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CONTROLLABILITY OF SECOND-ORDER STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY FRACTIONAL BROWNIAN MOTION

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Abstract. In this paper, we study the controllability of a class of second-order impulsive neutral stochastic differential equations driven by fractional Brownian motion (fBm) with infinite delay in Hilbert spaces. The Banach fixed point theorem and the theory of strongly continuous cosine families of operators are used to investigate the sufficient conditions for the controllability of the system considered. An example is provided to illustrate our results.

1. Introduction

Stochastic differential equations (SDEs) driven by Brownian motions are widely used in practice, such as physical systems, finance and economic areas [12, 20, 23]. Among them, several properties of SDEs such as existence, uniqueness, stability and controllability are studied for the first-order equations. But in many situations, it is useful to investigate the second-order abstract differential equations directly rather than to convert them to first-order systems introduced by Fitzgibbon [13]. The second-order stochastic differential equations are the right model in continuous time to account for integrated processes that can be made stationary. For instance, it is useful for engineers to model mechanical vibrations or charge on a capacitor or condenser subjected to white noise excitation by second-order stochastic differential equations [12].

Controllability, as a fundamental concept of control theory, plays an important role both in stochastic and deterministic control problems such as stabilization of unstable systems by feedback control. Control problems for stochastic distributed parameter systems, is, in our opinion, still at its very beginning stage. Recently, by using the cosine family of operators, stochastic analysis techniques, Muthukumar and Rajivganthi [22] considered approximate controllability of second-order neutral stochastic differential equations with infinite delay and Poisson jumps. Huan [17] have studied approximate controllability of damped second-order impulsive neutral stochastic integro-differential system with state-dependent delay. In [3], Arthi and Balachandran established the controllability of damped second-order impulsive

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neutral functional differential systems with infinite delay by means of the Sadovskii fixed point theorem combined with a noncompact condition on the cosine family of operators. Balasubramaniam and Muthukumar [4] proved sufficient conditions for the approximate controllability of second-order stochastic distributed implicit functional differential systems with infinite. In [27], Sakthivel et al. studied approximate controllability of second-order stochastic differential equations with impulsive effects. Very recently, Huan et al. [19] established the approximate controllability of the time-dependent impulsive neutral SDEs with memory by using the Holder's inequality, stochastic analysis and fixed point strategy.

On the other hand, there has not been very much study of the controllability of impulsive neutral SDEs with delay and fBm, while these have begun to gain attention recently. To be more precise, in [10], Chen considered the approximate controllability for semilinear stochastic equations driven by fBm. By using stochastic analysis theory and operator theory, Cui and Yan [11] investigated the controllability for neutral stochastic evolution equations driven by fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$. Boudaoui and Lakhel [7] studied the controllability results of impulsive neutral stochastic functional differential equations with infinite delay driven by fBm in a real separable Hilbert space. Moreover, by using the Banach fixed point theorem, Ahmed [1] studied the controllability of impulsive neutral stochastic functional differential equations with finite delay and fBm in a Hilbert space. Up to now, to the best of the authors knowledge, no results about the controllability of stochastic second-order differential systems with fBm are available in the literature. The present paper is devoted to study the controllability of impulsive neutral SDEs with infinite delay and fBm in Hilbert spaces. More precisely, we consider the following form:

$$\begin{cases} d\big[x'(t) - g\big(t, x_t, \int_0^t \sigma_1(t,s,x_s)ds\big)\big] = \big[Ax(t) + f\big(t, x_t, \int_0^t \sigma_2(t,s,x_s)ds\big)\big]dt \\ \qquad\qquad\qquad\qquad\qquad\qquad + Bu(t)dt + \sigma(t)dB^H(t), \quad t \in J := [0,T], \\ \Delta x(t_k) = I_k^1(x_{t_k}), \quad k = 1, 2.., m, \\ \Delta x'(t_k) = I_k^2(x_{t_k}), \quad k = 1, 2.., m, \\ x'(0) = x_1 \in \mathbb{H}, \\ x_0 = \varphi \in \mathcal{B}, \quad \text{for a.e. } s \in J_0 := (-\infty, 0], \end{cases} \tag{1}$$

