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SOME REMARKS ON THE PAPER "FIXED POINT OF α-GERAGHTY CONTRACTION WITH APPLICATIONS"

K. ANTHONY SINGH

Abstract. Muhammad Arshad, Aftab Hussain and Akbar Azam (Fixed point of α-Geraghty contraction with applications, U.P.B. Sci. Bull., Series A, Vol. 78, Iss. 2, 2016) improved the notion of α-Geraghty contraction type mappings and established some common fixed point theorems for a pair of α-admissible mappings under the improved notion of α-Geraghty contraction type condition in a complete metric space. But we observe some gaps in the proof of some theorems in the paper due to some insufficiency in the defining condition of the associated contraction mappings. The aim of this paper is to attempt to bridge the gaps by proposing some modifications in the contraction condition of the mappings.

1. Introduction

The Banach contraction principle which is a remarkable result in metric fixed point theory is indeed a powerful tool in nonlinear analysis having very useful applications in many disciplines. Due to this, several authors have improved, generalized and extended this basic result of Banach by defining new contractive conditions and replacing the metric space by more general abstract spaces. In 1973, Geraghty [4] generalized the Banach contraction principle by considering an auxilliary function. Among other results in this direction, the works of Amini-Harandi and Emami [1], Caballero et al. [2], Gordji et al. [5], Samet et al. [11], Karapinar and Samet [7], T. Abdeljawad [12] and P. Salimi et al. [10] can be mentioned. Recently, Cho et al. [3] defined the concept of α-Geraghty contraction type maps and proved the existence and uniqueness of a fixed point of such maps in the context of a complete metric space. Then, Muhammad Arshad et al. [9] improved the notion of α-Geraghty contraction type mappings and established some common fixed point theorems for a pair of α-admissible mappings under the improved notion of α-Geraghty contraction type condition in a complete metric space.

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2. Remarks on some results of [9]

Muhammad Arshad et al. [9] introduced an improved notion of α-Geraghty contraction type mappings and proved some common fixed point results.

Definition 2.1. [9] Let (X,d) be a metric space and $\alpha : X\times X \to \mathbb{R}$ be a function. Two mappings $S,T : X \to X$ is called a pair of generalized α-Geraghty contraction type mappings if there exists $\beta \in \Omega$ such that for all $x,y\in X$,

$$\alpha(x,y)d(Sx,Ty) \le \beta(M(x,y))M(x,y)$$

where

$$M(x,y) = max\left\{d(x,y), d(x,Sx), d(y,Ty), \frac{d(y,Sx)+d(x,Ty)}{2}\right\}.$$

Here Ω is the family of all functions $\beta : [0,+\infty) \to [0,1)$ which satisfy the condition: $\beta(t_n) \to 1$ implies $t_n \to 0$.

Theorem 2.2. [9] Let (X,d) be a complete metric space, $\alpha : X\times X \to \mathbb{R}$ be a function. Let $S,T : X \to X$ be two mappings. Suppose that the following holds:

(1) (S,T) is a pair of generalized $\alpha-$ Geraghty contraction type mappings,
(2) (S,T) is triangular $\alpha-$ admissible,
(3) there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \ge 1$,
(4) S and T are continuous.

Then, (S,T) have common fixed point.
The above Theorem 2.2 is indeed Theorem 3.1 (page 70, [9]) by Muhammad Arshad et al.

Remark 1. We reproduce the following two relations from the proof of the Theorem 3.1(page 70, [9]).
That is

$$d(x_{2i+1}, x_{2i+2}) < d(x_{2i}, x_{2i+1}). \tag{1}$$

This, implies that

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}), \text{for all n} \in \mathbb{N}\cup\{0\}\,. \tag{2}$$

The above implication combines the derived relation (1) with the relation

$$d(x_{2i+2}, x_{2i+3}) < d(x_{2i+1}, x_{2i+2}), \mathrm{i} \in \mathbb{N}\cup\{0\} \tag{3}$$

which is assumed to be similarly derivable.
But we observe that the assumed relation (3) is not derivable due to some insufficiency in the defining condition of the pair of generalized α-Geraghty contraction type mappings and the conditions given in the theorem. Indeed we have

$$d\left(x_{2i+2}, x_{2i+3}\right) = d\left(Tx_{2i+1}, Sx_{2i+2}\right) \le \alpha\left(x_{2i+1}, x_{2i+2}\right) d\left(Tx_{2i+1}, Sx_{2i+2}\right)$$

to which the contraction condition of the pair of generalized α-Geraghty contraction type mappings cannot be applied so as to arrive at the relation (3).
In order to rectify this, we propose to modify the defining condition of the pair of generalized α-Geraghty contraction type mappings as follows:

