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UNIQUENESS OF MEROMORPHIC FUNCTIONS SHARING VALUES WITH THEIR n-TH DERIVATIVES

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Abstract. In this paper, we prove some results on the uniqueness of meromorphic functions which share some values with their n-th derivatives. Our results improve and generalizes the results due to Gopalakrishna and Bhoosnurmath; Yang; Chen, Chen and Tsai; Lahiri and Pal; R. S. Dyavanal.

1. Introduction and main results

In the paper, by meromorphic functions we always mean meromorphic functions in the open complex plane $\mathbb{C}$. Let f be a non-constant meromorphic function. By $S(r,f)$ we denote any quantity satisfying $S(r,f) = \circ(T(r,f))$ as $r \to \infty$, possibly outside a set of finite linear measure. A meromorphic function $a = a(z)$ is said to be a small function of f if either $a \equiv \infty$ or $T(r,a) = S(r,f)$. We denote by $S(f)$ the collection of all small functions of f. Clearly $\mathbb{C} \cup \{\infty\} \in S(f)$ and $S(f)$ is a field over the set of complex numbers.

For a positive integer p and $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_{p)}(a;f)$ the set of those zeros of $f - a$ whose multiplicities do not exceed p, each zero is counted according to its multiplicities and $\overline{E}_{p)}(a;f)$ the set of those distinct zeros of $f - a$ whose multiplicities do not exceed p, where we mean by a zero of $f - \infty$ a pole of f. Also by $E_{\infty)}(a;f)(\overline{E}_{\infty)}(a;f))$ we denote the set of all zeros of $f - a$ counted with multiplicities(ignoring multiplicities). If $E_{\infty)}(a;f) = E_{\infty)}(a;g)$ $(\overline{E}_{\infty)}(a;f) = \overline{E}_{\infty)}(a;g))$, we say that f and g share a CM(IM). Also we say that a meromorphic function $f(z)$ partially shares a with a meromorphic function $g(z)$ if $\overline{E}_{\infty)}(a;f) \subseteq \overline{E}_{\infty)}(a;g)$.

For $A \subset \mathbb{C}$ we denote by $\overline{N}_A(r,a;f)$ the reduced counting function of those zeros of $f - a$ which belong to the set A, where $a \in \mathbb{C} \cup \{\infty\}$. Clearly if $A = \mathbb{C}$, then $\overline{N}_A(r,a;f) = \overline{N}(r,a;f)$.

For a positive integer p and $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N_{p)}(r,a;f)(\overline{N}_{p)}(r,a;f))$ the counting function (reduced counting function) of those zeros of $f-a$ whose multiplicities do not exceed p. Similarly we define $N_{(p}(r,a;f)(\overline{N}_{(p}(r,a;f))$ the counting

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function (reduced counting function) of those zeros of $f-a$ whose multiplicities greater than equal to p. Also we write $\overline{N}_{\infty)}(r,a;f)=\overline{N}(r,a;f)$.

For standard definitions and notations of Nevanlinna theory we refer the reader to [4, 6]. The modern theory of uniqueness of entire and meromorphic functions was initiated by R. Nevanlinna with his two famous theorems: The Five Value Theorem and The Four Value Theorem. The five value theorem of Nevanlinna may be stated as follows:

Theorem 1[[4], p. 48] Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions and $a_j\in\mathbb{C}\cup\{\infty\}$ be distinct for $j=1,2,...,5$. If $\overline{E}_{\infty)}(a_j;f)=\overline{E}_{\infty)}(a_j;g)$ for $j=1,2,...,5$, then $f(z)\equiv g(z)$.

Gopalakrishna and S. S. Bhoosnurmath [3] improved the above theorem in the following manner.

Theorem 2[3] Let f, g be distinct non-constant meromorphic functions. If there exist distinct elements $a_1,a_2,...,a_k\in\mathbb{C}\cup\{\infty\}$ such that $\overline{E}_{p_j)}(a_j;f)=\overline{E}_{p_j)}(a_j;g)$ for $j=1,2,...,k$, where $p_1,p_2,...,p_k$ are positive integers or ∞ with $p_1\geq p_2\geq...\geq p_k$, then

$$\sum_{j=2}^{k}\frac{p_j}{1+p_j}\leq 2+\frac{p_1}{1+p_1}.$$

C. C. Yang [[6],Theorem 3.2, p. 157] improved Theorem 1 by considering partial sharing of values and proved the following theorem.

