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DENUMERABLY MANY SYMMETRIC POSITIVE SOLUTIONS FOR SYSTEM OF EVEN ORDER SINGULAR BOUNDARY VALUE PROBLEMS ON TIME SCALES

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Abstract. In this paper, we study system of even order two-point singular boundary value problems with integral boundary conditions on time scales and establish the existence of denumerably many symmetric positive solutions. The proofs of our main results are based on the Hölder's inequality and Krasnoselskii's fixed point theorem.

1. Introduction

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of real numbers $\mathbb{R}$ with the subspace topology inherited from the standard topology of $\mathbb{R}$. The theory of time scales was first introduced by Stefan Hilger in 1988 in his PhD dissertation [12]. The main aim of time scales theory is to unify various definitions and results from the theories of discrete and continuous dynamical systems, and to extend such theories to more general classes of dynamical systems. The ime scales theory is widely applied to different situations, like, in the study of neural networks, insect population models, heat transfer and epidemic models [1, 4, 5]. For more details on time scale calculus we refer to the books by Bohner and Peterson [5, 6].

Recently, researchers shown much interest to establish existence of positive solutions of boundary value problems on time scales, for details [7–10, 15, 16, 18–20, 23–25, 28] and reference therein. However, there are few papers are available for the existence of symmetric positive solutions of boundary value problems with integral boundary conditions on time scales. In 2010, Hamal and Yoruk [14] established the existence of a symmetric positive solution of the following boundary value problem,

$$\begin{gathered}\big(q(t)\phi\big(p(t)u^{\Delta\nabla}\big)\big)^{\Delta\nabla}(t) = \lambda f\big(t, u(t)\big),\ t \in (0,1)_{\mathbb{T}},\\ u(0) = u(1) = \int_0^1 g(s)u(s)\nabla s,\\ q(0)\phi(p(0)u^{\Delta\nabla}(0)) = q(1)\phi(p(1)u^{\Delta\nabla}(1)) = \int_0^1 h(s)q(s)\phi(p(s)u^{\Delta\nabla}(s))\nabla s,\end{gathered} \tag{1}$$

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by using a fixed point index theory. In 2016, Oguz and Topal [21] considered the following boundary value problem on time scales,

$$\begin{aligned} u^{\Delta\nabla}(t) + f\big(t, u(t), u^{\Delta}(t)\big) = 0,\ t \in (a,b)_{\mathbb{T}}, \\ \alpha u(a) - \beta \lim_{t\to a^+} u^{\Delta}(t) = \int_a^b h_1(s)u(s)\nabla s, \\ \alpha u(b) + \beta \lim_{t\to b^-} u^{\Delta}(t) = \int_a^b h_2(s)u(s)\nabla s, \end{aligned} \tag{2}$$

and established symmetric positive solutions by using monotone iterative technique. In the same year, Topal and Denk [29] established sufficient conditions for the existence of at least one symmetric positive solution to the boundary value problem on time scales,

$$\begin{aligned} \big(g(t)u^{\Delta}(t)\big)^{\nabla} + \lambda f\big(t, u(t)\big) = 0,\ t \in (a,b)_{\mathbb{T}}, \\ \alpha u(a) - \beta \lim_{t\to a^+} g(t)u^{\Delta}(t) = \int_a^b h_1(s)u(s)\nabla s, \\ \alpha u(b) + \beta \lim_{t\to b^-} g(t)u^{\Delta}(t) = \int_a^b h_2(s)u(s)\nabla s, \end{aligned} \tag{3}$$

by using the Krasnoselskii fixed point theorem in cones. Motivated by aforementioned works, in this article, we establish the existence of denumerably many symmetric positive solutions for the system of even order boundary value problems with mixed derivatives on time scales

$$\left.\begin{aligned} (-1)^m u_1^{(\Delta\nabla)^m}(t) =& \psi_1(t) f_1\big(u_1(t), u_2(t)\big),\ t \in [0,T]_{\mathbb{T}}, \\ (-1)^n u_2^{(\Delta\nabla)^n}(t) =& \psi_2(t) f_2\big(u_1(t), u_2(t)\big),\ t \in [0,T]_{\mathbb{T}}, \end{aligned}\right\} \tag{4}$$

satisfying integral boundary conditions

$$\left.\begin{aligned} u_1^{(\Delta\nabla)^i}(0) &= \int_0^T a_{i+1}(s) u_1^{(\Delta\nabla)^i}(s)\nabla s,\ 0 \le i \le m-1, \\ u_1^{(\Delta\nabla)^i}(T) &= \int_0^T a_{i+1}(s) u_1^{(\Delta\nabla)^i}(s)\nabla s,\ 0 \le i \le m-1, \\ u_2^{(\Delta\nabla)^j}(0) &= \int_0^T b_{j+1}(s) u_2^{(\Delta\nabla)^j}(s)\nabla s,\ 0 \le j \le n-1, \\ u_2^{(\Delta\nabla)^j}(T) &= \int_0^T b_{j+1}(s) u_2^{(\Delta\nabla)^j}(s)\nabla s,\ 0 \le j \le n-1, \end{aligned}\right\} \tag{5}$$

where $n, m \in \mathbb{Z}^+$(positive integers), $\mathbb{T}$ is a symmetric time scale, $T \in \mathbb{T}$, $f_k \in C\big(\mathbb{R}\times\mathbb{R}, \mathbb{R}\big)$, $\psi_k(t) \in L^p_{\nabla}[0,1]_{\mathbb{T}} (k = 1, 2)$ for some $p \ge 1$ and have denumerably many singularities in $(0, \frac{T}{2})_{\mathbb{T}}$ and establish existence of denumerably many symmetric positive solutions by using the Hölder's inequality and Krasnoselskii's cone fixed point theorem in a Banach Space.

The rest of the paper is organized in the following fashion. In Section 2, we make certain assumptions. In Section 3, we construct the kernel for the homogeneous problem corresponding to (4)-(5), estimate bounds for the kernel, and some lemmas which are needed in establishing our main results are provided. In Section 4, we establish a criteria for the existence of denumerably many symmetric positive solutions for the boundary value problem (4)-(5) by applying Hölder's inequality and Krasnoselskii's fixed point theorem in cones. Finally, we provide an example of a family of functions $\psi(t)$ that satisfy required conditions.

2. Assumptions

For convenience, let $J_0 := [0, T]_{\mathbb{T}}$ and we make the following assumptions throughout the paper:

(H1) $f_1, f_2 : [0, \infty) \times [0, \infty) \to [0, \infty)$ are continuous and symmetric.
(H2) $\psi_1, \psi_2 \in L^p_\nabla(J_0)$ for some $1 \leq p \leq +\infty$, are symmetric on J_0 and there exist δ_1, $\delta_2 > 0$ such that $\psi_1(t) \geq \delta_1$, $\psi_2(t) \geq \delta_2$ a.e. on J_0,
(H3) $a_i \in L^1_\nabla(J_0)$ for all $1 \leq i \leq m$, are nonnegative, symmetric on J_0 and $\alpha_i \in (0, 1)$ for all $1 \leq i \leq m$, where $\alpha_i = \int_0^1 a_i(t)\nabla t$ for all $1 \leq i \leq m$.
(H4) $b_j \in L^1_\nabla(J_0)$ for all $1 \leq j \leq n$, are nonnegative, symmetric on J_0 and $\beta_j \in (0, 1)$ for all $1 \leq j \leq n$, where $\beta_j = \int_0^1 b_j(t)\nabla t$ for all $1 \leq j \leq n$.

3. kernel and It's Bounds

In this section, we construct the kernel for the homogeneous problem corresponding to (4)-(5) and estimate bounds for the kernel.

