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PERIODIC MILD SOLUTIONS OF INFINITE DELAY SECOND ORDER EVOLUTION EQUATIONS WITH IMPULSES

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Abstract. In this article, we study the existence of periodic mild solutions for a class of second order evolution equations with not instantaneous impulses. The techniques used are some fixed point theorems in Banach spaces (Darbo and Kuratowski fixed point theorems), the Poincaré operator and the measure of noncompactness.

1. Introduction

Functional evolution equations have recently been applied in various areas of engineering, mathematics, and other applied sciences. For some fundamental results in the theory of functional evolution equations we refer the reader to the monographs [1, 4, 16, 27, 29, 31] and the papers [2, 5, 6, 11, 25]. In [25], the authors considered a class of evolution equations on unbounded intervals by using the Tichonov's fixed point theorem. However in the previous papers some restrictions like, the compactness of the semigroup, the Lipschitz conditions on the nonlinear term or the boundedness of the obtained mild solutions, are supposed. Functional differential equations with non-instantaneous impulsive was studied in [3, 17, 26, 28].

In [21, 22, 23], the authors used the Poincaré operator and proved some results concerning the existence of periodic solutions of infinite delay evolution equations. In this paper, we discuss the existence of periodic mild solutions of the following class of second order evolution equations with infinite delay and not instantaneous impulses

$$\begin{cases} u''(t) + A(t)u(t) = f(t, u(t), u_t); \quad \text{if } t \in I_k;\ k = 0, 1, \dots, \\ u(t) = g_k(t, u(t_k^-)); \ \text{if } t \in J_k;\ k = 1, 2, \dots, \\ u(t) = \phi(t); \quad \text{if } t \in \mathbb{R}_- := (-\infty, 0], \\ u'(s_k) = \psi_k \in E;\ k = 0, \dots, m, \dots, \end{cases} \tag{1}$$

where $I_0 = [0, t_1],\ I_k := (s_k, t_{k+1}],\ J_k := (t_k, s_k],\ 0 = s_0 < t_1 \le s_1 \le t_2 < \cdots < s_{m-1} \le t_m \le s_m \le t_{m+1} = T \le s_{m+1} \le t_{m+2} \le \ldots < +\infty,\ f : I_k \times E \times \mathcal{B} \to$

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$E;\ k=0,\dots,\ g_k : J_k \times E \to E;\ k=1,2,\dots,$ are given functions $T-$periodic in $t,\ T>0,$ $\mathcal{B}$ is an abstract phase space to be specified later, $\phi : \mathbb{R}_- \to E$ is a given function, $\{A(t)\}_{t>0}$ is a $T-$periodic family of unbounded operators from E into E that generate an evolution system of operators $\{U(t,s)\}_{(t,s)\in\mathbb{R}_+\times\mathbb{R}_+}$; for $(t,s)\in\Lambda := \{(t,s)\in\mathbb{R}_+\times\mathbb{R}_+ : 0\le s\le t<+\infty\},\ \mathbb{R}_+ := [0,\infty)$, and $(E,\|\cdot\|_E)$ is a real Banach space.

For any continuous function u and any $t\in\mathbb{R}_+$, we denote by u_t the element of $\mathcal{B}$ defined by $u_t(\theta)=u(t+\theta)$ for $\theta\in\mathbb{R}_- := (-\infty,0]$. Here, $u_t(\cdot)$ represents the history of the state up to the present time t. We assume that the histories u_t belong to $\mathcal{B}$.

This paper initiates the existence of periodic mild solutions for evolution equations with infinite delay and not instantaneous impulses. We use the classical Darbo fixed point theorem, the Poincaré operator and the concept of measure of noncompactness in Banach spaces. This extends the study of deriving periodic solutions from bounded solutions to infinite delay differential equations in Banach spaces. The paper is organized as follows. In Section 2 some preliminary results are introduced. The main results is presented in Section 3, while the last section is devoted to an illustrative example.

2. Preliminaries

Let $I:=[0,T];\ T>0$. By $B(E)$ we denote the Banach space of all bounded linear operators from E into E, with the norm

$$\|N\|_{B(E)} = \sup_{\|u\|=1}\|N(u)\|.$$

Let $L^1(I,E)$ be the Banach space of measurable functions $u : I\to E$ which are Bochner integrable and normed by

$$\|u\|_{L^1} = \int_0^T \|u(t)\|dt.$$

Note that, a measurable function $u : I\to E$ is Bochner integrable if and only if $\|u\|$ is Lebesgue integrable. For properties of the Bochner integral, see for instance, Yosida [30].
As usual, $\mathcal{C} := C(I)$ denotes the Banach space of all continuous functions $u : I\to E$ with the norm