Theorem 3[[6],Theorem 3.2, p. 157] Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions such that $\overline{E}_{\infty)}(a_j;f)\subseteq\overline{E}_{\infty)}(a_j;g)$ for five distinct elements $a_1,a_2,...,a_5$ of $\mathbb{C}\cup\{\infty\}$.
If

$$\liminf_{r\to\infty}\frac{\sum\limits_{j=1}^{5}\overline{N}(r,a_j;f)}{\sum\limits_{j=1}^{5}\overline{N}(r,a_j;g)}>\frac{1}{2},$$

then $f(z)\equiv g(z)$.

In 2007 Chen, Chen and Tsai [1] extended Theorem 3 by considering $f(z)$ and $g(z)$ partially sharing more than five values proved the following theorem.

Theorem 4[1] Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions such that $\overline{E}_{\infty)}(a_j;f)\subseteq\overline{E}_{\infty)}(a_j;g)$ for k (≥ 5) distinct elements $a_1,a_2,...,a_k$ of $\mathbb{C}\cup\{\infty\}$.
If

$$\liminf_{r\to\infty}\frac{\sum\limits_{j=1}^{k}\overline{N}(r,a_j;f)}{\sum\limits_{j=1}^{k}\overline{N}(r,a_j;g)}>\frac{1}{k-3},$$

then $f(z)\equiv g(z)$.

In 2012 R. S. Dyavanal [2] improved Theorem 3 and Theorem 4 by considering uniqueness of n-th derivatives of meromorphic functions and proved the following theorem.

Theorem 5[2] Let f and g be two non-constant meromorphic functions and $a_j\in\mathbb{C}\cup\{\infty\}$ be distinct for $j=1,2,...,k$ (≥ 5) and for a non-negative integer

n, if $E_{\infty)}(a_j, f^{(n)}) \subseteq E_{\infty)}(a_j, g^{(n)})$ for $1 \le j \le k$, $E_{\infty)}(0, f) \subseteq E_{\infty)}(0, f^{(n)})$, $E_{\infty)}(0, g) \subseteq E_{\infty)}(0, g^{(n)})$ and

$$\liminf_{r\to\infty} \frac{\sum\limits_{j=1}^{k} N(r, a_j; f^{(n)})}{\sum\limits_{j=1}^{k} N(r, a_j; g^{(n)})} > \frac{n+1}{k-(n+3)},$$

then $f^{(n)} \equiv g^{(n)}$.

I. Lahiri and R. Pal[5] prove the following uniqueness theorem of meromorphic functions sharing k (≥ 5) small functions.

Theorem 6[5] Let f and g be two non-constant meromorphic functions and $a_j = a_j(z) \in S(f) \cap S(g)$ be distinct for $j = 1, 2, ...k$ $(k \ge 5)$. Suppose that $p_1 \ge p_2 \ge ... \ge p_k$ are positive integers or infinity and δ (≥ 0) is such that

$$\frac{1}{p_1} + (1 + \frac{1}{p_1}) \sum_{j=2}^{k} \frac{1}{1+p_j} + 1 + \delta < (k-2)(1 + \frac{1}{p_1}).$$

Let $A_j = \overline{E}_{p_j)}(a_j; f) \backslash \overline{E}_{p_j)}(a_j; g)$ for $j = 1, 2, ..., k$. If $\sum_{j=1}^{k} \overline{N}_{A_j}(r, a_j; f) \le \delta T(r, f)$ and

$$\liminf_{r\to\infty} \frac{\sum_{j=1}^{k} \overline{N}_{p_j)}(r, a_j; f)}{\sum_{j=1}^{k} \overline{N}_{p_j)}(r, a_j; g)} > \frac{p_1}{(k-2)(1+p_1) - (1+p_1)\sum_{j=2}^{k} \frac{1}{1+p_j} - 1 - (1+\delta)p_1},$$

then $f \equiv g$.