where $0 < t_1 < t_2 < \cdots < t_n < T$, $n \in \mathbb{N}$; $x(\cdot)$ is a stochastic process taking values in a real separable Hilbert space $\mathbb{H}$; $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ is the infinitesimal generator of a strongly continuous cosine family on $\mathbb{H}$. The history $x_t : J_0 \to \mathbb{H}$, $x_t(\theta) = x(t+\theta)$ for $t \geq 0$, belongs to the phase space $\mathcal{B}$, which will be described later. Assume that the mappings $f, g : J \times \mathcal{B} \times \mathbb{H} \to \mathbb{H}$, $\sigma : J \to \mathcal{L}_2^0$, $\sigma_i : J \times J \times \mathcal{B} \to \mathbb{H}$, $i = 1, 2$, $I_k^1, I_k^2 : \mathcal{B} \to \mathbb{H}$, $k = 1, 2, ..., m$ are appropriate functions to be specified later. The control function $u(\cdot)$ takes values in $\mathcal{L}_2(J, U)$ of admissible control functions for a separable Hilbert space U and B is a bounded linear operator from U into $\mathbb{H}$. Furthermore, let $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$ be prefixed points, and $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, represents the jump of the function x at time t_k with I_k determining the size of the jump, where $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively. Similarly $x'(t_k^+)$ and $x'(t_k^-)$ denote, respectively, the right and left limits of $x'(t)$ at t_k.

The structure of this paper is as follows: In Section 2, we briefly present some basic notations, preliminaries and assumptions. The main results in Section 3 are devoted to study the controllability for the system (1) with their proofs. An example is given in Section 4 to illustrate the theory.

2. Preliminaries

In this section, we introduce notations and preliminary results need to establish our results. For more details on this section, we refer the reader to [2, 12, 21, 25, 26, 16].

Let $(\mathbb{H}, \|\cdot\|_{\mathbb{H}}, \langle\cdot,\cdot\rangle)$ and $(\mathbb{K}, \|\cdot\|_{\mathbb{K}}, \langle\cdot,\cdot\rangle)$ denote two real separable Hilbert spaces, with their vectors norms and their inner products, respectively. We denote by $\mathcal{L}(\mathbb{K};\mathbb{H})$ the set of all linear bounded operators from $\mathbb{K}$ into $\mathbb{H}$, which is equipped with the usual operator norm $\|\cdot\|$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in J}, \mathbf{P})$ be a complete filtered probability space satisfying the usual condition (i.e., it is right continuous and $\mathcal{F}_0$ contains all $\mathbf{P}$-null sets). Denote $\{B^H(t)\}_{t\in J}$ an fBm to the filtration $\{\mathcal{F}_t\}_{t\in J}$.

An one-dimensional fBm with Hurst parameter $H \in (0,1)$ is a centered Gaussian process $\beta^H = \{\beta^H(t)\}_{t\in J}$ with covariance function

$$R(t,s) = \mathbf{E}[\beta^H(t)\beta^H(s)] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}).$$

We note that $\beta^{\frac{1}{2}}$ is a standard Brownian motion. It is known that $\beta^H(t)$ with $H \in (\frac{1}{2}, 1)$ has the following Volterra representation:

$$\beta^H(t) = \int_0^t K_H(t,s)d\beta(s), \tag{2}$$

where $\beta = \{\beta(t)\}_{t\in J}$ is a Wiener process and the Volterra kernel $K_H(t,s)$ is given by

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_0^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du$$

where

$$c_H = \sqrt{\frac{H(2H-1)}{B(2-2H, H-\frac{1}{2})}}$$

with $B(\cdot,\cdot)$ being the Beta function for $t > s$. We put $K_H(t,s) = 0$ if $t \le s$.