$$\|u\|_\infty = \sup_{t\in I}\|u(t)\|.$$

Consider the space

$$\begin{aligned}\tilde{C}((-\infty,0],E) = \{u : (-\infty,0]\to E :\ & u \text{ is continuous and there exist } \tau_k\in(-\infty,0);\\ & k=1,\dots,m, \text{ such that } u(\tau_k^-) \text{ and } u(\tau_k^+) \text{ exist with } u(\tau_k^-)=u(\tau_k)\Big\},\end{aligned}$$

and the Banach space

$$\begin{aligned}PC = \Big\{u : (-\infty,T]\to E : u|_{\mathbb{R}_-}\in\mathcal{B},\ u|_{J_k}=g_k;\ k=1,\dots,m,\ u|_{I_k};\ k=1,\dots,m\\ \text{is continuous and there exist } u(s_k^-),\ u(s_k^+),\ u(t_k^-) \text{ and } u(t_k^+)\\ \text{with } u(s_k^+)=g_k(s_k,u(s_k^-)) \text{ and } u(t_k^-)=g_k(t_k,u(t_k^-))\Big\},\end{aligned}$$

with the norm

$$\|u\|_{PC} = \max\{\|u\|_\infty, \|\phi\|_{\mathcal{B}}\}.$$

In what follows, let $\{A(t),\ t \geq 0\}$ be a family of closed linear operators on the Banach space E with domain $D(A(t))$ that is dense in E and independent of t. The existence of solutions to our problem is related to the existence of an evolution operator $U(t,s)$ for the homogeneous problem

$$u''(t) = A(t)u(t);\ t \in \mathbb{R}_+. \tag{2}$$

This concept of evolution operator has been developed by Kozak [19].

Definition 2.1. *A family $\mathcal{U}$ of bounded operators $\mathcal{U}(t,s) : E \to E;\ (t,s) \in \Lambda\}$, is called an evolution operator of the equation (2) if the following conditions hold;*

(P_1) *For any $u \in E$, the map $(t,s) \to \mathcal{U}(t,s)u$ is continuously differentiable and:*
 (a) *for any $t \in \mathbb{R}_+ : \mathcal{U}(t,t) = 0$;*
 (b) *for all $(t,s) \in \Lambda$ and for any $u \in E$, $\frac{\partial}{\partial t}\mathcal{U}(t,s)u|_{t=s} = u$ and $\frac{\partial}{\partial s}\mathcal{U}(t,s)u|_{t=s} = -u$.*

(P_2) *For all $(t,s) \in \Lambda$ if $u \in D(A(t))$, then $\frac{\partial}{\partial s}\mathcal{U}(t,s)u \in D(A(t))$, the map $(t,s) \to \mathcal{U}(t,s)u$ is of class C^2, and*
 (a) $\frac{\partial^2}{\partial t^2}\mathcal{U}(t,s)u = A(t)\mathcal{U}(t,s)u$;
 (b) $\frac{\partial^2}{\partial s^2}\mathcal{U}(t,s)u = \mathcal{U}(t,s)A(s)u$;
 (c) $\frac{\partial^2}{\partial t \partial s}\mathcal{U}(t,s)u|_{t=s} = 0$.

(P_3) *For all $(t,s) \in \Lambda$ if $u \in D(A(t))$, then the map $(t,s) \to A(t)\frac{\partial}{\partial s}\mathcal{U}(t,s)u$ is continuous, $\frac{\partial^3}{\partial t^2 \partial s}\mathcal{U}(t,s)u$ and $\frac{\partial^3}{\partial s^2 \partial t}\mathcal{U}(t,s)u$ exist and*
 (a) $\frac{\partial^3}{\partial t^2 \partial s}\mathcal{U}(t,s)u = A(t)\frac{\partial}{\partial s}\mathcal{U}(t,s)u$;
 (b) $\frac{\partial^3}{\partial s^2 \partial t}\mathcal{U}(t,s)u = A(t)\frac{\partial}{\partial t}\mathcal{U}(t,s)A(s)u$.

In this paper, we assume that the state space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a seminormed linear space of functions mapping $\mathbb{R}_-$ into E, and satisfying the following fundamental axioms introduced by Hale and Kato in [15].

(A_1): If $u \in PC$ and $u_0 \in \mathcal{B}$, then for every $t \in I$ the following conditions hold:
 (i) $u_t \in \mathcal{B}$
 (ii) $\|u_t\|_{\mathcal{B}} \leq K(t)\int_0^t \|u(s)\|ds + M(t)\|\phi\|_{\mathcal{B}}$,
 (iii) $\|u(t)\| \leq H\|u_t\|_{\mathcal{B}}$, where $H \geq 0$ is a constant, $K : I \to \mathbb{R}_+$ is continuous; $M : \mathbb{R}_+ \to \mathbb{R}_+$ is locally bounded and H, K, M, are independent of $u(\cdot)$.