In the paper we prove the following theorems:

Theorem 7 Let f and g be two non-constant meromorphic functions and a_j $(j = 1, 2, ..., k)$ be k (≥ 5) distinct complex numbers. For a non-negative integer n, let $A_j = E(a_j; f^{(n)}) \setminus E(a_j; g^{(n)})$ and $\sum_{j=1}^{k} N_{A_j}(r, a_j; f^{(n)}) \le \delta(T(r, f^{(n)}))$, for some δ such that $0 \le \delta \le \frac{kn}{kn+k-1}$. If

$$\lim_{r\to\infty} \inf \frac{\sum_{j=1}^{k} N\left(r, a_j; f^{(n)}\right)}{\sum_{j=1}^{k} N\left(r, a_j; g^{(n)}\right)} > \frac{1}{k-3+\frac{kn}{kn+k-1} - \delta},$$

then $f^{(n)}(z) \equiv g^{(n)}(z)$.

Theorem 8 Let f_1 and f_2 be two non-constant meromorphic functions and $a_j = a_j(z) \in S(f) \cap S(g)$ be distinct for $j = 1, 2, ...k$ $(k \ge 5)$. Suppose that m $(1 \le m \le k)$ is an integer; $p_1 \ge p_2 \ge ... \ge p_k$ are positive integers or infinity and δ (≥ 0) is such that

$$(1 + \frac{1}{p_m}) \sum_{j=m}^{k} \frac{1}{1+p_j} + 2 + \delta < (k-m-1)(1 + \frac{1}{p_m}) + m.$$

Let $A_j = \overline{E}_{p_j)}(a_j; f_1) \setminus \overline{E}_{p_j)}(a_j; f_2)$ for $j = 1, 2, ..., k$. If $\sum_{j=1}^{k} \overline{N}_{A_j}(r, a_j; f_1) \le \delta T(r, f_1)$ and

$$\liminf_{r\to\infty} \frac{\sum_{j=1}^{k} \overline{N}_{p_j)}(r, a_j; f_1)}{\sum_{j=1}^{k} \overline{N}_{p_j)}(r, a_j; f_2)} > \frac{p_m}{(1+p_m)\sum_{j=m}^{k} \frac{p_j}{1+p_j} + (m-2-\delta)p_m - 2(1+p_m)},$$

then $f_1 \equiv f_2$.

Theorem 9 Let $f_1,\ f_2$ be two non-constant meromorphic functions and $a_j \in \mathbb{C}\cup\{\infty\}$ be distinct for $j=1,2,...,k$ $(k\geq 5)$. Suppose that $p_1\geq p_2\geq ...\geq p_k$ are positive integers or infinity and δ (≥ 0) is such that

$$\frac{1}{p_1}+\big(1+\frac{1}{p_1}\big)\sum_{j=2}^{k}\frac{1}{1+p_j}+1+\delta<\frac{k-2}{n+1}\big(1+\frac{1}{p_1}\big)$$

for a non-negative integer n. Let $A_j=\overline{E}_{p_j)}(a_j;f_1^{(n)})\setminus\overline{E}_{p_j)}(a_j;f_2^{(n)})$ for $j=1,2,...,k$ and $E(0;f_i)\subset E(0;f_i^{(n)})$ for $i=1,2$. If $\sum_{j=1}^{k}\overline{N}_{A_j}(r,a_j;f_1^{(n)})\leq \delta T(r,f_1^{(n)})$ and

$$\liminf_{r\to\infty}\frac{\sum_{j=1}^{k}\overline{N}_{p_j)}(r,a_j;f_1^{(n)})}{\sum_{j=1}^{k}\overline{N}_{p_j)}(r,a_j;f_2^{(n)})}>\frac{(n+1)p_1}{(k-2)(1+p_1)-(n+1)(1+p_1)\sum_{j=2}^{k}\frac{1}{1+p_j}-(n+1)\{(1+\delta)p_1+1\}},$$

then $f_1^{(n)}(z)\equiv f_2^{(n)}(z)$.

2. Lemma

In this section we prove some lemmas which is needed in the sequel.

Lemma 1[7] Let f be a non-constant meromorphic function and $a_j\in S(f)$ be distinct for $j=1,\ 2,..,k$. Then for any $\epsilon(>0)$

$$(k-2-\epsilon)T(r,f)\leq\sum_{j=1}^{k}\overline{N}(r,a_j;f)+S(r,f).$$

Lemma 2 ([6], Theorem 1.35, p-49) Let $f(z)$ be a transcendental meromorphic function in the complex plane and $a_1,a_2,...,a_k$ be k (≥ 2) distinct finite complex numbers. Then for any positive integer n, we have

$$\left(k-1-\frac{k-1}{kn+k-1}\right)T(r,f^{(n)})<\sum_{j=1}^{k}N\left(r,a_j;f^{(n)}\right)+\epsilon T(r,f^{(n)})+S(r,f^{(n)}),$$

where ϵ is any positive number.