For the deterministic function $\varphi \in \mathcal{L}^2(J)$, the fractional Wiener integral of φ with respect to β^H is defined by

$$\int_J \varphi(s)d\beta^H(s) = \int_J K_H^*\varphi(s)d\beta(s),$$

where

$$K_H^*\varphi(s) = \int_s^T \varphi(r)\frac{\partial K}{\partial r}(r,s)dr.$$

We assume that there exists a complete orthonormal system $\{e_k\}_{k\ge1}$ in $\mathbb{K}$, a sequence of nonnegative real numbers λ_k such that $Qe_k = \lambda_k e_k, k = 1,2,...$, where $Q \in \mathcal{L}(\mathbb{K};\mathbb{H})$ with finite trace $tr(Q) = \sum_{k=1}^{\infty}\lambda_k < \infty$. We define the infinite dimensional fractional Brownian motion on $\mathbb{K}$ with covariance Q as

$$B^H(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} e_k \beta_k^H(t),$$

where β_k^H are real, independent fBm's. This process is a $\mathbb{K}$-valued Gaussian starting from 0 with zero mean and covariance:

$$\mathbf{E}\langle B^H(t), x\rangle\langle B^H(s), y\rangle = R(t,s)\langle Q(x), y\rangle \quad \forall x, y \in \mathbb{K}, \quad \forall t, s \in J.$$

Let $\mathcal{L}_2^0 = \mathcal{L}_2(Q^{\frac{1}{2}}\mathbb{K}; \mathbb{H})$ be the space of all Hilbert-Schmidt operators from $Q^{\frac{1}{2}}\mathbb{K}$ into $\mathbb{H}$ with the inner product $\langle a, b\rangle_{\mathcal{L}_2^0} = Tr[aQb^*]$, where b^* is the adjoint of the operator b.

The fractional Wiener integral of $\Psi : J \to \mathcal{L}_2^0$ with respect to Q-fBm is defined by

$$\int_0^t \Psi(s) dB^H(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n} \Psi(s) e_n d\beta_n^H(s) \tag{3}$$
$$= \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n} K_H^*(\Psi e_n) d\beta_n(s)$$

where β_n is the standard Brownian motion used to present β_n^H as in (2).

We have the following inequality which is instrumental to prove our results. ([6], Lemma 2) If $\Psi : J \to \mathcal{L}_2^0$ satisfies $\int_J \|\Psi(s)\|^2_{\mathcal{L}_2^0} ds < \infty$ then the above sum in (2.2) is well defined as a $\mathbb{H}$-valued random variable, and we have

$$\mathbf{E}\Big\|\Psi(s) dB^H(s)\Big\|^2 \le 2Ht^{2H-1} \int_0^t \|\Psi(s)\|^2_{\mathcal{L}_2^0} ds.$$

Next, to be able to access controllability for the system (1), we need to introduce theory of cosine functions of operators and the second order abstract Cauchy problem.

(1) The one-parameter family $\{C(t)\}_{t\in\mathbb{R}} \subset \mathcal{L}(\mathbb{H})$ is said to be a strongly continuous cosine family if the following hold:

(i) $C(0) = I$, I is the identity operators in $\mathbb{H}$;

(ii) $C(t)x$ is continuous in t on $\mathbb{R}$ for any $x \in \mathbb{H}$;

(iii) $C(t+s) + C(t-s) = 2C(t)C(s)$ for all $t, s \in \mathbb{R}$.

(2) The corresponding strongly continuous sine family $\{S(t)\}_{t\in\mathbb{R}} \subset \mathcal{L}(\mathbb{H})$, associated to the given strongly continuous cosine family $\{C(t)\}_{t\in\mathbb{R}} \subset \mathcal{L}(\mathbb{H})$ is defined by

$$S(t)x = \int_0^t C(s)x ds, \quad t \in \mathbb{R}, x \in \mathbb{H}.$$

(3) The infinitesimal generator $A : \mathbb{H} \to \mathbb{H}$ of $\{C(t)\}_{t\in\mathbb{R}} \subset \mathcal{L}(\mathbb{H})$ is given by

$$Ax = \frac{d^2}{dt^2} C(t)x\Big|_{t=0},$$

for all $x \in D(A) = \{x \in \mathbb{H} : C(\cdot) \in \mathcal{C}^2(\mathbb{R}, \mathbb{H})\}$. It is well known that the infinitesimal generator A is a closed, densely defined operator on $\mathbb{H}$, and the following properties hold, see Travis and Webb [30]. Suppose that A is the infinitesimal generator of a cosine family of operators $\{C(t)\}_{t\in\mathbb{R}}$. Then, the following hold:

