2}) + d(w_{2m+1}, w_{2m+2})] \leq 0$ so that $z_{2m+1} = z_{2m+2}$ and $w_{2m+1} = w_{2m+2}$.
Hence $z_{2m} = z_{2m+1} = z_{2m+2}$ and $w_{2m} = w_{2m+1} = w_{2m+2}$.
In general, $z_{2m} = z_{2m+k}$ and $w_{2m} = w_{2m+k}$ for $k = 0, 1, 2, \ldots$.
Case (ii): n odd.
We write $n = 2m + 1, m \in \mathbb{N}$. Now we consider

$$\begin{aligned}
d(z_{2m+2}, z_{2m+3}) + d(w_{2m+2}, w_{2m+3}) &\leq s^4[d(z_{2m+2}, z_{2m+3}) + d(w_{2m+2}, w_{2m+3})] \\
&= s^4[d(F(x_{2m+2}, y_{2m+2}), G(x_{2m+3}, y_{2m+3})) \\
&\quad + d(F(y_{2m+2}, x_{2m+2}), G(y_{2m+3}, x_{2m+3}))] \\
&\leq k_1 d(fx_{2m+2}, gx_{2m+3}) + d(fy_{2m+2}, gy_{2m+3}) \\
&+ k_2 d(fx_{2m+2}, F(x_{2m+2}, y_{2m+2})) + d(fy_{2m+2}, F(y_{2m+2}, x_{2m+2})) \\
&+ k_3 d(gx_{2m+3}, G(x_{2m+3}, y_{2m+3})) + d(gy_{2m+3}, G(y_{2m+3}, x_{2m+3})) \\
&+ k_4 d(fx_{2m+2}, G(x_{2m+3}, y_{2m+3})) + d(fy_{2m+2}, G(y_{2m+3}, x_{2m+3})) \\
&+ k_5 d(gx_{2m+3}, F(x_{2m+2}, y_{2m+2})) + d(gy_{2m+3}, F(y_{2m+2}, x_{2m+2})) \\
&\leq k_3 d(z_{2m+2}, z_{2m+3}) + d(w_{2m+2}, w_{2m+3}) \\
&\quad + sk_4 d(z_{2m+2}, z_{2m+3}) + d(w_{2m+2}, w_{2m+3})
\end{aligned}$$

which implies that $(1 - k_3 - sk_4)[d(z_{2m+2}, z_{2m+3}) + d(w_{2m+2}, w_{2m+3})] \leq 0$ so that $z_{2m+2} = z_{2m+3}$ and $w_{2m+2} = w_{2m+3}$.
Hence $z_{2m+1} = z_{2m+2} = z_{2m+3}$ and $w_{2m+1} = w_{2m+2} = w_{2m+3}$.
In general, $z_{2m+1} = z_{2m+k}$ and $w_{2m+1} = w_{2m+k}$ for $k = 0, 1, 2, \ldots$.
From Case (i) and Case (ii), we have $z_{n+k} = z_n$ and $w_{n+k} = w_n$ for $k = 0, 1, 2, \ldots$.
Therefore, $\{z_{n+k}\}$ and $\{w_{n+k}\}$ are constant sequences and hence $\{z_{n+k}\}$ and $\{w_{n+k}\}$

are Cauchy sequences.
Now we assume that $z_n \neq z_{n+1}$ and $w_n \neq w_{n+1}$ for all $n \in \mathbb{N}$.
If n is odd, then $n = 2m+1, m \in \mathbb{N}$.
We now consider

$$\begin{aligned} d(z_{2m+1}, z_{2m+2}) + d(w_{2m+1}, w_{2m+2}) &\leq s^4[d(z_{2m+1}, z_{2m+2}) + d(w_{2m+1}, w_{2m+2})] \\ &= s^4[d(F(x_{2m+2}, y_{2m+2}), G(x_{2m+1}, y_{2m+1})) \\ &\quad + d(F(y_{2m+2}, x_{2m+2}), G(y_{2m+1}, x_{2m+1}))] \\ &\leq k_1 d(fx_{2m+2}, gx_{2m+1}) + d(fy_{2m+2}, gy_{2m+1}) \end{aligned}$$

$$\begin{aligned} &+ k_2 d(fx_{2m+2}, F(x_{2m+2}, y_{2m+2})) + d(fy_{2m+2}, F(y_{2m+2}, x_{2m+2})) \\ &+ k_3 d(gx_{2m+1}, G(x_{2m+1}, y_{2m+1})) + d(gy_{2m+1}, G(y_{2m+1}, x_{2m+1})) \\ &+ k_4 d(fx_{2m+2}, G(x_{2m+1}, y_{2m+1})) + d(fy_{2m+2}, G(y_{2m+1}, x_{2m+1})) \\ &+ k_5 d(gx_{2m+1}, F(x_{2m+2}, y_{2m+2})) + d(gy_{2m+1}, F(y_{2m+2}, x_{2m+2})) \end{aligned}$$

which implies that
$(1-k_2-sk_5)[d(z_n, z_{n+1})+d(w_n, w_{n+1})] \leq (k_1+k_3+sk_5)[d(z_{n-1}, z_n)+d(w_{n-1}, w_n)]$
and hence