Lemma 3 Let f be a non-constant meromorphic function and $a_1,a_2,...,a_k$ be k (≥ 3) distinct complex numbers. If for a non-negative integer n, $E(0;f)\subseteq E(0;f^{(n)})$, then $(k-2+o(1))T(r,f)\leq\sum_{j=1}^{k}\overline{N}(r,a_j;f^{(n)})$.

Proof. By the Nevanlinna's first fundamental theorem, we have

$$\begin{aligned}T(r,f) &= T(r,\frac{1}{f})+O(1)\\ &\leq N(r,0;f)+m(r,\frac{f^{(n)}}{f})+m(r,\frac{1}{f^{(n)}})+O(1)\\ &\leq N(r,0;f)+T(r,f^{(n)})-N(r,0;f^{(n)})+S(r,f) \qquad (1)\end{aligned}$$

By the Nevanlinna's second fundamental theorem, we get

$$(k-1)T(r,f^n)\leq\overline{N}(r,\infty;f^{(n)})+\sum_{j=1}^{k-1}\overline{N}(r,a_j;f^{(n)})+\overline{N}(r,0;f^{(n)})+S(r,f).$$

Without loss of generality, we may assume that $a_k = 0$. Otherwise a suitable linear transformation is done. Then the above inequality reduces to

$$(k-1)T(r,f^n) \leq \overline{N}(r,\infty;f^{(n)}) + \sum_{j=1}^{k} \overline{N}(r,a_j;f^{(n)}) + S(r,f) \tag{2}$$

Using (2) in (1), we obtain

$$(k-1)T(r,f) \leq (k-1)N(r,0;f) + \overline{N}(r,\infty;f^n) + \sum_{j=1}^{k} \overline{N}(r,a_j;f^{(n)}) - (k-1)N(r,0;f^{(n)}) + S(r,f)$$

$$\Rightarrow (k-1)T(r,f) \leq (k-1)N(r,0;f) + \overline{N}(r,\infty;f) + \sum_{j=1}^{k} \overline{N}(r,a_j;f^{(n)}) \\ -(k-1)N(r,0;f^{(n)}) + S(r,f). \tag{3}$$

Since $E(0;f) \subseteq E(0;f^{(n)})$, we have from (3)

$$(k-1)T(r,f) \leq \overline{N}(r,\infty;f) + \sum_{j=1}^{k} \overline{N}(r,a_j;f^{(n)}) + S(r,f)$$

$$\Rightarrow (k-2+o(1))T(r,f) \leq \sum_{j=1}^{k} \overline{N}(r,a_j;f^{(n)}).$$

This complete the proof of the lemma. □

3. Proof of Main Theorems

Proof of Theorem 7:

Proof. Let us assume that $f^{(n)}(z) \not\equiv g^{(n)}(z)$. By Lemma 2, we have

$$(k-1-\frac{k-1}{kn+k-1}-\epsilon)T(r,f^{(n)}) < \sum_{j=1}^{k} N\left(r,a_j;f^{(n)}\right) + S(r,f^{(n)}),$$

$$\Rightarrow \left(k-1-\frac{k-1}{kn+k-1}-\epsilon+o(1)\right)T(r,f^{(n)}) < \sum_{j=1}^{k} N\left(r,a_j;f^{(n)}\right) \tag{4}$$

Similarly,

$$\left(k-1-\frac{k-1}{kn+k-1}-\epsilon+o(1)\right)T(r,g^{(n)}) < \sum_{j=1}^{k} N\left(r,a_j;g^{(n)}\right) \tag{5}$$

Now, let $B_j = E(a_j;f^{(n)}) \setminus A_j$, for $j = 1,2,...,k$. Then,

$$\begin{aligned} \sum_{j=1}^{k} N(r,a_j;f^n) &= \sum_{j=1}^{k} N_{A_j}(r,a_j;f^{(n)}) + \sum_{j=1}^{k} N_{B_j}(r,a_j;f^{(n)}) \\ &\leq \delta T(r,f^{(n)}) + N(r,0;f^{(n)}-g^{(n)}) \\ &\leq (1+\delta)T(r,f^{(n)}) + T(r,g^{(n)}). \end{aligned}$$