(A_2): For the function $u(\cdot)$ in (A_1), u_t is a $\mathcal{B}$-valued continuous function on I.

(A_3): The space $\mathcal{B}$ is complete.

Denote $K_b = \sup\{K(t) : t \in I\}$ and $M_b = \sup\{M(t) : t \in I\}$.

Remark 2.2. *Axiom (A_1)(ii) is equivalent to $\|\phi(0)\| \leq H\|\phi\|_{\mathcal{B}}$; for every $\phi \in \mathcal{B}$. From this equivalence; we can see that for all $\phi,\ \psi \in \mathcal{B}$ such that $\|\phi - \psi\|_{\mathcal{B}} = 0$, we necessarily have $\phi(0) = \psi(0)$.*

Lemma 2.3. *(Lemma 2.1 in [22]) There exists an integer $k_0 > 1$ such that*

$$\left(\frac{1}{2}\right)^{k_0-1} M < 1,$$

where $M = \sup_{(t,s)\in\Lambda} \|U(t,s)\|_{B(E)}$ is finite, and there exists a function h on $\mathbb{R}_-$ such that $h(0) = 1$, $h(-\infty) = +\infty$, h is decreasing on $\mathbb{R}_-$, and for $d \geq w_0 := \frac{T}{K_0}$ one has $\sup_{s\in(-\infty,0]} \frac{h(s)}{h(s-d)} \leq \frac{1}{2}$.

In all what follows, we consider the phase space

$$\mathcal{B} := \left\{\phi \in \tilde{C}((-\infty,0]),E) : \sup_{s\in(-\infty,0]} \frac{\|\phi(s)\|}{h(s)} < \infty\right\},$$

where $h : \mathbb{R}_- \to \mathbb{R}_+$ is the function given in Lemma 2.3. We have that the space $\mathcal{B}$ satisfies the condition (A_3). Also; $\mathcal{B}$ satisfies conditions (A_1) and (A_2) if

$$\sup_{t\in I} \sup_{-\infty<\theta\leq -t} \frac{\phi(t+\theta)}{h(\theta)} < \infty.$$

The space $\mathcal{B}$ endowed with the norm

$$\|\phi\|_{\mathcal{B}} = \sup_{s\in(-\infty,0]} \frac{\|\phi(s)\|}{h(s)},$$

is a Banach space [9].

Now, we recall the Kuratowski measure of noncompactness.

Definition 2.4. [7, 20] *Let X be a Banach space and Ω_X the bounded subsets of X. The Kuratowski measure of noncompactness is the map $\alpha : \Omega_X \to [0,\infty]$ defined by*

$$\alpha(B) = \inf\{\epsilon > 0 : B \subseteq \cup_{i=1}^{n} B_i \text{ and } diam(B_i) \leq \epsilon\}; \text{ here } B \in \Omega_E,$$

where

$$diam(B_i) = sup\{\|u-v\|_E : u,v \in B_i\}.$$

The Kuratowski measure of noncompactness satisfies the following properties:

Lemma 2.5. [7, 18] *Let A and B bounded sets.*

(a) $\alpha(B) = 0 \Leftrightarrow \overline{B}$ *is compact (B is relatively compact), where $\overline{B}$ denotes the closure of B.*
(b) *nonsingularity : α is equal to zero on every one element-set.*
(c) *If B is a finite set, then $\alpha(B) = 0$.*
(d) $\alpha(B) = \alpha(\overline{B}) = \alpha(convB)$, *where $convB$ is the convex hull of B.*
(e) *monotonicity: $A \subset B \Rightarrow \alpha(A) \leq \alpha(B)$.*
(f) *algebraic semi-additivity : $\alpha(A+B) \leq \alpha(A) + \alpha(B)$, where*

$$A + B = \{x + y : x \in A, \quad y \in B\}.$$

(g) *semi-homogencity: $\alpha(\lambda B) = |\lambda|\alpha(B)$; $\lambda \in \mathbb{R}$. where $\lambda(B) = \{\lambda x : x \in B\}$.*
(h) *semi-additivity : $\alpha(A \bigcup B) = max\{\alpha(A), \alpha(B)\}$.*
(i) $\alpha(A \bigcap B) = min\{\alpha(A), \alpha(B)\}$.
(j) *invariance under translations: $\alpha(B + x_0) = \alpha(B)$ for any $x_0 \in X$.*

In all what follows, by α we denote the Kuratowski measure of noncompactness.