Definition 2.1. [Modified] (X,d) be a metric space and $\alpha : X\times X \to \mathbb{R}$ be a function. Two mappings $S,T : X \to X$ is called a pair of generalized α-Geraghty contraction type mappings if there exists $\beta \in \Omega$ such that for all $x,y\in X$,

$$\alpha(x,y)d(Sx,Ty) \le \beta(M(x,y))M(x,y)$$

and

$$\alpha(y,x)d(Ty,Sx) \leq \beta(M(x,y))M(x,y)$$

where

$$M(x,y) = max\left\{d(x,y), d(x,Sx), d(y,Ty), \frac{d(y,Sx)+d(x,Ty)}{2}\right\}.$$

With this modified definition, we can now complete the proof of Theorem 2.2 with the derivation of the relation (3) as follows:
We have

$$\begin{aligned} d\left(x_{2i+2}, x_{2i+3}\right) &= d\left(Tx_{2i+1}, Sx_{2i+2}\right) \\ &\leq \alpha\left(x_{2i+1}, x_{2i+2}\right) d\left(Tx_{2i+1}, Sx_{2i+2}\right) \\ &\leq \beta\left(M(x_{2i+2}, x_{2i+1})\right) M(x_{2i+2}, x_{2i+1}), \end{aligned}$$

for all $i \in \mathbb{N} \cup \{0\}$. Now

$$\begin{aligned} M(x_{2i+2}, x_{2i+1}) &= max\big\{d(x_{2i+2}, x_{2i+1}), d(x_{2i+2}, Sx_{2i+2}), d(x_{2i+1}, Tx_{2i+1}), \\ &\quad \frac{d(x_{2i+2}, Tx_{2i+1}) + d(x_{2i+1}, Sx_{2i+2})}{2}\big\} \\ &= max\big\{d(x_{2i+2}, x_{2i+1}), d(x_{2i+2}, x_{2i+3}), d(x_{2i+1}, x_{2i+2}), \\ &\quad \frac{d(x_{2i+2}, x_{2i+2}) + d(x_{2i+1}, x_{2i+3})}{2}\big\} \\ &= max\big\{d(x_{2i+1}, x_{2i+2}), d(x_{2i+2}, x_{2i+3}), \frac{d(x_{2i+1}, x_{2i+3})}{2}\big\} \\ &= max\big\{d(x_{2i+1}, x_{2i+2}), d(x_{2i+2}, x_{2i+3})\big\} \end{aligned}$$

Thus, we have

$$\begin{aligned} d\left(x_{2i+2}, x_{2i+3}\right) &= d\left(Tx_{2i+1}, Sx_{2i+2}\right) \\ &\leq \beta\left(M(x_{2i+2}, x_{2i+1})\right) M(x_{2i+2}, x_{2i+1}) \\ &= \beta\left(d(x_{2i+1}, x_{2i+2})\right) d(x_{2i+1}, x_{2i+2}) \\ &< d(x_{2i+1}, x_{2i+2}) \end{aligned}$$

That is

$$d\left(x_{2i+2}, x_{2i+3}\right) < d(x_{2i+1}, x_{2i+2}), \text{which is relation (3).}$$

Muhammad Arshad et al. [9] also introduced the notion of $\alpha-\eta-$Geraghty contraction type mappings and proved some common fixed point results (page 74, [9]).
Definition 2.3. [9] Let (X,d) be a metric space and $\alpha, \eta : X \times X \to \mathbb{R}$ be two functions. Two mappings $S, T : X \to X$ is called a pair of generalized $\alpha-\eta-$Geraghty contraction type mappings if there exists $\beta \in \Omega$ such that for all $x, y \in X$,

$$\alpha(x,y) \geq \eta(x,y) \Rightarrow d(Sx,Ty) \leq \beta(M(x,y))M(x,y)$$

where

$$M(x,y) = max\left\{d(x,y), d(x,Sx), d(y,Ty), \frac{d(y,Sx)+d(x,Ty)}{2}\right\}.$$

Theorem 2.4. [9] Let (X,d) be a complete metric space. Let S be α-admissible mapping with respect to η such that the following holds:

(1) (S, T) is a pair of generalized $\alpha - \eta-$ Geraghty contraction type mappings,
(2) (S, T) is triangular $\alpha-$ admissible,
(3) there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq \eta(x_0, Sx_0)$,
(4) S and T are continuous.