$$\begin{aligned} [d(z_n, z_{n+1}) + d(w_n, w_{n+1})] &\leq \tfrac{(k_1+k_3+sk_5)}{(1-k_2-sk_5)}[d(z_{n-1}, z_n) + d(w_{n-1}, w_n)] \\ &= h_1[d(z_{n-1}, z_n) + d(w_{n-1}, w_n)] \end{aligned} \tag{16}$$

where $h_1 = \frac{k_1+k_3+sk_5}{1-k_2-sk_5} < 1$.
On the similar lines, if n is even, it follows that

$$\begin{aligned} [d(z_n, z_{n+1}) + d(w_n, w_{n+1})] &\leq \tfrac{(k_1+k_2+sk_4)}{(1-k_3-sk_4)}[d(z_{n-1}, z_n) + d(w_{n-1}, w_n)] \\ &= h_2[d(z_{n-1}, z_n) + d(w_{n-1}, w_n)] \end{aligned} \tag{17}$$

where $h_2 = \frac{k_1+k_2+sk_4}{1-k_3-sk_4} < 1$.
We take $h = \max\{h_1, h_2\}$, from (16) and (17), we have that
$[d(z_n, z_{n+1}) + d(w_n, w_{n+1})] \leq h[d(z_{n-1}, z_n) + d(w_{n-1}, w_n)]$
By Lemma 2.3, it follows that
$\{z_n\}$ and $\{w_n\}$ are Cauchy sequences in b-metric space (X, d).
Therefore $\{z_{2n+1}\}$ and $\{w_{2n+1}\}$ are Cauchy sequences in the subspace $(f(X), d)$.
Suppose that $f(X)$ is complete.
Since $\{z_{2n+1}\} \subseteq f(X)$ and $\{w_{2n+1}\} \subseteq f(X)$, it follows that
the sequences $\{z_{2n+1}\}$ and $\{w_{2n+1}\}$ are convergent in $(f(X), d)$.
Hence, there exist $u, v \in f(X)$ such that
$\lim_{n\to\infty} d(z_{2n+1}, u) = 0$ and $\lim_{n\to\infty} d(w_{2n+1}, v) = 0$.
Since $u, v \in f(X)$, there exist $s, t \in X$ such that $u = fs$ and $v = ft$.
Since $\{z_n\}$ and $\{w_n\}$ are Cauchy sequences in X and $\{z_{2n+1}\} \to u$ and
$\{w_{2n+1}\} \to v$ as $n \to \infty$, it follows that
$\{z_{2n}\} \to u$ and $\{w_{2n}\} \to v$ as $n \to \infty$.
Therefore $\lim_{n\to\infty} d(z_{2n}, u) = 0$ and $\lim_{n\to\infty} d(w_{2n}, v) = 0$.
By Lemma 1.7, we have
$\frac{1}{s} d(F(s,t), u) \leq \liminf_{n\to\infty} d(F(s,t), z_{2n+1}) \leq \limsup_{n\to\infty} d(F(s,t), z_{2n+1}) \leq s\ d(F(s,t), u)$
and
$\frac{1}{s} d(F(t,s), v) \leq \liminf_{n\to\infty} d(F(t,s), w_{2n+1}) \leq \limsup_{n\to\infty} d(F(t,s), w_{2n+1}) \leq s\ d(F(t,s), v)$.
We now prove that $F(s,t) = u = fs$ and $F(t,s) = v = ft$.
Suppose that $F(s,t) \neq u \neq fs$ and $F(t,s) \neq v \neq ft$.

Now we consider

$$\begin{cases} d(F(s,t),z_{2n+1})+d(F(t,s),w_{2n+1}) = d(F(s,t),G(x_{2n+1},y_{2n+1})) \\ \qquad +d(F(t,s),G(y_{2n+1},x_{2n+1})) \\ \leq s^4[d(F(s,t),G(x_{2n+1},y_{2n+1})) \\ \qquad +d(F(t,s),G(y_{2n+1},x_{2n+1}))] \\ \leq k_1[d(fs,gx_{2n+1})+d(ft,gy_{2n+1})] \\ \quad +k_2[d(fs,F(s,t))+d(ft,F(t,s))] \\ \quad +k_3[d(gx_{2n+1},G(x_{2n+1},y_{2n+1})) \\ \qquad +d(gy_{2n+1},G(y_{2n+1},x_{2n+1}))] \\ \quad +k_4[d(fs,G(x_{2n+1},y_{2n+1})) \\ \qquad +d(ft,G(y_{2n+1},x_{2n+1}))] \\ \quad +k_5[d(gx_{2n+1},F(s,t))+d(gy_{2n+1},F(t,s))] \\ = k_1[d(u,z_{2n})+d(v,w_{2n})] \\ \qquad +k_2[d(u,F(s,t))+d(v,F(t,s))] \\ \quad +k_3[d(z_{2n},z_{2n+1})+d(w_{2n},w_{2n+1})] \\ \quad +k_4[d(u,z_{2n+1})+d(v,w_{2n+1})] \\ \qquad +k_5[d(z_{2n},F(s,t))+d(w_{2n},F(t,s))] \\ \leq k_1[d(u,z_{2n})+d(v,w_{2n})] \\ \qquad +k_2[d(u,F(s,t))+d(v,F(t,s))] \\ \quad +k_3[d(z_{2n},z_{2n+1})+d(w_{2n},w_{2n+1})] \\ \quad +k_4[d(u,z_{2n+1})+d(v,w_{2n+1})] \\ \quad +sk_5[d(z_{2n},z_{2n+1})+d(z_{2n+1},F(s,t))] \\ \quad +sk_5[d(w_{2n},w_{2n+1})+d(w_{2n+1},F(t,s))] \end{cases} \tag{18}$$