Lemma 3.1. *Suppose that $\alpha_j \in (0, 1)$ for all $1 \leq j \leq m$. Then for any $q(t) \in C(J_0)$, boundary value problem,*

$$-u_1^{\Delta\nabla}(t) = q(t),\ t \in J_0, \tag{6}$$

$$u_1(0) = u_1(T) = \int_0^T a_j(x)u_1(x)\nabla x,\ for\ 1 \leq j \leq m, \tag{7}$$

has a unique solution

$$u_1(t) = \int_0^T \mathcal{K}_j(t, s)q(s)\nabla s,\ for\ 1 \leq j \leq m, \tag{8}$$

where

$$\mathcal{K}_j(t, s) = G(t, s) + \frac{1}{1 - \alpha_j}\int_0^T G(x, s)a_j(x)\nabla x,\ for\ 1 \leq j \leq m, \tag{9}$$

and

$$G(t, s) = \begin{cases} \dfrac{t}{T}(T - s),\ t \leq s, \\ \dfrac{s}{T}(T - t),\ s \leq t. \end{cases} \tag{10}$$

Proof. Suppose that u_1 is a solution of the problem (6), thereafter integrating twice, we get

$$\begin{aligned} u_1(t) &= -\int_0^t \int_0^s q(r)\nabla r\Delta s + C_0 t + C_1 \\ &= -\int_0^t (t - s)q(s)\nabla s + C_0 t + C_1, \end{aligned}$$

where $C_0 = \lim_{t\to 0^+} u_1^\Delta(t)$ and $C_1 = u_1(0)$. Using the boundary conditions (7) we have

$$C_0 = \frac{1}{T}\int_0^T (T - s)q(s)\nabla s$$

and

$$
\begin{aligned}
C_1 &= \int_0^T a_j(x)u_1(x)\nabla x \\
&= \int_0^T a_j(x)\Big[-\int_0^x (x-s)q(s)\nabla s + C_0x + C_1\Big]\nabla x \\
&= \int_0^T a_j(x)\Big[-\int_0^x (x-s)q(s)\nabla s + \frac{x}{T}\int_0^T (T-s)q(s)\nabla s\Big]\nabla x + C_1\alpha_j \\
&= \int_0^T a_j(x)\Big[\int_0^x \frac{s}{T}(T-x)q(s)\nabla s + \int_x^T \frac{x}{T}(T-s)q(s)\nabla s\Big]\nabla x + C_1\alpha_j \\
&= \int_0^T a_j(x)\Big[\int_0^1 G(x,s)q(s)\nabla s\Big]\nabla x + C_1\alpha_j \\
&= \int_0^T \Big[\int_0^T G(x,s)a_j(x)\nabla x\Big]q(s)\nabla s + C_1\alpha_j \\
&= \frac{1}{1-\alpha_j}\int_0^T \Big[\int_0^T G(x,s)a_j(x)\nabla x\Big]q(s)\nabla s.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
u(t) &= -\int_0^t (t-s)q(s)\nabla s + \int_0^T \frac{t}{T}(T-s)q(s)\nabla s \\
&\qquad + \frac{1}{1-\alpha_j}\int_0^T \Big[\int_0^T G(x,s)a_j(x)\nabla x\Big]q(s)\nabla s \\
&= \int_0^t \frac{s}{T}(T-t)q(s)\nabla s + \int_t^T \frac{t}{T}(T-s)q(s)\nabla s \\
&\qquad + \frac{1}{1-\alpha_j}\int_0^T \Big[\int_0^T G(x,s)a_j(x)\nabla x\Big]q(s)\nabla s \\
&= \int_0^T G(t,s)q(s)\nabla s + \frac{1}{1-\alpha_j}\int_0^T \Big[\int_0^T G(x,s)a_j(x)\nabla x\Big]q(s)\nabla s \\
&= \int_0^T \Big[G(t,s) + \frac{1}{1-\alpha_j}\int_0^T G(x,s)a_j(x)\nabla x\Big]q(s)\nabla s \\
&= \int_0^T \mathcal{K}_j(t,s)q(s)\nabla s,
\end{aligned}
$$

where $\mathcal{K}_j(t,s)$ is given in (9). This completes the proof. □

Lemma 3.2. *Assume that* $(H3)$ *holds and let* $\tau \in \big(0, \frac{T}{2}\big)_{\mathbb{T}}$ *and* $\alpha_j^*(\tau) = \int_\tau^{T-\tau} a_j(t)\nabla t$. *Then* $G(t,s)$ *and* $\mathcal{K}_j(t,s)$ *for* $1 \le j \le m$, *have the following properties:*

(i) $G(t,s) > 0$ *and* $\mathcal{K}_j(t,s) > 0$ *for all* $t, s \in J_0$,

(ii) $G(t,s) \le G(s,s)$, $\mathcal{K}_j(t,s) \le \mathcal{K}_j(s,s) \le \frac{1}{1-\alpha_j}G(s,s)$ *for all* $t, s \in J_0$,

(iii) $G(T-t, T-s) = G(t,s)$, $\mathcal{K}_j(T-t, T-s) = \mathcal{K}_j(t,s)$ *for all* $t, s \in J_0$,

(iv) $G(t,s) \ge \frac{\tau}{T}G(s,s)$ *for all* $t \in [\tau, T-\tau]_{\mathbb{T}}$ *and* $s \in J_0$

(v) $\mathcal{K}_j(t,s) \geq \rho_j(\tau)G(s,s)$ *where* $\rho_j(\tau) = \frac{\tau}{T}\left(1 + \frac{\alpha_j^*(\tau)}{1-\alpha_j}\right)$, $(1 \leq j \leq m)$ *for all* $t \in [\tau, T-\tau]_{\mathbb{T}}$ *and* $s \in J_0$.

Proof. Inequalities (i), (ii) and (iii) are obvious. To prove the inequality (iv) and (v), let $t \in [\tau, T-\tau]_{\mathbb{T}}$. Then, for $0 < t < s < T$,

$$\frac{G(t,s)}{G(s,s)} = \frac{t}{s} \geq \frac{\tau}{T},$$

and for $0 < s < t < T$,

$$\frac{G(t,s)}{G(s,s)} = \frac{T-t}{T-s} \geq \frac{\tau}{T}.$$

Next, for $1 \leq j \leq m$,

$$\begin{aligned}
\mathcal{K}_j(t,s) &= G(t,s) + \frac{1}{1-\alpha_j}\int_0^T G(x,s)a_j(x)\nabla x \\
&\geq \frac{\tau}{T}G(s,s) + \frac{1}{1-\alpha_j}\int_\tau^{T-\tau} G(x,s)a_j(x)\nabla x \\
&\geq \frac{\tau}{T}G(s,s) + \frac{1}{1-\alpha_j}\int_\tau^{T-\tau} \frac{\tau}{T}G(s,s)a_j(x)\nabla x \\
&\geq \frac{\tau}{T}G(s,s) + \frac{\alpha_j^*(\tau)}{1-\alpha_j}\frac{\tau}{T}G(s,s) \\
&= \rho_j(\tau)G(s,s).
\end{aligned}$$

This completes the proof. □

Lemma 3.3. *Assume that the condition (H3) is satisfied and* $\mathcal{K}_j(t,s)$ *for* $1 \leq j \leq m$, *is given in* (9). *Let* $\mathcal{G}_1(t,s) = \mathcal{K}_1(t,s)$ *and recursively define*

$$\mathcal{G}_j(t,s) = \int_0^T \mathcal{G}_{j-1}(t,r)\mathcal{K}_j(r,s)\nabla r, \quad \textit{for} \quad 2 \leq j \leq m. \tag{11}$$

Then $\mathcal{G}_n(t,s)$ *is the kernel for the homogeneous boundary value problem*

$$(-1)^m u^{(\Delta\nabla)^m}(t) = 0,\ t \in J_0,$$

$$u^{(\Delta\nabla)^i}(0) = u^{(\Delta\nabla)^i}(T) = \int_0^T a_{i+1}(s)u^{(\Delta\nabla)^i}(s)\nabla s,\ 0 \leq i \leq m-1.$$

Lemma 3.4. *Assume that the condition (H3) is satisfied. Define*

$$k(\tau) = \int_\tau^{T-\tau} G(s,s)\nabla s,\ k_j = \int_0^T \mathcal{K}_j(s,s)\nabla s,$$

$$\mathfrak{L} = \prod_{j=1}^{m-1} k_j,\ \mathfrak{M}(\tau) = \prod_{j=1}^{m-1} \rho_j(\tau)[k(\tau)]^j,$$

for $1 \leq j \leq m$. *Then the kernel* $\mathcal{G}_m(t,s)$ *satisfies the following inequalities:*

(i) $0 \leq \mathcal{G}_m(t,s) \leq \mathfrak{L}\mathcal{K}_m(s,s)$, *for all* $t, s \in J_0$ *and*
(ii) $\mathcal{G}_m(t,s) \geq \rho_m(\tau)\mathfrak{M}(\tau)G(s,s)$, *for* $t \in [\tau, T-\tau]_{\mathbb{T}}$ *and* $s \in J_0$.