Then (S, T) have common fixed point.
The above Theorem 2.4 is indeed Theorem 3.3 (page 74, [9]) by Muhammad Arshad et al.
Remark 2. In the initial part of the proof of the above theorem leading to the derivation of the relation $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$, we observe that the conditions on S and (S, T) in the theorem i.e. S is α-admissible mapping with respect to η and (S, T) is triangular α-admissible, are not enough to arrive at the said relation. Indeed, in the proof, the pair (S, T) is mentioned to be α-admissible with respect to η, which is not defined anywhere in the paper [9] and thus is not a condition of the theorem. We introduce the following definitions and also a lemma (the proof of which is quite obvious).
Definition 2.5. Let $S, T : X \to X$ be two mappings and $\alpha, \eta : X \times X \to \mathbb{R}$ be two functions. Then the pair (S, T) is said to be α-admissible with respect to η if $\alpha(x, y) \geq \eta(x, y)$ implies $\alpha(Sx, Ty) \geq \eta(Sx, Ty)$ and $\alpha(Tx, Sy) \geq \eta(Tx, Sy), x, y \in X$.
Definition 2.6. Let $S, T : X \to X$ be two mappings and $\alpha, \eta : X \times X \to \mathbb{R}$ be two functions. Then the pair (S, T) is said to be triangular α-admissible with respect to η if
(T1) $\alpha(x, y) \geq \eta(x, y)$ implies $\alpha(Sx, Ty) \geq \eta(Sx, Ty)$ and $\alpha(Tx, Sy) \geq \eta(Tx, Sy)$,
(T2) $\alpha(x, z) \geq \eta(x, z)$ and $\alpha(z, y) \geq \eta(z, y)$ imply $\alpha(x, y) \geq \eta(x, y), x, y, z \in X$.
Lemma 2.7. Let $S, T : X \to X$ be a pair of triangular α-admissible mappings with respect to η. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq \eta(x_0, Sx_0)$. Define a sequence $\{x_n\}$ by $x_{2i+1} = Sx_{2i}$ and $x_{2i+2} = Tx_{2i+1}$, where $i = 0, 1, 2, \ldots$. Then we have $\alpha(x_n, x_m) \geq \eta(x_n, x_m)$ for all $m, n \in \mathbb{N} \cup \{0\}$ with $n < m$.
Remark 3. In the later part (page 74, [9]) of the proof of the Theorem 2.4, the implication $d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$, for all $n \in \mathbb{N} \cup \{0\}$, combines the derived relation $d(x_{2i+1}, x_{2i+2}) < d(x_{2i}, x_{2i+1})$, with the relation

$$d(x_{2i+2}, x_{2i+3}) < d(x_{2i+1}, x_{2i+2}), \mathrm{i} \in \mathbb{N} \cup \{0\} \tag{4}$$

which is again assumed to be similarly derivable.
But we observe that the assumed relation (4) is not derivable due to some insufficiency in the defining condition of the pair of generalized $\alpha - \eta-$Geraghty contraction type mappings and the conditions given in the theorem (as in the case of Remark 1).
In order to rectify this, we propose to modify the defining condition of the pair of generalized $\alpha - \eta-$Geraghty contraction type mappings as follows:
Definition 2.3.[Modified] Let (X, d) be a metric space and $\alpha, \eta : X \times X \to \mathbb{R}$ be two functions. Two mappings $S, T : X \to X$ is called a pair of generalized $\alpha - \eta-$Geraghty contraction type mappings if there exists $\beta \in \Omega$ such that for all $x, y \in X$,

$$\alpha(x, y) \geq \eta(x, y) \Rightarrow d(Sx, Ty) \leq \beta(M(x, y))M(x, y)$$

and

$$\alpha(y, x) \geq \eta(y, x) \Rightarrow d(Ty, Sx) \leq \beta(M(x, y))M(x, y)$$

where

$$M(x,y) = max\left\{d(x,y), d(x,Sx), d(y,Ty), \frac{d(y,Sx)+d(x,Ty)}{2}\right\}.$$

Relation (4) can now be derived using the modified definition of a pair of generalized $\alpha - \eta-$Geraghty contraction type mappings (as we do in the case of Remark 1), thus completing the proof of Theorem 2.4.

With the proposed modifications (as in Remark 2 and Remark 3), Theorem 2.4 can now be restated as follows:

Theorem 2.4. [Modified] Let (X, d) be a complete metric space and $\alpha, \eta : X \times X \to \mathbb{R}$ be two functions. Let $S, T : X \to X$ be two mappings. Suppose that the following holds:

(1) (S, T) is a pair of generalized $\alpha - \eta-$ Geraghty contraction type mappings,
(2) (S, T) is triangular $\alpha-$ admissible with respect to η,
(3) there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq \eta(x_0, Sx_0)$,
(4) S and T are continuous.

Then (S, T) have common fixed point.

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K. Anthony Singh
Department of Mathematics, D.M. College of Science, Dhanamanjuri University, Imphal, Manipur, India
E-mail address: anthonykumam@manipuruniv.ac.in