Lemma 2.6. [14] *Let $V \subset C(I, E)$ be a bounded and equicontinuous set, then*

(i) *the function $t \to \alpha(V(t))$ is continuous on I, and*

$$\alpha_c(V) = \sup_{t \in I} \alpha(V(t)).$$

(ii) $\alpha \left(\int_0^T u(s)ds : u \in V \right) \leq \int_0^T \alpha(V(s))ds,$

where

$$V(t) = \{u(t) : u \in V\}; \ t \in I.$$

Lemma 2.7. [8] *If Y is a bounded subset of a Banach space $\mathcal{X}$, then for each $\epsilon > 0$, there is a sequence $\{y_k\}_{k=1}^{\infty} \subset Y$ such that*

$$\alpha(Y) \leq 2\alpha(\{y_k\}_{k=1}^{\infty}) + \epsilon.$$

Lemma 2.8. [24] *If $\{u_k\}_{k=1}^{\infty} \subset L^1(I)$ is uniformly integrable, then $\alpha(\{u_k\}_{k=1}^{\infty})$ is measurable and*

$$\alpha \left(\left\{ \int_0^t u_k(s)ds \right\}_{k=1}^{\infty} \right) \leq 2 \int_0^t \alpha(\{u_k(s)\}_{k=1}^{\infty})ds.$$

For our purpose we will need the following fixed point theorem.

Theorem 2.9. *(Darbo's Fixed Point Theorem)* [12, 13]*] Let X be a Banach space and C be a bounded, closed, convex and nonempty subset of X. Suppose a continuous mapping $N : C \to C$ is such that for all closed subsets D of C,*

$$\alpha(T(D)) \leq k\alpha(D), \tag{3}$$

where $0 \leq k < 1$. Then T has a fixed point in C.

Remark 2.10. *Mappings satisfying the Darbo-condition (3) have subsequently been called k-set contractions.*

Definition 2.11. *Let X be a Banach space and α be a mesure of noncompactness. An operator $P : X \to X$ is said to be condensing if P is continuous and takes bounded sets into bounded sets, and $\alpha(P(B)) \leq \alpha(B)$ for every bounded set B of X with $\alpha(B) > 0$.*

Theorem 2.12. *(Sadovskii's fixed point theorem)* [22]*] Let X be a Banach space, α be a mesure of noncompactness, and $P : X \to X$ be a condensing operator. If $P(H) \subset H$ for a convex, closed, and bounded set H of X then P has a fixed point in H.*

3. Existence of Periodic Mild Solutions

Definition 3.1. *By a periodic mild solution of problem (1) we mean a measurable and T-periodic function u that satisfies*

$$u(t)=\begin{cases}-\frac{\partial}{\partial s}U(t,0)\phi(0)+U(t,0)\psi_0+\displaystyle\int_0^t U(t,s)\ f(s,u(s),u_s)ds.; & \text{if } t\in I_0,\\ -\frac{\partial}{\partial s}U(t,s_k)g_k(s_k,u(s_k^-))+U(t,s_k)\psi_k & \\ +\displaystyle\int_{s_k}^t U(t,s)\ f(s,u(s),u_s)ds.; \quad \text{if } t\in I_k;\ k=1,\dots,m, & \\ g_k(t,u(t_k^-));\ \text{if } t\in J_k;\ k=1,\dots,m, & \\ \phi(t);\quad \text{if } t\in\mathbb{R}_-. & \end{cases}$$

The following hypotheses will be used in the sequel.

(H_1) The functions f and g_k are continuous in their variables, and they map bounded sets into bounded sets,

(H_2) The function $t\mapsto f(t,u,v)$ is measurable on $I_k,\ k=0,\dots,m$, for each $u,v\in E\times\mathcal{B}$, and the functions $u\mapsto f(t,u,v)$ and $v\mapsto f(t,u,v)$ are continuous on $E\times\mathcal{B}$ for a.e. $t\in I_k;\ k=0,\dots,m$,

(H_3) For a constant $T>0,\ f(t+T,u,v)=f(t,u,v),\ A(t+T)=A(t);\ t\in I_k;\ k=0,\dots,m,\ (u,v)\in E\times\mathcal{B}$, and $g_k(t+T,z)=g_k(t,z)\ t\in J_k;\ k=1,\dots,m, z\in E$,

(H_4) There exist continuous functions $p:I_k\to\mathbb{R}_+,\ q:J_k\to\mathbb{R}_+$, such that

$$\|f(t,u,v)|\leq p(t),\ for\ a.e.\ t\in I_k;\ k=0,\dots,m,\ and\ each\ u,v\in E\times\mathcal{B},$$

and

$$\|g_k(t,z)|\leq q(t),\ for\ a.e.\ t\in J_k,\ and\ each\ z\in E,\ k=0,\dots,m,$$