Proof. We can easily establish the result by induction on m. □

Lemma 3.5. *The kernel $\mathcal{G}_j(t,s)$ for $1 \leq j \leq m$, satisfies the following condition*

$$\mathcal{G}_j(t,s) = \mathcal{G}_j(T-t, T-s)\,\forall\, t,s \in J_0. \tag{12}$$

Proof. The proof is by induction. For $j = 1$, the equation (12) is obvious and assume that the equation (11) is true for fixed $j \geq 2$. Then from (11) and using transformation $x = T - r$, we have

$$\begin{aligned}\mathcal{G}_{j+1}(t,s) &= \int_0^T \mathcal{G}_j(t,r)\mathcal{K}_{j+1}(r,s)\nabla r\\ &= \int_0^T \mathcal{G}_j(T-t,T-r)\mathcal{K}_{j+1}(T-r,T-s)\nabla r\\ &= \int_0^T \mathcal{G}_j(T-t,x)\mathcal{K}_{j+1}(x,T-s)\nabla x\\ &= \mathcal{G}_{j+1}(T-t,T-s).\end{aligned}$$

This completes the proof. □

We can also formulate results similar to the Lemmas 3.2–3.7 for the problem,

$$-\,u_2^{\Delta\nabla}(t) = p(t),\ t \in J_0, \tag{13}$$

$$\left.\begin{aligned} u_2(0) &= \int_0^T b_j(s)u_2(s)\nabla s,\ 1 \leq j \leq n,\\ u_2(T) &= \int_0^T b_j(s)u_2(s)\nabla s,\ 1 \leq j \leq n \end{aligned}\right\} \tag{14}$$

as follows.

Lemma 3.6. *Let $(H3), (H4)$ hold. Then for any $p(t) \in C(J_0)$, the boundary value problem* (13)-(14) *has a unique solution*

$$u_2(t) = \int_0^T \mathcal{H}_j(t,s)p(s)\nabla s,\ for\ 1 \leq j \leq n, \tag{15}$$

where

$$\mathcal{H}_j(t,s) = G(t,s) + \frac{1}{1-\beta_j}\int_0^T G(r,s)b_j(r)\nabla r, \tag{16}$$

for $1 \leq j \leq n$,

Lemma 3.7. *Assume that $(H3)$ holds and let $\tau \in \left(0, \frac{T}{2}\right)_{\mathbb{T}}$ and $\beta_j^*(\tau) = \int_\tau^{T-\tau} b_j(t)\nabla t$. Then $\mathcal{H}_j(t,s)$ for $1 \leq j \leq n$, have the following properties:*

(i) $\mathcal{H}_j(t,s) > 0$ *for all* $t,s \in J_0$,

(ii) $\mathcal{H}_j(t,s) \leq \mathcal{H}_j(s,s) \leq \frac{1}{1-\beta_j}G(s,s)$ *for all* $t,s \in J_0$,

(iii) $\mathcal{H}_j(T-t,T-s) = \mathcal{H}_j(t,s)$ *for all* $t,s \in J_0$,

(iv) $\mathcal{H}_j(t,s) \geq \varrho_j(\tau)G(s,s)$ *where* $\varrho_j(\tau) = \dfrac{\tau}{T}\left(1 + \dfrac{\beta_j^*(\tau)}{1-\beta_j}\right)$, $(1 \leq j \leq n)$ *for all* $t \in [\tau, T-\tau]_{\mathbb{T}}$ *and* $s \in J_0$.

Lemma 3.8. *Assume that the condition (H3) is satisfied and $\mathcal{H}_j(t,s)$ for $1 \le j \le n$, is given in* (9). *Let $\mathcal{J}_1(t,s) = \mathcal{H}_1(t,s)$ and recursively define*

$$\mathcal{J}_j(t,s) = \int_0^T \mathcal{J}_{j-1}(t,r)\mathcal{H}_j(r,s)\nabla r, \quad for \quad 2 \le j \le n. \tag{17}$$

Then $\mathcal{J}_n(t,s)$ is the kernel for the homogeneous boundary value problem

$$(-1)^n u^{(\Delta\nabla)^n}(t) = 0, \ t \in J_0,$$

$$u^{(\Delta\nabla)^i}(0) = u^{(\Delta\nabla)^i}(T) = \int_0^1 a_{i+1}(s)u^{(\Delta\nabla)^i}(s)\nabla s, \ 0 \le i \le n-1.$$

Lemma 3.9. *Assume that the condition (H3) is satisfied. Define*

$$\kappa(\tau) = \int_\tau^{T-\tau} G(s,s)\nabla s, \ \kappa_j = \int_0^T \mathcal{H}_j(s,s)\nabla s,$$

$$\mathfrak{J} = \prod_{j=1}^{n-1} \kappa_j, \ \mathfrak{N}(\tau) = \prod_{j=1}^{n-1} \varrho_j(\tau)[\kappa(\tau)]^j,$$

for $1 \le j \le n$. Then the kernel $\mathcal{J}_n(t,s)$ satisfies the following inequalities:

(i) $0 \le \mathcal{J}_n(t,s) \le \mathfrak{J}\mathcal{H}_n(s,s)$, *for all $t, s \in J_0$ and*
(ii) $\mathcal{J}_n(t,s) \ge \varrho_n(\tau)\mathfrak{N}(\tau)G(s,s)$, *for $t \in [\tau, T-\tau]_{\mathbb{T}}$ and $s \in J_0$.*

Lemma 3.10. *The kernel $\mathcal{J}_j(t,s)$ for $1 \le j \le n$, satisfies the following condition*

$$\mathcal{J}_j(t,s) = \mathcal{J}_j(T-t, T-s) \ \forall \, t, s \in J_0. \tag{18}$$

4. Existence of denumerably Many Positive Solutions

In this section, we establish the existence of denumerably many symmetric positive solutions to the system (4)-(5) by applying Hölder's inequality and Krasnoselskii's fixed point theorem in cones.

Theorem 4.1. *(Krasnoselskii fixed point theorem, [11]). Let $\mathcal{B}$ be a Banach space and let $P \subset \mathcal{B}$ be a cone in $\mathcal{B}$. Assume that Ω_1, Ω_2 are open with $0 \in \Omega_1, \bar{\Omega}_1 \subset \Omega_2$, and let $T : P \cap (\bar{\Omega}_2 \backslash \Omega_1) \to P$ be a completely continuous operator such that either*

(i) $\|Tu\| \le \|u\|, u \in P \cap \partial\Omega_1$, *and* $\|Tu\| \ge \|u\|, u \in P \cap \partial\Omega_2$, *or*
(ii) $\|Tu\| \ge \|u\|, u \in P \cap \partial\Omega_1$, *and* $\|Tu\| \le \|u\|, u \in P \cap \partial\Omega_2$.