On taking limit superior as $n \to \infty$ in (18), we get

$$\begin{aligned} s^4\tfrac{1}{s}[d(u,F(s,t))+d(v,F(t,s))] &\leq \limsup_{n\to\infty} s^4[d(F(s,t),G(x_{2n+1},y_{2n+1})) \\ &\qquad +d(F(t,s),G(y_{2n+1},x_{2n+1}))] \\ &\leq k_2[d(u,F(s,t))+d(v,F(t,s))]+s^2k_5[d(u,F(s,t)) \\ &\qquad +d(v,F(t,s))] \\ &\leq (k_2+s^2k_5)[d(u,F(s,t))+d(v,F(t,s))] \\ &\leq (sk_2+s^2k_5)[d(u,F(s,t))+d(v,F(t,s))] \\ &< s[d(u,F(s,t))+d(v,F(t,s))] \end{aligned}$$

which implies that
$(s^3-s)[d(u,F(s,t))+d(v,F(t,s))]<0$,
a contracdiction.
Therefore $d(u,F(s,t))+d(v,F(t,s))=0$
which implies that
$F(s,t)=u=fs$ and $F(t,s)=v=ft$.
Hence (s,t) is a coincidence point of F and f.
Since the pair (F,f) is w-compatible, we have
$fu=f(F(s,t))=F(fs,ft)=F(u,v)$ and
$fv=f(F(t,s))=F(ft,fs)=F(v,u)$.
We now prove that $fu=u$ and $fv=v$.
Suppose that $fu\neq u$ and $fv\neq v$.

We now consider

$$\left\{\begin{aligned} s^4[d(fu,u)+d(fv,v)] &\le s^5[d(fu,z_{2n+1})+d(fv,w_{2n+1})]\\ &\quad +s^5[d(z_{2n+1},u)+d(w_{2n+1},v)]\\ &= s(s^4[d(F(u,v),G(x_{2n+1},y_{2n+1}))\\ &\quad +d(F(v,u),G(y_{2n+1},x_{2n+1}))])\\ &\quad +s^5[d(z_{2n+1},u)+d(w_{2n+1},v)]\\ &\le s[k_1[d(fu,gx_{2n+1})+d(fv,gy_{2n+1})]\\ &\quad +k_2[d(fu,F(u,v))+d(fv,F(v,u))]\\ &\quad +k_3[d(gx_{2n+1},G(x_{2n+1},y_{2n+1}))\\ &\quad +d(gy_{2n+1},G(y_{2n+1},x_{2n+1}))]\\ &\quad +k_4[d(fu,G(x_{2n+1},y_{2n+1}))+d(fv,G(y_{2n+1},x_{2n+1}))]\\ &\quad +k_5[d(gx_{2n+1},F(u,v))+d(gy_{2n+1},F(v,u))]]\\ &\quad +s^5[d(z_{2n+1},u)+d(w_{2n+1},v)]\\ &= s[k_1[d(fu,z_{2n})+d(fv,w_{2n})]\\ &\quad +k_2[d(fu,F(u,v))+d(fv,F(v,u))]\\ &\quad +k_3[d(z_{2n},z_{2n+1})+d(w_{2n},w_{2n+1})]\\ &\quad +k_4[d(fu,z_{2n+1})+d(fv,w_{2n+1})]\\ &\quad +k_5[d(z_{2n},fu)+d(w_{2n},fv)]]\\ &\quad +s^5[d(z_{2n+1},u)+d(w_{2n+1},v)]. \end{aligned}\right. \tag{19}$$

On taking limit superior as $n \to \infty$ in (19), we get

$$\begin{aligned} s^4[d(fu,u)+d(fv,v)] &\le s(sk_1+k_2+sk_4+sk_5)[d(fu,F(u,v))+d(fv,F(v,u))]\\ &\le s(sk_1+sk_2+s^2k_4+s^2k_5)[d(fu,F(u,v))+d(fv,F(v,u))]\\ &\le s^2[d(fu,F(u,v))+d(fv,F(v,u))] \end{aligned}$$

which implies that $(s^2-1)[d(fu,F(u,v))+d(fv,F(v,u))] \le 0$ so that $d(fu,F(u,v))+d(fv,F(v,u))=0$.
Therefore $fu=u$ and $fv=v$.
Thus $F(u,v)=fu=u$ and $F(v,u)=fv=v$.
Hence (u,v) is a common coupled fixed point of F and f.
By Proposition 2.2, we have (u,v) is a unique common coupled fixed point of F,G,f and g. □