(i) There exist a pair of constants $M_A \ge 1$ and $\alpha \ge 0$ such that $\|C(t)\| \le M_A e^{\alpha|t|}$ and hence, $\|S(t)\| \le M_A e^{\alpha|t|}$;

(ii) $A\int_s^r S(u)xdu = [C(r)-C(s)]x$, for all $0 \le s \le r < \infty$;

(iii) There exist $N \ge 1$ such that $\|S(s)-S(r)\| \le N\big|\int_s^r e^{\alpha|s|}ds\big|$, $0 \le s \le r < \infty$.
Thanks to the Proposition 2.1 and the uniform boundedness principle, as a direct consequence we see that both $\{C(t)\}_{t\in J}$ and $\{S(t)\}_{t\in J}$ are uniformly bounded by $\widetilde{M} = M_A e^{\alpha|T|}$.

The existence of solutions for the second order linear abstract Cauchy problem

$$\begin{cases} x''(t) &= Ax(t)+h(t), \quad t\in J,\\ x(0) &= z, \quad x'(0)=w, \end{cases} \tag{4}$$

where $h: J \to \mathbb{H}$ is an integrable function has been discussed in reference [28]. Similarly, the existence of solutions of the semilinear second order abstract Cauchy problem it has been treated in reference [30]. The function $x(\cdot)$ given by

$$x(t) = C(t)z + S(t)w + \int_0^t S(t-s)h(s)ds, \quad t\in J,$$

is called a mild solution of (4), and that when $z \in \mathbb{H}$, $x(\cdot)$ is continuously differentiable and

$$x'(t) = AS(t)z + C(t)w + \int_0^t C(t-s)h(s)ds, \quad t \in J.$$

For additional details about cosine function theory, we refer to the reader to references [14, 28, 30].

The collection of all strongly-measurable, square-integrable $\mathbb{H}$-valued random variables, denoted by $\mathcal{L}_2(\Omega,\mathbb{H})$, is a Banach space equipped with norm $\|x\|_{\mathcal{L}_2} = \big(\mathbf{E}\|x\|^2\big)^{\frac{1}{2}}$. Let $\mathcal{C}(J,\mathcal{L}_2(\Omega,\mathbb{H}))$ be the Banach space of all continuous map from J to $\mathcal{L}_2(\Omega,\mathbb{H})$ satisfying the condition $\sup_{t\in J}\mathbf{E}\|x(t)\|^2 < \infty$. An important subspace is given by $\mathcal{L}_2^0(\Omega,\mathbb{H}) = \{f \in \mathcal{L}_2(\Omega,\mathbb{H}) : \text{f is } \mathcal{F}_0\text{-measurable}\}$. Further, let $\mathcal{L}_2^{\mathbb{F}}(0,T;\mathbb{H}) =$

$$\Big\{g : J\times\Omega \to \mathbb{H} : g(\cdot) \text{ is } \mathbb{F}\text{-progressively measurable and } \mathbf{E}\Big(\int_J \|g(t)\|_{\mathbb{H}}^2 dt\Big) < \infty\Big\}.$$

Since the system (1) has impulsive effects, the phase space used in Balasubramaniam and Ntouyas [5] and Park et al. [24] cannot be applied to these systems. So, we need introduce an abstract phase space $\mathcal{B}$ as follows.

Assume that $l : J_0 \to (0,+\infty)$ is a continuous function with $l_0 = \int_{J_0} l(t)dt < \infty$. For any $a > 0$, we define

$\mathcal{B} := \Big\{\psi : J_0 \to \mathbb{H} : (\mathbf{E}\|\psi(\theta)\|^2)^{\frac{1}{2}}$ is a bounded and measurable function on $[-a,0]$ and

$\int_{J_0} l(s)\sup_{\theta\in[s,0]}(\mathbf{E}\|\psi(\theta)\|^2)^{\frac{1}{2}}ds < +\infty\Big\}$.