Then T has a fixed point in $P \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

Theorem 4.2. *(Hölder's inequality, [2, 22]) Let $f \in L^p_\nabla(I)$ with $p > 1, g \in L^q_\nabla(I)$ with $q > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$. Then $fg \in L^1_\nabla(I)$ and $\|fg\|_{L^1_\nabla} \le \|f\|_{L^p_\nabla}\|g\|_{L^q_\nabla}$, where*

$$\|f\|_{L^p_\nabla} := \begin{cases} \Big[\displaystyle\int_I |f|^p(s)\nabla s\Big]^{\frac{1}{p}}, & p \in \mathbb{R}, \\ inf\Big\{K \in \mathbb{R} \,/\, |f| \le K \ \nabla - a.e., \, on \, I\Big\}, & p = \infty, \end{cases}$$

and $I = [a,b]_{\mathbb{T}}$. Moreover, if $f \in L^1_\nabla(I)$ and $g \in L^\infty_\nabla(I)$. Then $fg \in L^1_\nabla(I)$ and $\|fg\|_{L^1_\nabla} \le \|f\|_{L^1_\nabla}\|g\|_{L^\infty_\nabla}$.

Let $\mathscr{C}$ denote the Banach space $C_{ld}(J_0,\mathbb{R})$ with norm $\|u\| = \max_{t\in J_0}|u(t)|$. Then $\mathfrak{B} = \mathscr{C} \times \mathscr{C}$ is a Banach space with norm $\|(u_1, u_2)\|_{\mathfrak{B}} = \|u_1\| + \|u_2\|$.

For $\tau \in (0, \frac{T}{2})_{\mathbb{T}}$, define the cone $\mathcal{P}_\tau \subset \mathfrak{B}$ by

$$\mathcal{P}_\tau = \Big\{(u_1, u_2) \in \mathfrak{B} : u_1(t) \geq 0, u_2(t) \geq 0 \text{ are symmetric and } \min_{t\in[\tau,\,T-\tau]_{\mathbb{T}}} \big(u_1(t) + u_2(t)\big) \geq \frac{M_\tau}{\eta}\|\big(u_1(t), u_2(t)\big)\|_{\mathfrak{B}}\Big\},$$

where $M_\tau = \min\{\rho_m(\tau)\mathfrak{M}(\tau),\ \varrho_n(\tau)\mathfrak{N}(\tau)\}$ and $\eta = \max\{\dfrac{\mathfrak{L}}{1-\alpha_m}, \dfrac{\mathfrak{J}}{1-\beta_n}\}$.
For any $(u_1, u_2) \in \mathcal{P}_\tau$, define an operator $\mathcal{F} : \mathcal{P}_\tau \to \mathfrak{B}$ by

$$\mathcal{F}(u_1, u_2)(t) = \big(\mathcal{F}_m(u_1, u_2), \mathcal{F}_n(u_1, u_2)\big),$$

where

$$\mathcal{F}_m(u_1, u_2) = \int_0^T \mathcal{G}_m(t, s)\omega_1(s)f_1(u_1, u_2)\nabla s$$

and

$$\mathcal{F}_n(u_1, u_2) = \int_0^T \mathcal{J}_n(t, s)\omega_2(s)f_2(u_1, u_2)\nabla s.$$

Lemma 4.3. *Assume that (H1)-(H4) hold. Then $\mathcal{F}(\mathcal{P}_\tau) \subset \mathcal{P}_\tau$ and $\mathcal{F} : \mathcal{P}_\tau \to \mathcal{P}_\tau$ is completely continuous for each $\tau \in (0, \frac{T}{2})_{\mathbb{T}}$.*

Proof. Fix $\tau \in (0, \frac{T}{2})_{\mathbb{T}}$. First note that $(u_1, u_2) \in \mathcal{P}_\tau$ implies that $\mathcal{F}_m(u_1, u_2)(t) \geq 0$ and $\mathcal{F}_n(u_1, u_2)(t) \geq 0$ for all $t \in J_0$. On the other hand, by Lemma 3.5 and Lemma 3.9 we obtain

$$\begin{aligned}
&\mathcal{F}_m(u_1, u_2)(t) + \mathcal{F}_n(u_1, u_2)(t)\\
&\quad= \int_0^T \mathcal{G}_m(t, s)\psi_1(s)f_1(u_1, u_2)\nabla s + \int_0^T \mathcal{J}_n(t, s)\psi_2(s)f_2(u_1, u_2)\nabla s\\
&\quad\leq \mathfrak{L}\int_0^T \mathcal{K}_m(s, s)\psi_1(s)f_1(u_1, u_2)\nabla s + \mathfrak{J}\int_0^T \mathcal{H}_n(s, s)\psi_2(s)f_2(u_1, u_2)\nabla s\\
&\quad\leq \frac{\mathfrak{L}}{1-\alpha_m}\int_0^T G(s, s)\psi_1(s)f_1(u_1, u_2)\nabla s + \frac{\mathfrak{J}}{1-\beta_n}\int_0^T G(s, s)\psi_2(s)f_2(u_1, u_2)\nabla s\\
&\quad\leq \eta\Big(\int_0^T G(s, s)\psi_1(s)f_1(u_1, u_2)\nabla s + \int_0^T G(s, s)\psi_2(s)f_2(u_1, u_2)\nabla s\Big)
\end{aligned}$$

and

$$\begin{aligned}
&\min_{t\in[\tau,\,T-\tau]_{\mathbb{T}}} \big(\mathcal{F}_m(u_1,u_2)(t)+\mathcal{F}_n(u_1,u_2)(t)\big)\\
&=\min_{t\in[\tau,\,T-\tau]_{\mathbb{T}}} \bigg(\int_0^T \mathcal{G}_m(t,s)\psi_1(s)f_1(u_1,u_2)\nabla s+\int_0^T \mathcal{J}_n(t,s)\psi_2(s)f_2(u_1,u_2)\nabla s\bigg)\\
&=\rho_m(\tau)\mathfrak{M}(\tau)\int_0^T G(s,s)\psi_1(s)f_1(u_1,u_2)\nabla s+\varrho_n(\tau)\mathfrak{N}(\tau)\int_0^T G(s,s)\psi_2(s)f_2(u_1,u_2)\nabla s\\
&=M_\tau\bigg(\int_0^T G(s,s)\psi_1(s)f_1(u_1,u_2)\nabla s+\int_0^T G(s,s)\psi_2(s)f_2(u_1,u_2)\nabla s\bigg)\\
&\geq \frac{M_\tau}{\eta}\|\big(\mathcal{F}_m(u_1,u_2),\mathcal{F}_n(u_1,u_2)\big)\|_{\mathfrak{B}}\\
&\geq \frac{M_\tau}{\eta}\|\mathcal{F}\big(u_1,u_2\big)\|_{\mathfrak{B}}.
\end{aligned}$$

So, $\mathcal{F}(u_1,u_2)\in\mathcal{P}_\tau$ and then $\mathcal{F}(\mathcal{P}_\tau)\subset\mathcal{P}_\tau$. Next, by standard methods and the Arzela-Ascoli theorem, one can easily prove that the operator $\mathcal{F}$ is completely continuous. The proof is complete. □

We consider three possible cases for $\psi_1,\,\psi_2\in L^p_\nabla(J_0)$: $p>1, p=1, p=\infty$. When $p>1$ we have the following theorem.