Theorem 2.6. Let (X,d) be a b-metric space with coefficient $s \ge 1$. Let $F,G : X\times X \to X$, $f,g : X \to X$ be four mappings. Suppose that there exist k_1,k_2,k_3,k_4 and k_5 in $[0,1)$ with
$k_1+k_2+k_3+k_4+k_5+k_6+2sk_7+2sk_8+2sk_9+2sk_{10} < 1$ such that

$$\begin{aligned} s^4 d(F(x,y),G(u,v)) &\le k_1 d(fx,gu)+k_2 d(fy,gv)\\ &\quad +k_3 d(fx,F(x,y))+k_4 d(fy,F(y,x))+k_5 d(gu,G(u,v))\\ &\quad +k_6 d(gv,G(v,u))+k_7 d(fx,G(u,v))+k_8 d(fy,G(v,u))\\ &\quad +k_9 d(gu,F(x,y))+k_{10} d(gv,F(y,x)) \end{aligned} \tag{20}$$

for all $x,y,u,v \in X$. Also, suppose the following hypotheses:

(*i*) $F(X\times X) \subset g(X)$ and $G(X\times X) \subset f(X)$,
(*ii*) either $f(X)$ or $g(X)$ is a complete subspace of X,
(*iii*) (F,f) and (G,g) are w-compatible.

Then F,G,f and g have a unique common coupled fixed point in $X\times X$.

Proof. Let $x, y, u, v \in X$ be arbitrary. Then from the inequality (20), we have

$$\begin{aligned} s^4 d(F(x,y), G(u,v)) \leq\ & k_1 d(fx, gu) + k_2 d(fy, gv) + k_3 d(fx, F(x,y)) \\ & + k_4 d(fy, F(y,x)) + k_5 d(gu, G(u,v)) + k_6 d(gv, G(v,u)) \\ & + k_7 d(fx, G(u,v)) + k_8 d(fy, G(v,u)) + k_9 d(gu, F(x,y)) \\ & + k_{10} d(gv, F(y,x)) \end{aligned} \tag{21}$$

and

$$\begin{aligned} s^4 d(F(y,x), G(v,u)) \leq\ & k_1 d(fy, gv) + k_2 d(fx, gu) + k_3 d(fy, F(y,x)) \\ & + k_4 d(fx, F(x,y)) + k_5 d(gv, G(v,u)) + k_6 d(gu, G(u,v)) \\ & + k_7 d(fy, G(v,u)) + k_8 d(fx, G(u,v)) + k_9 d(gv, F(y,x)) \\ & + k_{10} d(gu, F(x,y)). \end{aligned} \tag{22}$$

From (21) and (22), we get

$$\begin{aligned} d(F(x,y), G(u,v)) + d(F(y,x), G(v,u)) \leq\ & (k_1 + k_2)[d(fx, gu) + d(fy, gv)] \\ & + (k_3 + k_4)[d(fx, F(x,y)) + d(fy, F(y,x))] \\ & + (k_5 + k_6)[d(gu, G(u,v)) + d(gv, G(v,u))] \\ & + s(k_7 + k_8)[d(fx, G(u,v)) + d(fy, G(v,u))] \\ & + s(k_9 + k_{10})[d(gu, F(x,y)) + d(gv, F(y,x))]. \end{aligned}$$

Therefore proof follows from Theorem 2.5. □

3. Examples and corollaries

The following is an example in support of Theorem 2.4.

Example 3.1. Let $X = [0, \infty)$ and let $d : X \times X \to \mathbb{R}^+$ defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 4 & \text{if } x, y \in [0,1), \\ 5 + \frac{1}{x+y} & \text{if } x, y \in [1, \infty), \\ \frac{27}{10} & \text{otherwise.} \end{cases}$$

Then clearly (X, d) is a complete b-metric space with coefficient $s = \frac{489}{480} (> 1)$.

We define $F, G : X \times X \to X$ and $f, g : X \to X$ by

$$F(x,y) = \begin{cases} 2 & \text{if } x, y \in [0,1) \\ \frac{x^2+y^2}{2} & \text{if } x, y \in [1, \infty) \\ 0 & \text{otherwise} \end{cases} \quad G(x,y) = \begin{cases} xy & \text{if } x, y \in [0,1) \\ \frac{2}{x^3+y^3} & \text{if } x, y \in [1, \infty) \\ 0 & \text{otherwise} \end{cases}$$

$$f(x) = \begin{cases} \frac{x(5-x)}{4} & \text{if } x \in [0,1) \\ \frac{1+x}{2} & \text{if } x \in [1, \infty) \end{cases} \quad \text{and } g(x) = \begin{cases} x(2-x) & \text{if } x \in [0,1) \\ 2x - 1 & \text{if } x \in [1, \infty). \end{cases}$$

Clearly $F(X \times X) \subseteq g(X)$ and $G(X \times X) \subseteq f(X)$. The pairs (F, f) and (G, g) are w-compatible.

Without loss of generality, we assume that $x \geq y \geq u \geq v$.

Case (i). $x, y, u, v \in [0, 1)$.