If $\mathcal{B}$ is endowed with the norm

$$\|\psi\|_{\mathcal{B}} = \int_{J_0} l(s) \sup_{\theta\in[s,0]} (\mathbf{E}\|\psi(\theta)\|^2)^{\frac{1}{2}}ds, \quad \forall\psi\in\mathcal{B},$$

then it is clear that $(\mathcal{B},\|\cdot\|_{\mathcal{B}})$ is a Banach space (see, e.g. [15]).

Let $J_T = (-\infty, T]$. We consider the space

$\mathcal{B}_T := \Big\{x : J_T \to \mathbb{H}$ such that $x_k \in \mathcal{C}(J_k,\mathbb{H})$ and there exist $x(t_k^-)$ and $x(t_k^+)$ with

$$x(t_k^-) = x(t_k^+),\ x(0) = \varphi \in \mathcal{B},\ k = 1, 2, ..., m\Big\},$$

where x_k is the restriction of x to $J_k = (t_k, t_{k+1}]$, $k = 1, 2, ..., m$. Set $\|\cdot\|_T$ be a seminorm in $\mathcal{B}_T$ defined by

$$\|x\|_T = \|\varphi\|_{\mathcal{B}} + \sup_{s\in J}(\mathbf{E}\|x(s)\|^2)^{\frac{1}{2}}, \quad x \in \mathcal{B}_T.$$

Now, we recall the following useful lemma appeared in reference [9]. ([9]) Assume that $x \in \mathcal{B}_T$, then for $t \in J$, $x_t \in \mathcal{B}$. Moreover,

$$l_0\big(\mathbf{E}\|x(t)\|^2\big)^{\frac{1}{2}} \le \|x_t\|_{\mathcal{B}} \le \|x_0\|_{\mathcal{B}} + l_0 \sup_{s\in[0,t]}(\mathbf{E}\|x(s)\|^2)^{\frac{1}{2}}.$$

Next, we give the definition of mild solution for (1). An $\mathcal{F}_t$-adapted stochastic process $x : J_T \to \mathbb{H}$ is called a mild solution of (1) on J_T if $x_0 = \varphi \in \mathcal{B}$ and $x'(0) = x_1 \in \mathbb{H}$ satisfying $\varphi, x_1 \in \mathcal{L}_2^0(\Omega, \mathbb{H})$, the functions $C(t-s)g(s, x_s, \int_0^s \sigma_1(s, \tau, x_\tau)d\tau)$ and $S(t-s)f(s, x_s, \int_0^s \sigma_2(s, \tau, x_\tau)d\tau)$ are integrable on $[0, T)$ such that the following conditions hold:

(i) $\{x_t : t \in J\}$ is a $\mathcal{B}$-valued stochastic process;

(ii) For arbitrary $t \in J$, $x(t)$ satisfies the following integral equation:

$$\begin{aligned} x(t) =& C(t)x_0) + S(t)[x_1 - g(0, x_0, 0)] + \int_0^t C(t-s)g(s, x_s, \int_0^s \sigma_1(s, \tau, x_\tau)d\tau)ds \\ &+ \int_0^t S(t-s)f(s, x_s, \int_0^s \sigma_2(s, \tau, x_\tau)d\tau)ds + \int_0^t S(t-s)Bu(s)ds \\ &+ \int_0^t S(t-s)\sigma(s)dB^H(s) + \sum_{0<t_k<t} C(t-t_k)I_k^1(x_{t_k}) + \sum_{0<t_k<t} S(t-t_k)I_k^2(x_{t_k}); \end{aligned} \tag{5}$$

(iii) $\Delta x(t_k) = I_k^1(x_{t_k})$, $\Delta x'(t_k) = I_k^2(x_{t_k})$, $k = 1, 2, ..., m$. The system (1) is said to be controllable on the interval J_T, if for every initial stochastic process $\varphi \in \mathcal{B}$ defined on J_0, $x'(0) = x_1 \in \mathbb{H}$ and $y_1 \in \mathbb{H}$, there exists a stochastic control $u \in L^2(J, U)$ which is adapted to the filtration $\{\mathcal{F}_t\}_{t\in J}$ such that the solution $x(\cdot)$ of the system (1.1) satisfies $x(T) = y_1$, where y_1 and T are preassigned the terminal state and time respectively. To prove our main results, we list the following basic assumptions of this paper.