Theorem 4.4. *Assume that* $(H1)-(H4)$ *hold, let* $\{\tau_k\}_{k=1}^\infty$ *be such that* $t_{k+1}<\tau_k<t_k$, $k=1,2,3,\cdots$. *Let* $\{S_k\}_{k=1}^\infty$ *and* $\{r_k\}_{k=1}^\infty$ *be such that*

$$S_{k+1}<\frac{M_{\tau_k}}{\eta}r_k<Rr_k<S_k,\;\; k\in\mathbb{N},$$

where

$$R=\max\bigg\{\frac{1}{\delta_1\rho_m(\tau_1)\mathfrak{M}(\tau_1)\int_{\tau_1}^{1-\tau_1}G(s,s)\nabla s},\;\frac{1}{\delta_2\varrho_n(\tau_1)\mathfrak{N}(\tau_1)\int_{\tau_1}^{1-\tau_1}G(s,s)\nabla s},\;1\bigg\}.$$

Assume that f *satisfies*

- (A1) $f_1(u_1,u_2)\leq\frac{M_1S_k}{2}$ *and* $f_2(u_1,u_2)\leq\frac{M_1^{'}S_k}{2}$ *for all* $t\in J_0, 0\leq u_1+u_2\leq S_k$, *where*
$$M_1<\frac{1-\alpha_m}{\mathfrak{L}\|G\|_{L^q_\nabla}\|\psi_1\|_{L^p_\nabla}}\;\text{ and } M_1^{'}<\frac{1-\beta_n}{\mathfrak{I}\|G\|_{L^q_\nabla}\|\psi_2\|_{L^p_\nabla}}$$
- (A2) $f_1(u_1,u_2)\geq Rr_k$ *or* $f_2(u_1,u_2)\geq Rr_k$ *for all* $t\in[\tau_k,T-\tau_k]_{\mathbb{T}}$, $\frac{M_{\tau_k}}{\eta}r_k\leq u_1+u_2\leq r_k$.

Then the system (4)-(5) *has denumerably many symmetric positive solutions* $\{(u_1^{[k]},u_2^{[k]})\}_{k=1}^\infty$. *Furthermore,* $r_k\leq\|(u_1^{[k]},u_2^{[k]})\|\leq S_k$ *for each* $k\in\mathbb{N}$.

Proof. Consider the sequences $\{\Omega_{1,k}\}_{k=1}^\infty$ and $\{\Omega_{2,k}\}_{k=1}^\infty$ of open subsets of $\mathfrak{B}$ defined by

$$\Omega_{1,k}=\{(u_1,u_2)\in\mathfrak{B}:\|(u_1,u_2)\|_{\mathfrak{B}}<S_k\},$$

$$\Omega_{2,k}=\{(u_1,u_2)\in\mathfrak{B}:\|(u_1,u_2)\|_{\mathfrak{B}}<r_k\}.$$

Let $\{\tau_k\}_{k=1}^{\infty}$ be as in the hypothesis and note that $t^* < t_{k+1} < \tau_k < t_k < \frac{T}{2}$, for all $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, define the cone $\mathcal{P}_{\tau_k}$ by

$$\mathcal{P}_{\tau_k} = \Big\{(u_1, u_2) \in \mathfrak{B} : u_1(t) \geq 0,\ u_2(t) \geq 0 \text{ are symmetric and} \\ \min_{t\in[\tau_k,\, 1-\tau_k]_{\mathbb{T}}} \big(u_1(t) + u_2(t)\big) \geq \frac{M_{\tau_k}}{\eta} \|\big(u_1(t), u_2(t)\big)\|_{\mathfrak{B}}\Big\}.$$

Let $(u_1, u_2) \in \mathcal{P}_{\tau_k} \cap \partial\Omega_{1,k}$. Then,

$$u_1(s) + u_2(s) \leq S_k = \|(u_1, u_2)\|_{\mathfrak{B}}$$

for all $s \in J_0$. By $(A1)$,

$$\begin{aligned}
\|\mathcal{F}_m(u_1, u_2)\| &= \max_{t\in J_0} \int_0^T \mathcal{G}_m(t, s)\psi_1(s) f_1\big(u_1(s), u_2(s)\big)\nabla s \\
&\leq \mathfrak{L} \int_0^T \mathcal{K}_m(s, s)\psi_1(s) f_1(u_1, u_2)\nabla s \\
&\leq \frac{\mathfrak{L}}{1 - \alpha_m} \int_0^T G(s, s)\psi_1(s) f_1(u_1, u_2)\nabla s \\
&\leq \frac{\mathfrak{L}}{1 - \alpha_m} \|G\|_{L^q_\nabla} \|\psi_1\|_{L^p_\nabla} \frac{M_1 S_k}{2} \\
&\leq \frac{S_k}{2} = \frac{\|(u_1, u_2)\|_{\mathfrak{B}}}{2}.
\end{aligned}$$

Thus we have $\|\mathcal{F}_m(u_1, u_2)\| \leq \dfrac{\|(u_1, u_2)\|_{\mathfrak{B}}}{2}$. Similarly we can see that

$$\|\mathcal{F}_n(u_1, u_2)\| \leq \frac{\|(u_1, u_2)\|_{\mathfrak{B}}}{2}.$$

Therefore, for $(u_1, u_2) \in \mathcal{P}_{\tau_k} \cap \partial\Omega_{1,k}$, and $t \in J_0$ we get

$$\begin{aligned}
\|\mathcal{F}(u_1, u_2)\|_{\mathfrak{B}} &= \|\big(\mathcal{F}_m(u_1, u_2), \mathcal{F}_n(u_1, u_2)\big)\|_{\mathfrak{B}} \\
&= \|\mathcal{F}_m(u_1, u_2)\| + \|\mathcal{F}_n(u_1, u_2)\| \\
&\leq \|(u_1, u_2)\|_{\mathfrak{B}}.
\end{aligned} \tag{19}$$

Let $t \in [\tau_k, 1 - \tau_k]_{\mathbb{T}}$. Then, for $(u_1, u_2) \in \mathcal{P}_{\tau_k} \cap \partial\Omega_{2,k}$,

$$\begin{aligned}
r_k = \|(u_1, u_2)\| &\geq u_1(t) + u_2(t) \\
&\geq \min_{t\in[\tau_k,\, 1-\tau_k]_{\mathbb{T}}} \big(u_1(t) + u_2(t)\big) \\
&\geq \frac{M_{\tau_k}}{\eta} \|(u_1, u_2)\| \\
&\geq \frac{M_{\tau_k}}{\eta} r_k.
\end{aligned}$$

By $(A2)$,

$$\begin{aligned}
\|\mathcal{F}(u_1,u_2)\| &= \|\mathcal{F}_m(u_1,u_2)\| + \|\mathcal{F}_n(u_1,u_2)\| \geq \|\mathcal{F}_m(u_1,u_2)\| \\
&= \max_{t\in J_0} \int_0^T \mathcal{G}_m(t,s)\psi_1(s) f_1\big(u_1(s),u_2(s)\big)\nabla s \\
&\geq \max_{t\in J_0} \int_{\tau_k}^{T-\tau_k} \mathcal{G}_m(t,s)\psi_1(s) f_1\big(u_1(s),u_2(s)\big)\nabla s \\
&\geq \max_{t\in J_0} \int_{\tau_k}^{T-\tau_k} \mathcal{G}_m(t,s)\psi_1(s)\nabla s R r_k \\
&\geq R r_k \delta_1 \max_{t\in[\tau_1,\,1-\tau_1]_{\mathbb{T}}} \int_{\tau_1}^{T-\tau_1} \mathcal{G}_m(t,s)\nabla s \\
&\geq R r_k \delta_1 \rho_m(\tau_1)\mathfrak{M}(\tau_1) \int_{\tau_1}^{1-\tau_1} G(s,s)\nabla s \\
&\geq r_k = \|(u_1,u_2)\|_{\mathfrak{B}}.
\end{aligned}$$

Thus, if $(u_1,u_2)\in \mathcal{P}_{\tau_k}\cap \partial\Omega_{2,k}$, then

$$\|\mathcal{F}(u_1,u_2)\| \geq \|(u_1,u_2)\|_{\mathfrak{B}}. \tag{20}$$

It is obvious that $0\in \Omega_{2,k}\subset \bar{\Omega}_{2,k}\subset \Omega_{1,k}$. By (19),(20), it follows from Theorem 4.1 that the operator $\mathcal{F}$ has a fixed point $(u_1^{[k]},u_2^{[k]})\in \mathcal{P}_{\tau_k}\cap \big(\bar{\Omega}_{1,k}\backslash\Omega_{2,k}\big)$ such that $r_k\leq \|(u_1^{[k]},u_2^{[k]})\|\leq S_k$. Since $k\in\mathbb{N}$ was arbitrary, the proof is complete. □

Now we deal with the case $p=1$.