Using (4) and (5) we have,

$$\Rightarrow \left(k-1-\frac{k-1}{kn+k-1}-\epsilon+o(1)\right)\sum_{j=1}^{k}N(r,a_j;f^{(n)}) \leq (1+\delta)\sum_{j=1}^{k}N(r,a_j;f^{(n)})+\sum_{j=1}^{k}N(r,a_j;g^{(n)}).$$

Therefore,

$$\{k-1-\frac{k-1}{kn+k-1}-\epsilon-(1+\delta)+o(1)\}\sum_{j=1}^{k}N(r,a_j;f^{(n)}) \leq \sum_{j=1}^{k}N(r,a_j;g^{(n)})$$

$$\Rightarrow \frac{\sum_{j=1}^{k}N(r,a_j;f^{(n)})}{\sum_{j=1}^{k}N(r,a_j;g^{(n)})} \leq \frac{1}{k-1-\frac{k-1}{kn+k-1}-\epsilon-(1+\delta)+o(1)}.$$

Since ϵ is arbitrary, taking limit as $r\to\infty$, we have

$$\begin{aligned}\liminf_{r\to\infty}\frac{\sum_{j=1}^{k}N(r,a_j;f^{(n)})}{\sum_{j=1}^{k}N(r,a_j;g^{(n)})} &\leq \frac{1}{k-1-\frac{k-1}{kn+k-1}-(1+\delta)}\\ &= \frac{1}{k-3+\frac{kn}{kn+k-1}-\delta},\end{aligned}$$

which is a contradiction.
Hence $f^{(n)}(z)\equiv g^{(n)}(z)$.

□

Corollary 1 In Theorem 7 if $E_{\infty)}(a_j;f^{(n)})\subseteq E_{\infty)}(a_j;g^{(n)})$, for $j=1,2,...,k$, then $A_j=\phi$. So we can choose $\delta=0$. Then

$$\liminf_{r\to\infty}\frac{\sum_{j=1}^{k}N(r,a_j;f^{(n)})}{\sum_{j=1}^{k}N(r,a_j;g^{(n)})} > \frac{1}{k-3+\frac{kn}{kn+k-1}}.$$

Since $\frac{1}{k-3+\frac{kn}{kn+k-1}} \leq \frac{1}{k-(n+3)}$, therefore Theorem 7 is an improvement of Theorem 5.

Proof of Theorem 8.

Proof. Suppose$f_1\not\equiv f_2$. Then by Lemma 1 we have

$$\begin{aligned}(k-2-\epsilon)T(r,f_1) &< \sum_{j=1}^{k}\overline{N}(r,a_j;f_1)+S(r,f_1)\\ &\leq \sum_{j=1}^{k}\left\{\overline{N}_{p_j)}(r,a_j;f_1)+\overline{N}_{(p_j+1}(r,a_j;f_1)\right\}+S(r,f_1)\\ &\leq \sum_{j=1}^{k}\left\{\overline{N}_{p_j)}(r,a_j;f_1)+\frac{1}{1+p_j}N_{(p_j+1}(r,a_j;f_1)\right\}+S(r,f_1)\\ &\leq \sum_{j=1}^{k}\left\{\frac{p_j}{1+p_j}\overline{N}_{p_j)}(r,a_j;f_1)+\frac{1}{1+p_j}N(r,a_j;f_1)\right\}+S(r,f_1)\\ &\leq \sum_{j=1}^{k}\frac{p_j}{1+p_j}\overline{N}_{p_j)}(r,a_j;f_1)+\sum_{j=1}^{k}\frac{1}{1+p_j}T(r,f_1)+S(r,f_1)(6)\end{aligned}$$