(**H**1) There exists positive constants M_C, M_S such that for all $t \in J$,

$$\|C(t)\|^2 \le M_C, \quad \|S(t)\|^2 \le M_S.$$

(**H**2) There exists a positive constant M_{σ_1} such that for all $t, s \in J$, $x, y \in \mathcal{B}$

$$\mathbf{E}\Big\|\int_0^t [\sigma_1(t, s, x) - \sigma_1(t, s, y)]ds\Big\|^2 \le M_{\sigma_1}\|x - y\|_{\mathcal{B}}^2.$$

(**H**3) The function $g : J \times \mathcal{B} \times \mathbb{H} \to \mathbb{H}$ is continuous and there exists a positive constant M_g such that for all $t \in J$, $x_1, x_2 \in \mathcal{B}$, $y_1, y_2 \in \mathcal{L}_2(\Omega, \mathbb{H})$

$$\mathbf{E}\|g(t, x_1, y_1) - g(t, x_2, y_2)\|^2 \le M_g(\|x_1 - x_2\|_{\mathcal{B}}^2 + \mathbf{E}\|y_1 - y_2\|^2).$$

(**H**4) For each $(t, s) \in J \times J$, the function $\sigma_2 : J \times J \times \mathcal{B} \to \mathbb{H}$ is continuous and there exists a positive constant M_{σ_2} such that for all $t, s \in J$, $x, y \in \mathcal{B}$

$$\mathbf{E}\Big\|\int_0^t [\sigma_2(t, s, x) - \sigma_2(t, s, y)]ds\Big\|^2 \le M_{\sigma_2}\|x - y\|_{\mathcal{B}}^2.$$

(**H**5) The function $f : J \times \mathcal{B} \times \mathbb{H} \to \mathbb{H}$ is continuous and there exists a positive constant M_f such that for all $t \in J$, $x_1, x_2 \in \mathcal{B}$, $y_1, y_2 \in \mathcal{L}_2(\Omega, \mathbb{H})$

$$\mathbf{E}\|f(t, x_1, y_1) - f(t, x_2, y_2)\|^2 \le M_f(\|x_1 - x_2\|^2_{\mathcal{B}} + \mathbf{E}\|y_1 - y_2\|^2).$$

(**H**6) The functions $I^1_k,\ I^2_k \in \mathcal{C}(\mathcal{B}, \mathbb{H}),\ k = 1, 2, ..., m$ and there exist positive constants $M_{I^1_k}, \overline{M}_{I^1_k},\ M_{I^2_k}, \overline{M}_{I^2_k}$ such that for all $x, y \in \mathcal{B}$

$$\mathbf{E}\|I^1_k(x)\|^2 \le M_{I^1_k}, \quad \mathbf{E}\|I^2_k(x)\|^2 \le M_{I^2_k};$$

$$\mathbf{E}\|I^1_k(x) - I^1_k(y)\|^2 \le \overline{M}_{I^1_k}\|x - y\|^2_{\mathcal{B}}, \quad \mathbf{E}\|I^2_k(x) - I^2_k(y)\|^2 \le \overline{M}_{I^2_k}\|x - y\|^2_{\mathcal{B}}.$$

(**H**7) The function $\sigma : J \to \mathcal{L}^0_2$ satisfies the following conditions: $\int_J \|\sigma(s)\|^2_{\mathcal{L}^0_2} < \infty$.