In this case,

$d(F(x,y), G(u,v)) = \frac{27}{10}, d(F(y,x), G(v,u)) = \frac{27}{10}, d(fx, gu) = 4, d(fy, gv) = 4$
$d(fx, F(x,y)) = \frac{27}{10}, d(fy, F(y,x)) = \frac{27}{10}, d(gu, G(u,v)) = 4, d(gv, G(v,u)) = 4,$
$d(fx, G(u,v)) = 4, d(fy, G(v,u)) = 4, d(gu, F(x,y)) = \frac{27}{10}, d(gv, F(y,x)) = \frac{27}{10}$ and

$$\begin{aligned} & \max\{d(fx, gu) + d(fy, gv), d(fx, F(x,y)) + d(fy, F(y,x)), d(gu, G(u,v)) + d(gv, G(v,u)), \\ & \quad cm \frac{d(fx, G(u,v)) + d(fy, G(v,u))}{2s}, \frac{d(gu, F(x,y)) + d(gv, F(y,x))}{2s^2}\} \\ & = \max\{8, \tfrac{27}{5}, 8, (\tfrac{240}{489})(8), (\tfrac{230400}{478242})(\tfrac{27}{5})\} = 8. \end{aligned}$$

Now we consider

$$\begin{aligned} s^4[d(F(x,y),G(u,v))+d(F(y,x),G(v,u))] &= (\tfrac{489}{480})^4[\tfrac{27}{10}+\tfrac{27}{10}] \\ &\leq (\tfrac{4}{5})\ 8 \\ &\leq k\max\{d(fx,gu)+d(fy,gv), \\ &\quad d(fx,F(x,y))+d(fy,F(y,x)), \\ &\quad d(gu,G(u,v))+d(gv,G(v,u)), \\ &\quad \tfrac{d(fx,G(u,v))+d(fy,G(v,u))}{2s}, \\ &\quad \tfrac{d(gu,F(x,y))+d(gv,F(y,x))}{2s^2}\}. \end{aligned}$$

Case (ii). $x,y,u,v\in(1,\infty)$.

In this case, $d(F(x,y),G(u,v))=\frac{27}{10}, d(F(y,x),G(v,u))=\frac{27}{10}, d(fx,gu)=5+\frac{1}{x+y}$, $d(fy,gv)=5+\frac{1}{x+y}, d(fx,F(x,y))=5+\frac{1}{x+y}, d(fy,F(y,x))=5+\frac{1}{x+y}$, $d(gu,G(u,v))=\frac{27}{10}, d(gv,G(v,u))=\frac{27}{10}, d(fx,G(u,v))=\frac{27}{10}, d(fy,G(v,u))=\frac{27}{10}$, $d(gu,F(x,y))=5+\frac{1}{x+y}, d(gv,F(y,x))=5+\frac{1}{x+y}$ and

$$\begin{aligned} &\max\{d(fx,gu)+d(fy,gv), d(fx,F(x,y))+d(fy,F(y,x)), d(gu,G(u,v))+d(gv,G(v,u)), \\ &\quad \tfrac{d(fx,G(u,v))+d(fy,G(v,u))}{2s}, \tfrac{d(gu,F(x,y))+d(gv,F(y,x))}{2s^2}\} \\ &= \max\{10+\tfrac{2}{x+y}, 10+\tfrac{2}{x+y}, \tfrac{27}{5}, (\tfrac{240}{489})(\tfrac{27}{5}), (\tfrac{230400}{478242})(10+\tfrac{2}{x+y})\} = 10+\tfrac{2}{x+y}. \end{aligned}$$

Now we consider

$$\begin{aligned} s^4[d(F(x,y),G(u,v))+d(F(y,x),G(v,u))] &= (\tfrac{489}{480})^4[\tfrac{27}{10}+\tfrac{27}{10}] \\ &\leq (\tfrac{4}{5})(10+\tfrac{2}{x+y}) \\ &\leq k\max\{d(fx,gu)+d(fy,gv), \\ &\quad d(fx,F(x,y))+d(fy,F(y,x)), \\ &\quad d(gu,G(u,v))+d(gv,G(v,u)), \\ &\quad \tfrac{d(fx,G(u,v))+d(fy,G(v,u))}{2s}, \\ &\quad \tfrac{d(gu,F(x,y))+d(gv,F(y,x))}{2s^2}\}. \end{aligned}$$

Case (iii). $x,y\in(1,\infty), u,v\in[0,1)$.

In this case,
$d(F(x,y),G(u,v))=\frac{27}{10}, d(F(y,x),G(v,u))=\frac{27}{10}, d(fx,gu)=\frac{27}{10}, d(fy,gv)=\frac{27}{10}$, $d(fx,F(x,y))=5+\frac{1}{x+y}, d(fy,F(y,x))=5+\frac{1}{x+y}, d(gu,G(u,v))=4$, $d(gv,G(v,u))=4, d(fx,G(u,v))=\frac{27}{10}, d(fy,G(v,u))=\frac{27}{10}, d(gu,F(x,y))=\frac{27}{10}$, $d(gv,F(y,x))=\frac{27}{10}$ and

$$\begin{aligned} &\max\{d(fx,gu)+d(fy,gv), d(fx,F(x,y))+d(fy,F(y,x)), d(gu,G(u,v))+d(gv,G(v,u)), \\ &\quad \tfrac{d(fx,G(u,v))+d(fy,G(v,u))}{2s}, \tfrac{d(gu,F(x,y))+d(gv,F(y,x))}{2s^2}\} \\ &= \max\{\tfrac{27}{5}, 10+\tfrac{2}{x+y}, 8, (\tfrac{240}{489})(\tfrac{27}{5}), (\tfrac{230400}{478242})(\tfrac{27}{5})\} = 10+\tfrac{2}{x+y}. \end{aligned}$$