Theorem 4.5. *Assume that $(H1)-(H4)$ hold, let $\{\tau_k\}_{k=1}^{\infty}$ be such that $t_{k+1}<\tau_k<t_k$, $k=1,2,3,\cdots$. Let $\{S_k\}_{k=1}^{\infty}$ and $\{r_k\}_{k=1}^{\infty}$ be such that*

$$S_{k+1} < \frac{M_{\tau_k}}{\eta} r_k < R r_k < S_k,\ \ k\in\mathbb{N},$$

where R is defined in Theorem 4.4. Also assume that f satisfies
$(B1)$ $f_1(u_1,u_2)\leq \frac{M_2 S_k}{2}$ and $f_2(u_1,u_2)\leq \frac{M_2' S_k}{2}$ for all $t\in J_0$, $0\leq u_1+u_2\leq S_k$, where

$$M_2 < \min\left\{\frac{1-\alpha_m}{\mathfrak{L}\|G\|_{L^\infty_\nabla}\|\psi_1\|_{L^1_\nabla}},\ R\right\},$$

$$M_2' < \min\left\{\frac{1-\beta_n}{\mathfrak{J}\|G\|_{L^\infty_\nabla}\|\psi_2\|_{L^1_\nabla}},\ R\right\}$$

and $(A2)$. Then the boundary value problem (4)–(5) has denumerably many symmetric positive solutions $\{(u_1^{[k]},u_2^{[k]})\}_{k=1}^{\infty}$. Furthermore, for each $k\in\mathbb{N}$, $r_k\leq \|(u_1^{[k]},u_2^{[k]})\|\leq S_k$.

Proof. For a fixed k, let $\Omega_{1,k}$ be as in the proof of Theorem 4.4 and let (u_1,u_2) be an element of $P_{\tau_k}\cap\partial\Omega_{1,k}$. Then

$$u_1(s)+u_2(s)\leq S_k = \|(u_1,u_2)\|_{\mathfrak{B}},$$

for all $s \in J$. By $(B1)$ and Theorem 4.4,

$$\begin{aligned}\|\mathcal{F}(u_1,u_2)\| &= \|\mathcal{F}_m(u_1,u_2)\| + \|\mathcal{F}_n(u_1,u_2)\| \\ &\le \max_{t\in J_0}\int_0^T \mathcal{G}_m(t,s)\psi_1(s)f_1\big(u_1(s),u_2(s)\big)\nabla s \\ &\qquad + \max_{t\in J_0}\int_0^T \mathcal{J}_n(t,s)\psi_2(s)f_2\big(u_1(s),u_2(s)\big)\nabla s \\ &\le \frac{\mathfrak{L}}{1-\alpha_m}\|G\|_{L^\infty_\nabla}\|\psi_1\|_{L^1_\nabla}\frac{M_2S_k}{2} + \frac{\mathfrak{J}}{1-\beta_n}\|G\|_{L^\infty_\nabla}\|\psi_2\|_{L^1_\nabla}\frac{M_2'S_k}{2} \\ &\le S_k.\end{aligned}$$

Thus,

$$\|F(u_1,u_2)\| \le \|(u_1,u_2)\|_{\mathfrak{B}},$$

for $(u_1,u_2) \in P_{\tau_k} \cap \partial\Omega_{1,k}$. Now define $\Omega_{2,k} = \{(u_1,u_2) \in \mathfrak{B} : \|(u_1,u_2)\|_{\mathfrak{B}} < r_k\}$. Let $(u_1,u_2) \in P_{\tau_k} \cap \partial\Omega_{2,k}$ and let $s \in [\tau_k,\, 1-\tau_k]_{\mathbb{T}}$. Then, the argument leading to (20) carries over to the present case and completes the proof. □

Finally we consider the case of $p = \infty$.

Theorem 4.6. *Assume that $(H1)-(H4)$ hold. Let $\{S_k\}_{k=1}^{\infty}$ and $\{r_k\}_{k=1}^{\infty}$ be such that*

$$S_{k+1} < \frac{M_\tau}{\eta} r_k < Cr_k < S_k,\ k \in \mathbb{N},$$

where C is defined in Theorem 4.4. Also assume that f satisfies
$(E1)$ $f_1(u_1,u_2) \le M_3S_k$ and $f_1(u_1,u_2) \le M_3'S_k$ for all $t \in J$, $0 \le u_1+u_2 \le S_k$, where

$$M_3 < \min\left\{\frac{1-\alpha_m}{\mathfrak{L}\|G\|_{L^1_\nabla}\|\psi_1\|_{L^\infty_\nabla}},\ R\right\},\ M_3' < \min\left\{\frac{1-\beta_n}{\mathfrak{J}\|G\|_{L^1_\nabla}\|\psi_2\|_{L^\infty_\nabla}},\ R\right\}$$

and $(A2)$. Then the boundary value problem (4)–(5) has denumerably many symmetric positive solutions $\{(u_1^{[k]},u_2^{[k]})\}_{k=1}^{\infty}$. Furthermore, for each $k \in \mathbb{N}$,
$r_k \le \|(u_1^{[k]},u_2^{[k]})\| \le S_k$.

Proof. By $(E1)$,

$$\begin{aligned}\|\mathcal{F}(u_1,u_2)\| &= \|\mathcal{F}_m(u_1,u_2)\| + \|\mathcal{F}_n(u_1,u_2)\| \\ &\le \max_{t\in J_0}\int_0^T \mathcal{G}_m(t,s)\psi_1(s)f_1\big(u_1(s),u_2(s)\big)\nabla s \\ &\qquad + \max_{t\in J_0}\int_0^T \mathcal{J}_n(t,s)\psi_2(s)f_2\big(u_1(s),u_2(s)\big)\nabla s \\ &\le \frac{\mathfrak{L}}{1-\alpha_m}\|G\|_{L^1_\nabla}\|\psi_1\|_{L^\infty_\nabla}\frac{M_3S_k}{2} + \frac{\mathfrak{J}}{1-\beta_n}\|G\|_{L^1_\nabla}\|\psi_2\|_{L^\infty_\nabla}\frac{M_3'S_k}{2} \\ &\le S_k.\end{aligned}$$

This shows that if $(u_1,u_2) \in P_{\tau_k} \cap \partial\Omega_{1,k}$, where

$$\Omega_{1,k} = \{(u_1,u_2) \in \mathfrak{B} : \|(u_1,u_2)\| < S_k\}.$$

Then,

$$\|\mathcal{F}(u_1,u_2)\| \le \|(u_1,u_2)\|.$$

Define $\Omega_{2,k} = \{(u_1, u_2) \in \mathfrak{B} : \|(u_1, u_2)\| < r_k\}$ and let $(u_1, u_2) \in \mathcal{P}_{\tau_k} \cap \partial\Omega_{2,k}$. Then, the argument employed in the proof of Theorem 4.4 applies directly to yield $\|\mathcal{F}(u_1, u_2)\| \geq \|(u_1, u_2)\|$. By the Theorem 4.1, completes the proof. □

5. Example

In this section, we provide an example of a family of functions $\psi(t)$ that satisfy conditions $(H2)$ corresponding to the cases $p = 1$ and $p = 2$.