(H_5) For each bounded sets $B(t)\subset E$, and $B_t\subset\mathcal{B};\ t\in\mathbb{R}_+$, such that

$$B(t)=\{u(t):u\in C(I)\},\ and\ B_t=\{u_t:u_t\in\mathcal{B}\},$$

we have

$$\alpha(f(t,B(t),B_t))\leq p(t)\alpha(B);\ for\ a.e.\ t\in I_k;\ k=0,\dots,m,$$

and

$$\alpha(g_k(t,B))\leq q(t)\alpha(B);\ for\ a.e.\ t\in J_k;\ k=1,\dots,m.$$

Set

$$M_0=\sup_{(t,s)\in\Lambda}\left\|\frac{\partial}{\partial s}U(t,s)\right\|_{B(E)},\ p^*=\sup_{t\in I_k}p(t),\quad and\quad q^*=\sup_{t\in J_k}q(t).$$

Now, we shall prove the following theorem concerning the existence of periodic mild solutions of problem (1).

Theorem 3.2. *Assume that the hypotheses $(H_1)-(H_5)$ hold. If $\ell:=4MTp^*<1$, then the problem (1) has at least one T-periodic mild solution defined on $\mathbb{R}$.*

Proof. The proof will be given in two parts. Consider the problem

$$\begin{cases} u''(t) + A(t)u(t) = f(t, u(t), u_t); & \text{if } t \in I_k; \ k = 0, \dots, m, \\ u(t) = g_k(t, u(t_k^-)); \ \text{if } t \in J_k; \ k = 1, \dots, m, \\ u(t) = \phi(t); & \text{if } t \in \mathbb{R}_- := (-\infty, 0], \\ u'(s_k) = \psi_k; \ k = 0, \dots, m. \end{cases} \tag{4}$$

Part 1. *Existence of mild solutions.*

We prove that problem (4) has a mild solution $u \in PC$, with $\|u\|_{PC} \leq R$ where

$$R \geq \max\{\|\phi\|_{\mathcal{B}}, q^*, M_0\|\phi(0)\| + M\|\psi_0\| + Mp^*, M_0q^* + M\|\psi_k\| + Mp^* + Mp^*\}.$$

Consider the operator $N : PC \to PC$ defined by:

$$(Nu)(t) = \begin{cases} -\frac{\partial}{\partial s}U(t,0)\phi(0) + U(t,0)\psi_0 + \displaystyle\int_0^t U(t,s) \ f(s, u(s), u_s)ds.; & \text{if } t \in I_0, \\ -\frac{\partial}{\partial s}U(t, s_k)g_k(s_k, u(s_k^-)) + U(t, s_k)\psi_k & \\ + \displaystyle\int_{s_k}^t U(t,s) \ f(s, u(s), u_s)ds.; \ \text{if } t \in I_k; \ k = 1, \dots, m, & \\ g_k(t, u(t_k^-)); \ \text{if } t \in J_k; \ k = 1, \dots, m, & \\ \phi(t); \ \ \text{if } t \in \mathbb{R}_-. & \end{cases} \tag{5}$$

Clearly, the fixed points of the operator N are mild solutions of problem (4).
For any $u \in PC$ and each $t \in I_0$, we have

$$\begin{aligned} \|(Nu)(t)\| &\leq M_0\|\phi(0)\| + M\|\psi_0\| + M\int_0^t \|f(s, u(s), u_s)\|ds \\ &\leq M_0\|\phi(0)\| + M\|\psi_0\| + Mp^* \\ &\leq R. \end{aligned}$$

Next, for any $u \in PC$ and each $t \in I_k; \ k = 1, \dots, m$, we have

$$\begin{aligned} \|(Nu)(t)\| &\leq M_0q^* + M\|\psi_k\| + M\int_{s_k}^t \|f(s, u(s), u_s)\|ds \\ &\leq M_0q^* + M\|\psi_k\| + Mp^* \\ &\leq R. \end{aligned}$$

Also, for any $u \in PC$ and each $t \in J_k; \ k = 1, \dots, m$, we have

$$\|(Nu)(t)\| \leq q^* \leq R,$$

and for any $u \in PC$ and each $t \in \mathbb{R}_-$, we have

$$\|(Nu)(t)\| = \|\phi\|_{\mathcal{B}} \leq R.$$

This proves that N transforms the ball $B_R := \{w \in PC : \|w\|_{PC} \leq R\}$ into itself.
We shall show that the operator $N : B_R \to B_R$ satisfies all the assumptions of Theorem 2.9. The proof will be given in two steps.

Step 1. $N : B_R \to B_R$ *is continuous.*
Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in B_R.