Since $1 \geq \frac{p_1}{1+p_1} \geq \frac{p_2}{1+p_2} \geq ... \geq \frac{p_k}{1+p_k} \geq \frac{1}{2}$, we get from (6)

$$\begin{aligned}(k-2-\epsilon)T(r,f_1) &\leq \sum_{j=1}^{m-1}\{\frac{p_j}{1+p_j}-\frac{p_m}{1+p_m}\}\overline{N}_{p_j)}(r,a_j;f_1)+\sum_{j=1}^{k}\frac{1}{1+p_j}T(r,f_1)\\ &+ \sum_{j=1}^{k}\frac{p_m}{1+p_m}\overline{N}_{p_j)}(r,a_j;f_1)+S(r,f_1)\\ &\leq \sum_{j=1}^{k}\frac{p_m}{1+p_m}\overline{N}_{p_j)}(r,a_j;f_1)\\ &+ \left(m-1-\frac{(m-1)p_m}{1+p_m}+\sum_{j=m}^{k}\frac{1}{1+p_j}\right)T(r,f_1)+S(r,f_1)\end{aligned}$$

i.e.,

$$\left(\sum_{j=m}^{k}\frac{p_j}{1+p_j}+\frac{(m-1)p_m}{1+p_m}-2-\epsilon+o(1)\right)T(r,f_1)\leq\frac{p_m}{1+p_m}\sum_{j=1}^{k}\overline{N}_{p_j)}(r,a_j;f_1) \tag{7}$$

Similarly, we get

$$\left(\sum_{j=m}^{k}\frac{p_j}{1+p_j}+\frac{(m-1)p_m}{1+p_m}-2-\epsilon+o(1)\right)T(r,f_2)\leq\frac{p_m}{1+p_m}\sum_{j=1}^{k}\overline{N}_{p_j)}(r,a_j;f_2) \tag{8}$$

Let $B_j=\overline{E}_{p_j)}(a_j;f_1)\setminus A_j$ for $j=1,2,...,k$ and using (7), (8) we have

$$\begin{aligned}\sum_{j=1}^{k}\overline{N}_{p_j)}(r,a_j;f_1) &= \sum_{j=1}^{k}\overline{N}_{A_j}(r,a_j;f_1)+\sum_{j=1}^{k}\overline{N}_{B_j}(r,a_j;f_1)\\ &\leq \delta T(r,f_1)+N(r,0;f_1-f_2)\\ &\leq (1+\delta)T(r,f_1)+T(r,f_2)+O(1)\end{aligned}$$

i.e.,

$$\left(\sum_{j=m}^{k}\frac{p_j}{1+p_j}+\frac{(m-1)p_m}{1+p_m}-2-\epsilon+o(1)\right)\sum_{j=1}^{k}\overline{N}_{p_j)}(r,a_j;f_1)$$

$$\leq(1+\delta)\frac{p_m}{1+p_m}\sum_{j=1}^{k}\overline{N}_{p_j)}(r,a_j;f_1)+\{1+o(1)\}\frac{p_m}{1+p_m}\sum_{j=1}^{k}\overline{N}_{p_j)}(r,a_j;f_2)$$

$$\left(\sum_{j=m}^{k}\frac{p_j}{1+p_j}+\frac{(m-1)p_m}{1+p_m}-(1+\delta)\frac{p_m}{1+p_m}-2-\epsilon+o(1)\right)\sum_{j=1}^{k}\overline{N}_{p_j)}(r,a_j;f_1)$$

$$\leq\{1+o(1)\}\frac{p_m}{1+p_m}\sum_{j=1}^{k}\overline{N}_{p_j)}(r,a_j;f_2))$$

Since ϵ (>0) is arbitrary, we have

$$\liminf_{r\to\infty}\frac{\sum_{j=1}^{k}\overline{N}_{p_j)}(r,a_j;f_1)}{\sum_{j=1}^{k}\overline{N}_{p_j)}(r,a_j;f_2)}\le\frac{\frac{p_m}{1+p_m}}{\left(\sum_{j=m}^{k}\frac{p_j}{1+p_j}+\frac{(m-1)p_m}{1+p_m}-(1+\delta)\frac{p_m}{1+p_m}-2\right)},$$

which is a contradiction.
Therefore $f_1(z)\equiv f_2(z)$. This completes the proof. $\square$

Corollary 2 For $m=1$ in Theorem 8 we get Theorem 6. Hence Theorem 8 is a generalization of Theorem 6.
Proof of Theorem 9.