Let $\mathbb{T} = [0, \frac{1}{10}] \cup \{\frac{1}{9}, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}\} \cup [\frac{1}{5}, \frac{4}{5}] \cup \{\frac{8}{9}, \frac{7}{8}, \frac{6}{7}, \frac{5}{6}\} \cup [\frac{9}{10}, 1]$ be bounded symmetric time scale and consider the family of functions $\psi(t, \mathfrak{y}) : [0, 1]_{\mathbb{T}} \to (0, +\infty]$ given by

$$\psi(t, \mathfrak{y}) = \begin{cases} \dfrac{1}{|t - \frac{1}{2}|^{\mathfrak{y}}} & \text{for } t \in \left[0, \dfrac{1}{5}\right] \cup \left[\dfrac{4}{5}, 1\right], \\ \displaystyle\sum_{i=1}^{\infty} \frac{\chi[\mathfrak{z}_i, \mathfrak{z}_{i-1}]}{\left||t - \frac{1}{2}| + t_i - \frac{1}{2}\right|^{\mathfrak{y}}} & \text{for } t \in \left[\dfrac{1}{5}, \dfrac{4}{5}\right], \end{cases}$$

where

$$t_0 = \frac{3}{8}, \; t_i = t_0 - \sum_{k=0}^{i-1} \frac{1}{(k+2)^4}, \; i = 1, 2, 3, \cdots,$$

and

$$\mathfrak{z}_0 = 1, \; \mathfrak{z}_i = \frac{t_i + t_{i+1}}{2}, \; i = 1, 2, 3, \cdots.$$

At first, it can be seen that

$$\psi(t, \mathfrak{y}) \geq \psi(1, \mathfrak{y}) = \frac{1}{|1 - \frac{1}{2}|^{\mathfrak{y}}} = 2^{\mathfrak{y}},$$

$$t_1 = \frac{5}{16} < \frac{1}{2} \text{ and } t_i - t_{i+1} = \frac{1}{(i+2)^4}, \; i = 1, 2, 3, \cdots,$$

and note that $\displaystyle\sum_{i=1}^{\infty} \frac{1}{l^4} = \frac{\pi^4}{90}$. So,

$$t^* = \lim_{i \to \infty} t_i = \frac{3}{8} - \sum_{k=0}^{\infty} \frac{1}{(k+2)^4} = \frac{3}{8} - \left(\frac{\pi^4}{90} - 1\right) = \frac{11}{8} - \frac{\pi^4}{90} > \frac{1}{5}.$$

We claim that if $\mathfrak{y} = \frac{1}{2}$, then $\psi(t, \mathfrak{y}) \in L^1_\nabla[0, 1]$. Note that $\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$, we have

$$\begin{aligned}
\int_0^1 \psi(t,\mathfrak{y})\nabla t &= \int_0^{\frac{1}{10}} \psi(t,\mathfrak{y})\nabla t + \int_{\frac{1}{5}}^{\frac{4}{5}} \psi(t,\mathfrak{y})\nabla t + \int_{\frac{9}{10}}^{1} \psi(t,\mathfrak{y})\nabla t \\
&+ \Bigg[\left(\frac{1}{9}-\frac{1}{10}\right)\psi\left(\frac{1}{9},\mathfrak{y}\right) + \left(\frac{1}{8}-\frac{1}{9}\right)\psi\left(\frac{1}{8},\mathfrak{y}\right) + \left(\frac{1}{7}-\frac{1}{8}\right)\psi\left(\frac{1}{7},\mathfrak{y}\right) \\
&+ \left(\frac{1}{6}-\frac{1}{7}\right)\psi\left(\frac{1}{6},\mathfrak{y}\right) + \left(\frac{1}{5}-\frac{1}{6}\right)\psi\left(\frac{1}{5},\mathfrak{y}\right) + \left(\frac{8}{9}-\frac{4}{5}\right)\psi\left(\frac{8}{9},\mathfrak{y}\right) \\
&+ \left(\frac{7}{8}-\frac{8}{9}\right)\psi\left(\frac{7}{8},\mathfrak{y}\right) + \left(\frac{6}{7}-\frac{7}{8}\right)\psi\left(\frac{6}{7},\mathfrak{y}\right) + \left(\frac{5}{6}-\frac{6}{7}\right)\psi\left(\frac{5}{6},\mathfrak{y}\right) \\
&+ \left(\frac{9}{10}-\frac{5}{6}\right)\psi\left(\frac{9}{10},\mathfrak{y}\right)\Bigg] \\
&= \int_0^{\frac{1}{10}} \frac{1}{|t-\frac{1}{2}|^{\mathfrak{y}}}\nabla t + \int_{\frac{1}{5}}^{\frac{4}{5}} \sum_{i=1}^{\infty} \frac{\chi[\mathfrak{z}_i,\, \mathfrak{z}_{i-1}]}{\left||t-\frac{1}{2}|+t_l-\frac{1}{2}\right|^{\mathfrak{y}}}\nabla t + \int_{\frac{9}{10}}^{1} \frac{1}{|t-\frac{1}{2}|^{\mathfrak{y}}}\nabla t + 0.220567 \\
&= \sum_{i=1}^{\infty} \int_{\mathfrak{z}_i}^{\mathfrak{z}_{i-1}} \frac{1}{\left||t-\frac{1}{2}|+t_i-\frac{1}{2}\right|^{\mathfrak{y}}}\nabla t + \int_0^{\frac{1}{10}} \frac{1}{(\frac{1}{2}-t)^{\mathfrak{y}}}\nabla t \\
&\qquad + \int_{\frac{9}{10}}^{1} \frac{1}{(t-\frac{1}{2})^{\mathfrak{y}}}\nabla t + 0.220567 \\
&= \sum_{i=1}^{\infty} \left[\int_{\mathfrak{z}_i}^{t_i} \frac{1}{(t_i-t)^{\mathfrak{y}}}\nabla t + \int_{t_i}^{\mathfrak{z}_{i-1}} \frac{1}{(t-t_i)^{\mathfrak{y}}}\nabla t\right] \\
&\qquad + \frac{2}{1-\mathfrak{y}}\left[\left(\frac{1}{2}\right)^{1-\mathfrak{y}} - \left(\frac{2}{5}\right)^{1-\mathfrak{y}}\right] + 0.220567 \\
&= \sum_{i=1}^{\infty} \left[\int_{\frac{t_i+t_{i+1}}{2}}^{t_l} \frac{1}{(t_i-t)^{\mathbb{T}}heta}\nabla t + \int_{t_i}^{\frac{t_{i-1}+t_i}{2}} \frac{1}{(t-t_i)^{\mathbb{T}}heta}\nabla t\right] + 0.298605 + 0.220567 \\
&= \frac{1}{1-\mathfrak{y}} \sum_{i=1}^{\infty} \left[\left(\frac{t_i-t_{i+1}}{2}\right)^{1-\mathfrak{y}} + \left(\frac{t_{i-1}-t_i}{2}\right)^{1-\mathfrak{y}} + 0.519172\right. \\
&= \frac{1}{2^{1-\mathfrak{y}}(1-\mathfrak{y})} \sum_{i=1}^{\infty} \left[\frac{1}{(i+2)^{4(1-\mathfrak{y})}} + \frac{1}{(i+1)^{4(1-\mathfrak{y})}}\right] + 0.519172 \\
&= \sqrt{2}\sum_{i=1}^{\infty} \left[\frac{1}{(i+1)^2} + \frac{1}{(i+1)^2}\right] + 0.519172 \\
&= \sqrt{2}\left(\frac{\pi^2}{3} - \frac{9}{4}\right) + 0.519172.
\end{aligned}$$

This shows that $\psi(t, \mathfrak{y}) \in L^1_\nabla[0, 1]$.