For each $t \in \mathbb{R}_- \cup J_k;\ k = 1, \ldots, m$, we have

$$\|(Nu_n)(t) - (Nu)(t)\| = 0 \to 0 \quad \text{as } n \to \infty,$$

and for each $t \in I_k;\ k = 0, \ldots, m$, we have

$$\|(Nu_n)(t) - (Nu)(t)\| \leq M \int_0^t \|f(s, u_n(s), u_{sn}) - f(s, u(s), u_s)\| ds. \tag{6}$$

Since $u_n \to u$ as $n \to \infty$ and f is continuous, then by the Lebesgue dominated convergence theorem, equation (6) implies

$$\|(Nu_n)(t) - (Nu)(t)\| \to 0 \quad \text{as } n \to \infty.$$

Hence

$$\|N(u_n) - N(u)\|_{PC} \to 0 \quad \text{as } n \to \infty.$$

Step 2. *For each closed subset D of $C(I)$, $\alpha(N(D)) \leq \ell\alpha(D)$.*
From Lemmas 2.7 and 2.8, for any $D \subset B_R$ and any $\epsilon > 0$, there exists a sequence $\{u_k\}_{k=0}^{\infty} \subset D$, such that for all $t \in I_k;\ k = 0, \ldots, m$, we have

$$\begin{aligned}
\alpha((ND)(t)) &= \alpha\left(\left\{-\frac{\partial}{\partial s}U(t,0)\phi(0) + U(t,0)\psi_0\right.\right. \\
&+ \left.\left.\int_0^t U(t,s)\ f(s, u(s), u_s)ds;\ u \in D\right\}\right) \\
&\leq 2\alpha\left(\left\{\int_0^t U(t,s) f(s, u_k(s), u_{ks})ds\right\}_{k=1}^{\infty}\right) + \epsilon \\
&\leq 4\int_0^t \alpha\left(\|U(t,s)\|_{B(E)}\{f(s, u_k(s), u_{ks})\}_{k=1}^{\infty}\right) ds + \epsilon \\
&\leq 4M\int_0^t \alpha\left(\{f(s, u_k(s), u_{ks})\}_{k=1}^{\infty}\right) ds + \epsilon \\
&\leq 4M\int_0^t p(s)\alpha\left(\{u_k(s)\}_{k=1}^{\infty}\right) ds + \epsilon \\
&\leq 4Mp^*\int_0^t \alpha\left(\{u_k(s)\}_{k=1}^{\infty}\right) ds + \epsilon, \\
&\leq 4MTp^*\alpha_c(D) + \epsilon,
\end{aligned}$$

and, for all $t \in I_k;\ k = 1, \ldots, m$, we get

$$\begin{aligned}
\alpha((ND)(t)) &= \alpha\left(\left\{-\frac{\partial}{\partial s}U(t,s_k)g_k(s_k, u(s_k^-)) + U(t,s_k)\psi_k\right.\right. \\
&+ \left.\left.\int_{s_k}^t U(t,s)\ f(s, u(s), u_s)ds;\ u \in D\right\}\right) \\
&\leq 2\alpha\left(\left\{\int_0^t U(t,s) f(s, u_k(s), u_{ks})ds\right\}_{k=1}^{\infty}\right) + \epsilon \\
&\leq 4MTp^*\alpha_c(D) + \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, then

$$\alpha_c(ND) \leq \ell\alpha_c(D).$$

As a consequence of these two steps together with Theorem 2.9, we can conclude that N has a fixed point in $u \in B_R$ which is a mild solution of problem (1).

Part 2. *Periodic mild solutions.*
A standard approach in deriving $T-$periodic solutions is to define the Poincaré operator $P : \mathcal{B} \to \mathcal{B}$ given by $P(\phi) = u_T(\phi)$ such that

$$(P\phi)(s) = u_T(s, \phi) = u(T + s, \phi); \ s \in \mathbb{R}_-,$$

which maps an initial function (or value) ϕ along the unique mild solution $u(\phi)$ to our problem (1) by $T-$ units (i.e., T units along the unique solution $u(\cdot, \phi)$ determined by the initial function ϕ).
We show that P is a condensing operator with respect to Kuratowski's measure of non-compactness in the phase space $\mathcal{B}$, then the given conditions such that the fixed point theorem (Theorem 2.12) can be applied to get fixed points for the Poincaré operator, which give rise to periodic solutions. We do this in two steps.