Proof. By Lemma 3, we have

$$(k-2+o(1))T(r,f_1)<\sum_{j=1}^{k}\overline{N}(r,a_j;f_1^{(n)}) \tag{9}$$

and

$$(k-2+o(1))T(r,f_2)<\sum_{j=1}^{k}\overline{N}(r,a_j;f_2^{(n)}). \tag{10}$$

From (9) we have

$$\begin{aligned}(k-2+o(1))T(r,f_1) &\le \sum_{j=1}^{k}\{\overline{N}_{p_j)}(r,a_j;f_1^{(n)})+\overline{N}_{(p_j+1}(r,a_j;f_1^{(n)})\}\\ &\le \sum_{j=1}^{k}\{\overline{N}_{p_j)}(r,a_j;f_1^{(n)})+\frac{1}{1+p_j}N_{(p_j+1}(r,a_j;f_1^{(n)})\}\\ &\le \sum_{j=1}^{k}\{\frac{p_j}{1+p_j}\overline{N}_{p_j)}(r,a_j;f_1^{(n)})+\frac{1}{1+p_j}N(r,a_j;f_1^{(n)})\}\\ &\le \sum_{j=1}^{k}\frac{p_j}{1+p_j}\overline{N}_{p_j)}(r,a_j;f_1^{(n)})+\sum_{j=1}^{k}\frac{1}{1+p_j}T(r,f_1^{(n)})\\ &\le \sum_{j=1}^{k}\frac{p_j}{1+p_j}\overline{N}_{p_j)}(r,a_j;f_1^{(n)})+(n+1)\sum_{j=1}^{k}\frac{1}{1+p_j}T(r,f_1)\end{aligned}$$

i.e.,

$$\{(k-2)-(n+1)\sum_{j=1}^{k}\frac{1}{1+p_j}+o(1)\}T(r,f_1)\le\sum_{j=1}^{k}\frac{p_j}{1+p_j}\overline{N}_{p_j)}(r,a_j;f_1^{(n)})$$

Similarly from (10) we get

$$\{(k-2)-(n+1)\sum_{j=1}^{k}\frac{1}{1+p_j}+o(1)\}T(r,f_2)\le\sum_{j=1}^{k}\frac{p_j}{1+p_j}\overline{N}_{p_j)}(r,a_j;f_2^{(n)})$$

Let $B_j=\overline{E}_{p_j)}(a_j;f_1^{(n)})\setminus A_j$ for $j=1,2,...,k$.

Now

$$\begin{aligned}\sum_{j=1}^{k}\overline{N}_{p_j)}(r,a_j;f_1^{(n)}) &= \sum_{j=1}^{k}\overline{N}_{A_j}(r,a_j;f_1^{(n)})+\sum_{j=1}^{k}\overline{N}_{B_j}(r,a_j;f_1^{(n)})\\ &\leq \delta T(r,f_1^{(n)})+N(r,0;f_1^{(n)}-f_2^{(n)})\\ &\leq (1+\delta)(n+1)T(r,f_1)+(n+1)T(r,f_2)\end{aligned}$$

i.e.,

$$\Big\{(k-2)-(n+1)\sum_{j=1}^{k}\frac{1}{1+p_j}+o(1)\Big\}\sum_{j=1}^{k}\overline{N}_{p_j)}(r,a_j;f_1^{(n)})$$
$$\leq \quad (1+\delta)(n+1)\sum_{j=1}^{k}\frac{p_j}{1+p_j}\overline{N}_{p_j)}(r,a_j;f_1^{(n)})+(n+1)\sum_{j=1}^{k}\frac{p_j}{1+p_j}\overline{N}_{p_j)}(r,a_j;f_2^{(n)})$$

Since $1\geq\frac{p_1}{1+p_1}\geq\frac{p_2}{1+p_2}\geq...\geq\frac{p_k}{1+p_k}\geq\frac{1}{2}$, we get from above inequality

$$\Big\{(k-2)-(n+1)\sum_{j=1}^{k}\frac{1}{1+p_j}+o(1)\Big\}\sum_{j=1}^{k}\overline{N}_{p_j)}(r,a_j;f_1^{(n)})$$
$$\leq \quad (1+\delta)(n+1)\frac{p_1}{1+p_1}\sum_{j=1}^{k}\overline{N}_{p_j)}(r,a_j;f_1^{(n)})+(n+1)\frac{p_1}{1+p_1}\sum_{j=1}^{k}\overline{N}_{p_j)}(r,a_j;f_2^{(n)})$$

i.e.,

$$\Big\{(k-2)-(n+1)\sum_{j=1}^{k}\frac{1}{1+p_j}-(1+\delta)(n+1)\frac{p_1}{1+p_1}+o(1)\Big\}\sum_{j=1}^{k}\overline{N}_{p_j)}(r,a_j;f_1^{(n)})$$
$$\leq (n+1)\frac{p_1}{1+p_1}\sum_{j=1}^{k}\overline{N}_{p_j)}(r,a_j;f_2^{(n)})$$

.