Now we consider

$$\begin{aligned} s^4[d(F(x,y),G(u,v))+d(F(y,x),G(v,u))] &= (\tfrac{489}{480})^4[\tfrac{27}{10}+\tfrac{27}{10}] \\ &\leq (\tfrac{4}{5})(10+\tfrac{2}{x+y}) \\ &\leq k\max\{d(fx,gu)+d(fy,gv), \\ &\quad d(fx,F(x,y))+d(fy,F(y,x)), \\ &\quad d(gu,G(u,v))+d(gv,G(v,u)), \\ &\quad \tfrac{d(fx,G(u,v))+d(fy,G(v,u))}{2s}, \\ &\quad \tfrac{d(gu,F(x,y))+d(gv,F(y,x))}{2s^2}\}. \end{aligned}$$

Case (iv). $x=y=1, u,v\in[0,1)$.

In this case,
$d(F(x,y),G(u,v))=\frac{27}{10}, d(F(y,x),G(v,u))=\frac{27}{10}, d(fx,gu)=\frac{27}{10}, d(fy,gv)=\frac{27}{10}$, $d(fx,F(x,y))=0, d(fy,F(y,x))=0, d(gu,G(u,v))=4, d(gv,G(v,u))=4$,

$d(fx, G(u,v)) = \frac{27}{10}, d(fy, G(v,u)) = \frac{27}{10}, d(gu, F(x,y)) = \frac{27}{10}, d(gv, F(y,x)) = \frac{27}{10}$
and

$$\max\{d(fx,gu)+d(fy,gv), d(fx,F(x,y))+d(fy,F(y,x)), d(gu,G(u,v))+d(gv,G(v,u)), \frac{d(fx,G(u,v))+d(fy,G(v,u))}{2s}, \frac{d(gu,F(x,y))+d(gv,F(y,x))}{2s^2}\}$$
$$= \max\{\tfrac{27}{5}, 0, 8, (\tfrac{240}{489})(\tfrac{27}{5}), (\tfrac{230400}{478242})(\tfrac{27}{5})\} = 8.$$

Now we consider

$$\begin{aligned} s^4[d(F(x,y),G(u,v)) + d(F(y,x),G(v,u))] &= (\tfrac{489}{480})^4[\tfrac{27}{10} + \tfrac{27}{10}] \\ &\le (\tfrac{4}{5})(8) \\ &\le k\max\{d(fx,gu)+d(fy,gv), \\ &\quad d(fx,F(x,y))+d(fy,F(y,x)), \\ &\quad d(gu,G(u,v))+d(gv,G(v,u)), \\ &\quad \tfrac{d(fx,G(u,v))+d(fy,G(v,u))}{2s}, \\ &\quad \tfrac{d(gu,F(x,y))+d(gv,F(y,x))}{2s^2}\}. \end{aligned}$$

From all the above cases, F, G, f and g satisfy all the hypotheses of Theorem 2.4 with $k = \frac{4}{5}$ and $(1,1)$ is a unique common coupled fixed point of F, G, f and g. The following is an example in support of Theorem 2.5.

Example 3.2. Let $X = [0,\infty)$ and let $d : X \times X \to \mathbb{R}^+$ defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 4 & \text{if } x, y \in (0,1), \\ 5 + \frac{1}{x+y} & \text{if } x, y \in [1,\infty), \\ \frac{27}{10} & \text{otherwise.} \end{cases}$$

Then clearly (X, d) is a complete b-metric space with coefficient $s = \frac{489}{480} (> 1)$.
We define $F, G : X \times X \to X$ and $f, g : X \to X$ by

$$F(x,y) = \begin{cases} 2 & \text{if } x, y \in (0,1) \\ \frac{x+y}{2} & \text{if } x, y \in [1,\infty) \\ 0 & \text{otherwise} \end{cases} \quad G(x,y) = \begin{cases} xy & \text{if } x, y \in (0,1) \\ \frac{2}{x^2+y^2} & \text{if } x, y \in [1,\infty) \\ 0 & \text{otherwise} \end{cases}$$

$$f(x) = \begin{cases} x(1-x) & \text{if } x \in [0,1) \\ 3x-2 & \text{if } x \in [1,\infty) \end{cases} \quad \text{and } g(x) = \begin{cases} x & \text{if } x \in [0,1) \\ 2x^2-1 & \text{if } x \in [1,\infty). \end{cases}$$

Clearly $F(X \times X) \subseteq g(X)$ and $G(X \times X) \subseteq f(X)$. The pairs (F, f) and (G, g) are w-compatible.
Without loss of generality, we assume that $x \ge y \ge u \ge v$.
We choose $k_1 = k_2 = \frac{1}{11}, k_3 = \frac{4}{5}, k_4 = k_5 = \frac{60}{14181}$.
Then clearly $k_1 + k_2 + k_3 + 2sk_4 + 2sk_5 < 1$.