Step 1. *The fixed points of P give rise to periodic mild solutions of (1).*
Let $\phi \in \mathcal{B}$ be such that $p(\phi) = \phi$. Then for the solution $u(\cdot) = u(\cdot, \phi)$ with $u_0(\cdot, \phi) = \phi$, we can define $v(t) = u(t + T)$. Now, for $t > 0$, we can use the known properties of $U(t, s)$, and the fact that $A(t), f$ and g_k are T-periodic functions in t, to obtain that v is also a solution with $v_0(\cdot, \phi) = u_T(\phi) = u(\cdot, \phi)$. Indeed; we can obtain that

$$v(t) = \begin{cases} -\frac{\partial}{\partial s}U(t,0)\phi(0) + U(t,0)\psi_0 + \displaystyle\int_0^t U(t,s)\ f(s, v(s), v_s)ds.; \quad \text{if } t \in I_0, \\ -\frac{\partial}{\partial s}U(t,s_k)g_k(s_k, u(s_k^-)) + U(t,s_k)\psi_k \\ + \displaystyle\int_{s_k}^t U(t,s)\ f(s, v(s), v_s)ds.; \quad \text{if } t \in I_k; \ k = 1, \dots, m, \\ g_k(t, v(t_k^-)); \ \text{if } t \in J_k; \ k = 1, \dots, m, \\ \phi(t); \quad \text{if } t \in \mathbb{R}_-. \end{cases}$$

Then the uniqueness of $\{U(t,s)\}_{(t,s)\in\Lambda}$ implies that $v(t) = u(t)$, so that $u(t) = u(t + T)$ is a T-periodic solution.

Step 2. *P is condensing.*
Now, we prove that the operator $P : \mathcal{B} \to \mathcal{B}$ is condensing. Let $D \subset \mathcal{B}$ be bounded with $\alpha_c(D) > 0$. From Theorem 4.1 in [22], we get

$$\alpha_c(P(D)) \le \left(\frac{1}{2}\right)^{k_0 - 1} M\alpha_c(D) < \alpha_c(D).$$

Thus from Theorem 2.12, P has a fixed point which gives rise to a periodic mild solution of our problem (1).

4. An Example

Consider the following functional evolution problem

$$\begin{cases} \dfrac{\partial^2 z}{\partial t^2}(t,x) = a(t,x)\dfrac{\partial^2 z}{\partial x^2}(t,x) + Q(t, z(t,x), z_t(\cdot,x)); & x \in [0,\pi],\ t \in I_k;\ k = 0,\cdots, \\ z(t,x) = g_k(t,x); & x \in [0,\pi],\ t \in J_k;\ k = 1,\cdots, \\ z(t,0) = z(t,\pi) = 0; & t \in \mathbb{R}_+, \\ z(0,x) = \Phi(x); & x \in [0,\pi], \\ z(t,x) = \phi(t,x); & t \in \mathbb{R}_-,\ x \in [0,\pi], \end{cases} \tag{7}$$

where $a(t,x) : \mathbb{R}_+ \times [0,\pi] \to \mathbb{R}$ is a continuous function and is uniformly Hölder continuous in t, $Q : \mathbb{R}_+ \times \mathbb{R} \times \mathcal{B} \to \mathbb{R}$, $\Phi : [0,\pi] \to \mathbb{R}$ and $\phi : \mathbb{R}_- \times [0,\pi] \to \mathbb{R}$ are continuous functions such that $\Phi(x) = \phi(0,x);\ x \in [0,\pi]$.

Consider $E = L^2([0,\pi],\mathbb{R})$ and define $A(t)$ by $A(t)w = a(t,x)w''$ with domain

$$D(A) = \{w \in E : w, w' \text{ are absolutely continuous}, w'' \in E,\ w(0) = w(\pi) = 0\}.$$

Then $A(t)$ generates an evolution system $U(t,s)$ (see [10]).

For $x \in [0,\pi]$, we have

$$y(t)(x) = z(t,x); \quad t \in \mathbb{R}_+,$$
$$f(t, u(t), u_t, x) = Q(t, z(t,x), z_t(\cdot,x); \quad t \in \mathbb{R}_+,$$
$$u_0(x) = \Phi(x); \quad x \in [0,\pi],$$
$$u(t,x) = \phi(t,x);\ x \in [0,\pi],\ t \in \mathbb{R}_-.$$

Thus, under the above definitions of f, u_0 and $A(\cdot)$, the system (7) can be represented by the functional evolution problem (1). Furthermore, more appropriate conditions on Q ensure the hypotheses $(H_1) - (H_5)$. Consequently, Theorem 3.2 implies that the evolution problem (7) has at least one periodic mild solution.