Therefore

$$\begin{aligned}&\liminf_{r\to\infty}\frac{\sum_{j=1}^{k}\overline{N}_{p_j)}(r,a_j;f_1^{(n)})}{\sum_{j=1}^{k}\overline{N}_{p_j)}(r,a_j;f_2^{(n)})}\\ &\leq \frac{(n+1)p_1}{(k-2)(1+p_1)-(n+1)(1+p_1)\sum_{j=1}^{k}\frac{1}{1+p_j}-(n+1)(1+\delta)p_1}\\ &= \frac{(n+1)p_1}{(k-2)(1+p_1)-(n+1)(1+p_1)\sum_{j=2}^{k}\frac{1}{1+p_j}-(n+1)\{(1+\delta)p_1+1\}},\end{aligned} \tag{11}$$

which is a contradiction.
Therefore $f_1^{(n)}(z)\equiv f_2^{(n)}(z)$. This complete the proof. □

Corollary 3 Let $p_k=\infty$ and $L=\liminf_{r\to\infty}\frac{\sum_{j=1}^{k}\overline{N}_{\infty)}(r,a_j;f_1^{(n)})}{\sum_{j=1}^{k}\overline{N}_{\infty)}(r,a_j;f_2^{(n)})}>\frac{n+1}{k-(n+3)}$.

If $\sum_{j=1}^{k}\overline{N}_{A_j}(r,a_j;f_1^{(n)})\leq\delta T(r,f_1^{(n)})$, for some δ with $0\leq\delta<\frac{k-(n+3)}{n+1}-\frac{1}{L}$, then $f_1^{(n)}(z)\equiv f_2^{(n)}(z)$.

If we assume $E_{\infty)}(a_j; f_1^{(n)}) \subseteq E_{\infty)}(a_j; f_2^{(n)})$, then $A_j = \phi$ for $j = 1, 2, ..., k$ and so we can choose $\delta = 0$. Choosing $n = 0$ we get Theorem 4.

Corollary 4 For $k = 5$, then Corollary 1 is reduced to Theorem 3.

Corollary 5 Let $f_1 \not\equiv f_2$. For $n = 0$ and $\overline{E}_{p_j)}(a_j; f_1) = \overline{E}_{p_j)}(a_j; f_2)$ for $j = 1, 2, ..., k$, we have $A_j = \phi$, therefore we can choose $\delta = 0$. We have from (11)

$$1 \leq \frac{p_1}{(1 + p_1)(k - 2) - (1 + p_1) \sum_{j=2}^{k} \frac{1}{1+p_j} - (1 + p_1)}$$

$$\Rightarrow \sum_{j=2}^{k} \frac{p_j}{1 + p_j} \leq \frac{p_1}{(1 + p_1)} + 2.$$

hence Theorem 9 reduced to Theorem 2.

Example 1 Let $f(z) = e^z + a$ and $g(z) = e^z + b$ where a, b $(a \neq b)$ are constants. Then $E(a_j; f') = E(a_j; g')$ so, $A_j = \phi$ for $j = 1, 2, ..., 5$, we can choose $\delta = 0$ and

$$\lim_{r\to\infty} \inf \frac{\sum_{j=1}^{5} N(r, a_j; f')}{\sum_{j=1}^{5} N(r, a_j; g')} = 1 > \frac{9}{23}.$$

Therefore by Theorem 7 we have $f'(z) \equiv g'(z)$.

Example 2 Let $f(z) = \frac{i}{e^z+1}$ and $g(z) = \frac{-ie^z}{e^z+1}$. Clearly, $E(0; f) \subset E(0; f')$ and $E(0; g) \subset E(0; g')$. Since $\overline{E}(a_j; f') = \overline{E}(a_j; g')$ so, $A_j = \phi$ for $j = 1, 2, ..., 7$, we can choose $\delta = 0$ and

$$\liminf_{r\to\infty} \frac{\sum_{j=1}^{k} \overline{N}(r, a_j; f')}{\sum_{j=1}^{k} \overline{N}(r, a_j; g')} = 1 > \frac{2}{3}$$

Therefore by Theorem 9 we have $f'(z) \equiv g'(z)$.

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