Case (i). $x, y, u, v \in [0,1)$.
In this case,
$d(F(x,y), G(u,v)) = \frac{27}{10}, d(F(y,x), G(v,u)) = \frac{27}{10}, d(fx, gu) = 4, d(fy, gv) = 4$
$d(fx, F(x,y)) = \frac{27}{10}, d(fy, F(y,x)) = \frac{27}{10}, d(gu, G(u,v)) = 4, d(gv, G(v,u)) = 4,$
$d(fx, G(u,v)) = 4, d(fy, G(v,u)) = 4, d(gu, F(x,y)) = \frac{27}{10}, d(gv, F(y,x)) = \frac{27}{10}.$
Now we consider

$$\begin{aligned} s^4[d(F(x,y),G(u,v)) + d(F(y,x),G(v,u))] &= (\tfrac{489}{480})^4[\tfrac{27}{10} + \tfrac{27}{10}] \\ &\le (\tfrac{1}{11})(8)+(\tfrac{1}{11})(\tfrac{27}{5})+(\tfrac{4}{5})(8)+(\tfrac{60}{14181})(8) \\ &\quad + (\tfrac{60}{14181})(\tfrac{27}{5}) \\ &\le k_1[d(fx,gu)+d(fy,gv)] \\ &\quad + k_2[d(fx,F(x,y))+d(fy,F(y,x))] \\ &\quad + k_3[d(gu,G(u,v))+d(gv,G(v,u))] \\ &\quad + k_4[d(fx,G(u,v))+d(fy,G(v,u))] \\ &\quad + k_5[d(gu,F(x,y))+d(gv,F(y,x))]. \end{aligned}$$

Case (ii). $x, y, u, v \in (1, \infty)$.
In this case, $d(F(x,y), G(u,v)) = \frac{27}{10}, d(F(y,x), G(v,u)) = \frac{27}{10}, d(fx, gu) = 5 + \frac{1}{x+y}$,
$d(fy, gv) = 5 + \frac{1}{x+y}, d(fx, F(x,y)) = 5 + \frac{1}{x+y}, d(fy, F(y,x)) = 5 + \frac{1}{x+y}$,
$d(gu, G(u,v)) = \frac{27}{10}, d(gv, G(v,u)) = \frac{27}{10}, d(fx, G(u,v)) = \frac{27}{10}, d(fy, G(v,u)) = \frac{27}{10}$,
$d(gu, F(x,y)) = 5 + \frac{1}{x+y}, d(gv, F(y,x)) = 5 + \frac{1}{x+y}$.
Now we consider

$$\begin{aligned} s^4[d(F(x,y), G(u,v)) + d(F(y,x), G(v,u))] &= (\tfrac{489}{480})^4[\tfrac{27}{10} + \tfrac{27}{10}] \\ &\leq (\tfrac{1}{11})(10 + \tfrac{2}{x+y}) \\ &\quad + (\tfrac{1}{11})(10 + \tfrac{2}{x+y}) + (\tfrac{4}{5})(\tfrac{27}{5}) + (\tfrac{60}{14181})(\tfrac{27}{5}) \\ &\quad + (\tfrac{60}{14181})(10 + \tfrac{2}{x+y}) \\ &\leq k_1[d(fx, gu) + d(fy, gv)] \\ &\quad + k_2[d(fx, F(x,y)) + d(fy, F(y,x))] \\ &\quad + k_3[d(gu, G(u,v)) + d(gv, G(v,u))] \\ &\quad + k_4[d(fx, G(u,v)) + d(fy, G(v,u))] \\ &\quad + k_5[d(gu, F(x,y)) + d(gv, F(y,x))]. \end{aligned}$$

Case (iii). $x, y \in (1, \infty), u, v \in [0, 1)$.
In this case,
$d(F(x,y), G(u,v)) = \frac{27}{10}, d(F(y,x), G(v,u)) = \frac{27}{10}, d(fx, gu) = \frac{27}{10}, d(fy, gv) = \frac{27}{10}$,
$d(fx, F(x,y)) = 5 + \frac{1}{x+y}, d(fy, F(y,x)) = 5 + \frac{1}{x+y}, d(gu, G(u,v)) = 4$,
$d(gv, G(v,u)) = 4, d(fx, G(u,v)) = \frac{27}{10}, d(fy, G(v,u)) = \frac{27}{10}, d(gu, F(x,y)) = \frac{27}{10}$,
$d(gv, F(y,x)) = \frac{27}{10}$.
Now we consider

$$\begin{aligned} s^4[d(F(x,y), G(u,v)) + d(F(y,x), G(v,u))] &= (\tfrac{489}{480})^4[\tfrac{27}{10} + \tfrac{27}{10}] \\ &\leq (\tfrac{1}{11})(\tfrac{27}{5}) + (\tfrac{1}{11})(10 + \tfrac{2}{x+y}) + (\tfrac{4}{5})(8) \\ &\quad + (\tfrac{60}{14181})(\tfrac{27}{5}) + (\tfrac{60}{14181})(\tfrac{27}{5}) \\ &\leq k_1[d(fx, gu) + d(fy, gv)] \\ &\quad + k_2[d(fx, F(x,y)) + d(fy, F(y,x))] \\ &\quad + k_3[d(gu, G(u,v)) + d(gv, G(v,u))] \\ &\quad + k_4[d(fx, G(u,v)) + d(fy, G(v,u))] \\ &\quad + k_5[d(gu, F(x,y)) + d(gv, F(y,x))]. \end{aligned}$$