References

[1] S. Abbas and M. Benchohra, *Advanced Functional Evolution Equations and Inclusions*, Developments in Mathematics, **39**, Springer, Cham, 2015.
[2] S. Abbas, W. Albarakati and M. Benchohra, Successive approximations for functional evolution equations and inclusions, *J. Nonlinear Funct. Anal.*, Vol. 2017 (2017), Article ID 39, pp. 1-13.
[3] R. P. Agarwal, S. Hristova, and D. O'Regan, *Non-instantaneous Impulses in Differential Equations*. Springer, Cham, 2017.
[4] N. U. Ahmed, *Semigroup Theory with Applications to Systems and Control*, Pitman Research Notes in Mathematics Series, 246. Longman Scientific & Technical, Harlow; John Wiley & Sons, New York, 1991.
[5] S. Baghli and M. Benchohra, Global uniqueness results for partial functional and neutral functional evolution equations with infinite delay, *Differential Integral Equations*, **23** (2010), 31–50.
[6] S. Baghli and M. Benchohra, Multivalued evolution equations with infinite delay in Fréchet spaces, *Electron. J. Qual. Theo. Differ. Equ.* 2008, No. 33, 24 pp.
[7] J. Banaŝ and K. Goebel, *Measures of Noncompactness in Banach Spaces*, Marcel Dekker, New York, 1980.

[8] D. Bothe, Multivalued perturbation of m-accretive differential inclusions, *Isr. J. Math.* **108** (1998), 109-138.
[9] T. Burton, Stability and Periodic Solutions of Ordinary Differential Equations and Functional Differential Equations, *Academic Press, San Diego,* (1985), 197-308.
[10] A. Freidman, *Partial Differential Equations*, Holt, Rinehat and Winston, New York, 1969.
[11] M. Frigon and A. Granas, Résultats de type Leray-Schauder pour des contractions sur des espaces de Fréchet, *Ann. Sci. Math. Québec* **22** (2) (1998), 161-168.
[12] K. Goebel, *Concise Course on Fixed Point Theorems*, Yokohama Publishers, Japan, 2002.
[13] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2003.
[14] D.J. Guo, V. Lakshmikantham, X. Liu, *Nonlinear Integral Equations in Abstract Spaces,* Kluwer Academic Publishers, Dordrecht, 1996.
[15] J. Hale, J. Kato, Phase space for retarded equations with infinite delay, *Funkcial. Ekvac.*, **21** (1978), 11-41.
[16] S. Heikkila and V. Lakshmikantham, *Monotone Iterative Technique for Nonlinear Discontinuous Differential Equations*, Marcel Dekker Inc., New York, 1994.
[17] E. Hernández and D. O'Regan, On a new class of abstract impulsive differential equations, *Proc. Amer. Math. Soc.* **141** (2013), 1641-1649.
[18] W.A. Kirk and B. Sims, *Handbook of Metric Fixed Point Theory*, Springer-Science + Business Media, B.V, Dordrecht, 2001.
[19] M. Kozak, A fundamental solution of a second-order differential equation in a Banach space, *Univ. Iagel. Acta Math.*, **32** (1995), 275-289.
[20] K. Kuratowski, Sur les espaces complets, *Fund. Math.* **15** (1930), 301-309.
[21] J. Liang, J.H. Liu, M.V. Nguyen and T.J. Xiao, Periodic solutions of impulsive differential equations with infinite delay in Banach spaces, *J. Nonlinear Funct. Anal.*, 2019 (2019), Article ID 18, 1-10.
[22] J.H. Liu, Periodic solutions of infinite delay evolution equations, *J. Math. Anal. Appl.*, **247** (2000), 627-644.
[23] J.H. Liu, T. Naito and N.V. Minh, Bounded and periodic solutions of infinite delay evolution equations, *J. Math. Anal. Appl.*, **286** (2003) 705-712.
[24] H. Mönch, Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces, *Nonlinear Anal.* **4** (1980), 985-999.
[25] L. Olszowy and S. Wędrychowicz, Mild solutions of semilinear evolution equation on an unbounded interval and their applications, *Nonlinear Anal.* **72** (2010), 2119-2126.
[26] D. N. Pandey, S. Das and N. Sukavanam, Existence of solution for a second-order neutral differential equation with state dependent delay and non-instantaneous impulses, *Int. J. Nonlin. Sci.* **18**(2)(2014), 145-155.
[27] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.
[28] M. Pierri, D. O'Regan and V. Rolnik, Existence of solutions for semi-linear abstract differential equations with not instantaneous impulses, *Appl. Math. Comput.* **219** (2013), 6743-6749.
[29] J. Wu, *Theory and Applications of Partial Functional Differential Equations*, Applied Mathematical Sciences 119, Springer-Verlag, New York, 1996.
[30] K. Yosida, *Functional Analysis*, 6^{th} edn. Springer-Verlag, Berlin, 1980.
[31] Y. Zhou, R.N. Wang and L. Peng, *Topological Structure of the Solution Set for Evolution Inclusions*, Springer-Nature, Singapore, 2017.

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