Next, we claim that if $\mathfrak{y} = \frac{1}{4}$, then $\psi(t,\mathfrak{y}) \in L^2_\nabla[0,1]$. In this case, we need the Cauchy product,

$$\sum_{i=1}^{\infty} a_i \cdot \sum_{i=1}^{\infty} b_i = \sum_{i=1}^{\infty} c_i, \tag{21}$$

where

$$c_i = \sum_{k=1}^{l} a_k b_{i-k+1}. \tag{22}$$

Note that

$$\int_0^1 \psi^2(t,\mathfrak{y})\nabla t = \int_0^{\frac{1}{5}} \psi^2(t,\mathfrak{y})\nabla t + \int_{\frac{1}{5}}^{\frac{4}{5}} \left[\sum_{i=1}^{\infty} \frac{\chi[\mathfrak{z}_i,\, \mathfrak{z}_{i-1}]}{\left||t-\frac{1}{2}|+t_i-\frac{1}{2}\right|^{2\mathfrak{z}}}\right]^2 \nabla t + \int_{\frac{4}{5}}^{1} \psi^2(t,\mathfrak{y})\nabla t, \tag{23}$$

we use (21) and (22) and the fact that, if $\mathtt{A} \cap \mathtt{B} = \emptyset$, then $\chi[\mathtt{A}] \cdot \chi[\mathtt{B}] = 0$ to simplify the integrand,

$$\begin{aligned}\left[\sum_{i=1}^{\infty} \frac{\chi[\mathfrak{z}_i,\, \mathfrak{z}_{i-1}]}{|t-t_i|^{\mathfrak{y}}}\right]^2 &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \frac{\chi[\mathfrak{z}_k,\, \mathfrak{z}_{i-1}]}{|t-t_k|^{\mathfrak{y}}} \frac{\chi[\mathfrak{z}_{i-k+1},\, \mathfrak{z}_{i-k}]}{|t-t_{i-k+1}|^{\mathfrak{y}}} \\ &= \sum_{i=1}^{\infty} \frac{\chi[\mathfrak{z}_i,\, \mathfrak{z}_{i-1}]}{|t-t_i|^{2\mathfrak{y}}} \text{ a.e.,}\end{aligned}$$

and so (23) may be written as

$$\begin{aligned}\int_0^1 \psi^2(t,\mathfrak{y})\nabla t =& \sum_{i=1}^{\infty} \int_{\frac{1}{5}}^{\frac{4}{5}} \frac{\chi[\mathfrak{z}_l,\, \mathfrak{z}_{l-1}]}{\left||t-\frac{1}{2}|+t_l-\frac{1}{2}\right|^{2\mathfrak{y}}}\nabla t + \int_0^{\frac{1}{10}} \psi^2(t,\mathfrak{y})\nabla t + \int_{\frac{9}{10}}^{1} \psi^2(t,\mathfrak{y})\nabla t \\ &+ \left[\left(\frac{1}{9}-\frac{1}{10}\right)\psi^2\left(\frac{1}{9},\mathfrak{y}\right) + \left(\frac{1}{8}-\frac{1}{9}\right)\psi^2\left(\frac{1}{8},\mathfrak{y}\right) + \left(\frac{1}{7}-\frac{1}{8}\right)\psi^2\left(\frac{1}{7},\mathfrak{y}\right)\right. \\ &+ \left(\frac{1}{6}-\frac{1}{7}\right)\psi^2\left(\frac{1}{6},\mathfrak{y}\right) + \left(\frac{1}{5}-\frac{1}{6}\right)\psi^2\left(\frac{1}{5},\mathfrak{y}\right) + \left(\frac{8}{9}-\frac{4}{5}\right)\psi^2\left(\frac{8}{9},\mathfrak{y}\right) \\ &+ \left(\frac{7}{8}-\frac{8}{9}\right)\psi^2\left(\frac{7}{8},\mathfrak{y}\right) + \left(\frac{6}{7}-\frac{7}{8}\right)\psi^2\left(\frac{6}{7},\mathfrak{y}\right) + \left(\frac{5}{6}-\frac{6}{7}\right)\psi^2\left(\frac{5}{6},\mathfrak{y}\right) \\ &+ \left.\left(\frac{9}{10}-\frac{5}{6}\right)\psi^2\left(\frac{9}{10},\mathfrak{y}\right)\right] \\ =& \int_{\frac{1}{5}}^{\frac{4}{5}} \sum_{i=1}^{\infty} \frac{\chi[\mathfrak{z}_i,\, \mathfrak{z}_{i-1}]}{\left||t-\frac{1}{2}|+t_i-\frac{1}{2}\right|^{2\mathfrak{y}}}\nabla t + \int_0^{\frac{1}{10}} \frac{1}{|t-\frac{1}{2}|^{2\mathfrak{y}}}\nabla t \\ &+ \int_{\frac{9}{10}}^{1} \frac{1}{|t-\frac{1}{2}|^{2\mathfrak{y}}}\nabla t + 0.2432747 \text{ with } \mathfrak{y} = \frac{1}{4}.\end{aligned}$$

Then

$$\begin{aligned}\int_0^1 \psi^2(t,\mathfrak{y})\nabla t &= \sum_{i=1}^{\infty}\int_{\mathfrak{z}_i}^{\mathfrak{z}_{i-1}} \frac{1}{\big||t-\frac{1}{2}|+t_i-\frac{1}{2}\big|^{2\mathfrak{y}}}\nabla t + \int_0^{\frac{1}{10}} \frac{1}{(\frac{1}{2}-t)^{2\mathfrak{y}}}\nabla t \\
&\quad + \int_{\frac{1}{10}}^{1} \frac{1}{(t-\frac{1}{2})^{2\mathfrak{y}}}\nabla t + 0.2432747 \\
&= \sum_{i=1}^{\infty}\Big[\int_{\mathfrak{z}_i}^{t_i}\frac{1}{(t_i-t)^{2\mathfrak{y}}}\nabla t + \int_{t_i}^{\mathfrak{z}_{i-1}}\frac{1}{(t-t_i)^{2\mathfrak{y}}}\nabla t\Big] \\
&\quad + \frac{2}{1-2\mathfrak{y}}\Big[\Big(\frac{1}{2}\Big)^{1-2\mathfrak{y}} - \Big(\frac{2}{5}\Big)^{1-2\mathfrak{y}}\Big] + 0.2432747 \\
&= \sum_{i=1}^{\infty}\Big[\int_{\frac{t_i+t_{i+1}}{2}}^{t_i}\frac{1}{(t_i-t)^{2\mathfrak{y}}}\nabla t + \int_{t_i}^{\frac{t_{i-1}+t_i}{2}}\frac{1}{(t-t_i)^{2\mathfrak{y}}}\nabla t\Big] + 0.2986050 \\
&\quad + 0.2432747 \\
&= \frac{1}{1-2\mathfrak{y}}\sum_{i=1}^{\infty}\Big[\Big(\frac{t_i-t_{i+1}}{2}\Big)^{1-2\mathfrak{y}} + \Big(\frac{t_{i-1}-t_i}{2}\Big)^{1-2\mathfrak{y}}\Big] + 0.5418797 \\
&= \frac{1}{2^{1-2\mathfrak{y}}(1-2\mathfrak{y})}\sum_{i=1}^{\infty}\Big[\frac{1}{(i+2)^{4(1-2\mathfrak{y})}} + \frac{1}{(i+1)^{4(1-2\mathfrak{y})}}\Big] + 0.5418797 \\
&= \sqrt{2}\sum_{i=1}^{\infty}\Big[\frac{1}{(i+1)^2} + \frac{1}{(i+1)^2}\Big] + 0.5418797 \\
&= \sqrt{2}\Big(\frac{\pi^2}{3} - \frac{9}{4}\Big) + 0.5418797,\end{aligned}$$

which implies $\psi(t,\mathfrak{y}) \in L^2_{\nabla}[0,1]$.

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