Case (iv). $x = y = 1, u, v \in [0, 1)$.
In this case,
$d(F(x,y), G(u,v)) = \frac{27}{10}, d(F(y,x), G(v,u)) = \frac{27}{10}, d(fx, gu) = \frac{27}{10}, d(fy, gv) = \frac{27}{10}$,
$d(fx, F(x,y)) = 0, d(fy, F(y,x)) = 0, d(gu, G(u,v)) = 4, d(gv, G(v,u)) = 4$,
$d(fx, G(u,v)) = \frac{27}{10}, d(fy, G(v,u)) = \frac{27}{10}, d(gu, F(x,y)) = \frac{27}{10}, d(gv, F(y,x)) = \frac{27}{10}$.
Now we consider

$$\begin{aligned} s^4[d(F(x,y), G(u,v)) + d(F(y,x), G(v,u))] &= (\tfrac{489}{480})^4[\tfrac{27}{10} + \tfrac{27}{10}] \\ &\leq (\tfrac{1}{11})(\tfrac{27}{5}) + (\tfrac{4}{5})(8) + (\tfrac{60}{14181})(\tfrac{27}{5}) + (\tfrac{60}{14181})(\tfrac{27}{5}) \\ &\leq k_1[d(fx, gu) + d(fy, gv)] \\ &\quad + k_2[d(fx, F(x,y)) + d(fy, F(y,x))] \\ &\quad + k_3[d(gu, G(u,v)) + d(gv, G(v,u))] \\ &\quad + k_4[d(fx, G(u,v)) + d(fy, G(v,u))] \\ &\quad + k_5[d(gu, F(x,y)) + d(gv, F(y,x))] \end{aligned}$$

From all the above cases, F, G, f and g satisfy all the hypotheses of Theorem 2.5 and $(1, 1)$ is a unique common coupled fixed point of F, G, f and g.

Corollary 3.3. Let (X,d) be a b-metric space with coefficient $s \geq 1$. Let $F,G : X \times X \to X, g : X \to X$ be three mappings. Suppose that there exists with $k \in [0,1)$ such that

$s^4[d(F(x,y),G(u,v)) + d(F(y,x),G(v,u))] \leq kM(x,y,u,v)$ for all $x,y,u,v \in X$, where

$$M(x,y,u,v) = \max\{d(gx,gu) + d(gy,gv), d(gx,F(x,y)) + d(gy,F(y,x)), d(gu,G(u,v)) + d(gv,G(v,u)), \frac{d(gx,G(u,v))+d(gy,G(v,u))}{2s}, \frac{d(gu,F(x,y))+d(gv,F(y,x))}{2s^2}\}.$$

Also, suppose the following hypotheses:

(i) $F(X \times X) \subset g(X)$ and $G(X \times X) \subset g(X)$,
(ii) $g(X)$ is a complete subspace of X,
(iii) (F,g) and (G,g) are w-compatible.

Then F,G and g have a unique common coupled fixed point in $X \times X$.

Proof. Follows by choosing $f = g$ in Theorem 2.4. □

Corollary 3.4. Let (X,d) be a b-metric space with coefficient $s \geq 1$. Let $F,G : X \times X \to X, f,g : X \to X$ be four mappings. Suppose that there exists with $k \in [0,1)$ such that

$s^4[d(F(x,y),G(u,v)) + d(F(y,x),G(v,u))] \leq k[d(fx,gu) + d(fy,gv)]$

for all $x,y,u,v \in X$. Also, suppose the following hypotheses:

(i) $F(X \times X) \subset g(X)$ and $G(X \times X) \subset f(X)$,
(ii) either $f(X)$ or $g(X)$ is a complete subspace of X,
(iii) (F,f) and (G,g) are w-compatible.

Then F,G,f and g have a unique common coupled fixed point in $X \times X$.

Corollary 3.5. Let (X,d) be a b-metric space with coefficient $s \geq 1$. Let $F,G : X \times X \to X, g : X \to X$ be three mappings. Suppose that there exists with $k \in [0,\frac{1}{s})$ such that

$s^4[d(F(x,y),G(u,v)) + d(F(y,x),G(v,u))] \leq k[d(gu,F(x,y)) + d(gv,F(y,x))]$

for all $x,y,u,v \in X$. Also, suppose the following hypotheses:

(i) $F(X \times X) \subset g(X)$ and $G(X \times X) \subset g(X)$,
(ii) $g(X)$ is a complete subspace of X,
(iii) (F,g) and (G,g) are w-compatible.

Then F,G and g have a unique common coupled fixed point in $X \times X$.

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