(**H**8) The linear operator $W : \mathcal{L}_2(J, U) \to \mathcal{L}_2(\Omega, \mathbb{H})$ defined by

$$Wu = \int_J S(T - s)Bu(s)ds$$

has an induced inverse W^{-1} which takes values in $L^2(J, U)/KerW$ (see [8]) and there exist two positive constants M_B and M_W such that

$$\|B\|^2 \le M_B \quad and \quad \|W^{-1}\|^2 \le M_W.$$

(**H**9) Assume that the following relationship holds:

$$C_1 := T \sup_{(t,s)\in J\times J} \sigma^2_1(t, s, 0), \quad C_2 := \sup_{t\in J} \|g(t, 0, 0)\|^2,$$
$$C_3 := T \sup_{(t,s)\in J\times J} \sigma^2_2(t, s, 0), \quad C_4 := \sup_{t\in J} \|f(t, 0, 0)\|^2,$$

$$\Sigma := 56T^2 l^2_0\big(1 + 8T^2 M_B M_S M_W\big)\Big[M_C M_g(1 + 2M_{\sigma_1}) + M_S M_f(1 + 2M_{\sigma_2})\Big],$$

$$\Delta := \Bigg\{10l^2_0\big(1 + 4T^2 M_B M_S M_W\big) \times \Big[T^2 M_C M_g\big(1 + M_{\sigma_1}\big) + T^2 M_S M_f\big(1 + M_{\sigma_2}\big) + mM_C\sum_{k=1}^{m}\overline{M}_{I^1_k} + mM_S\sum_{k=1}^{m}\overline{M}_{I^2_k}\Big]\Bigg\}.$$

3. Main Theorems

In this section, we shall investigate the controllability of impulsive neutral SDEs with infinite delay and fBm in Hilbert spaces.

The main result of this paper is the following theorem.

Theorem 3.1. Assume that the assumptions (**H1**) − (**H9**) hold. If $\Sigma < 1$ and $\Delta < 1$, then the system (1) is controllable on J_T.

Proof. Using the hypothesis (**H**8) for an arbitrary function $x(\cdot)$, define the control process

$$\begin{aligned} u_x^T(t) =& W^{-1}\Big\{y_1 - C(T)x_0 - S(T)[x_1 - g(0,x_0,0)] \\ & - \int_0^T C(T-s)g(s,x_s,\int_0^s \sigma_1(s,\tau,x_\tau)d\tau)ds - \sum_{0<t_k<T} C(T-t_k)I_k^1(x_{t_k}) \\ & - \int_0^T S(T-s)f(s,x_s,\int_0^s \sigma_2(s,\tau,x_\tau)d\tau)ds - \sum_{0<t_k<T} S(T-t_k)I_k^2(x_{t_k}) \\ & - \int_0^T S(T-s)\sigma(s)dB^H(s)\Big\}(t). \end{aligned} \tag{6}$$

We transform (1) into a fixed point problem. Consider the operator $\Pi : \mathcal{B}_T \to \mathcal{B}_T$ defined by

$$\begin{aligned} \Pi x(t) =& \varphi(t), \quad t \in J_0; \\ \Pi x(t) =& C(t)x_0 + S(t)[x_1 - g(0,x_0,0)] \\ & + \int_0^t C(t-s)g(s,x_s,\int_0^s \sigma_1(s,\tau,x_\tau)d\tau)ds \\ & + \int_0^t S(t-s)f(s,x_s,\int_0^s \sigma_2(s,\tau,x_\tau)d\tau)ds + \int_0^t S(t-s)Bu_x^T(s)ds \\ & + \int_0^t S(t-s)\sigma(s)dB^H(s) \\ & + \sum_{0<t_k<t} C(t-t_k)I_k^1(x_{t_k}) + \sum_{0<t_k<t} S(t-t_k)I_k^2(x_{t_k}), \text{ for a.e. } t \in J. \end{aligned}$$

In what follows, we shall show that using the control $u_x^T(\cdot)$ the operator Π has a fixed point, which is then a mild solution for system (1).

Clearly, $\Pi x(T) = y_1$.

For $\varphi \in \mathcal{B}$, we defined $\widetilde{\varphi}$ by

$$\widetilde{\varphi}(t) = \begin{cases} \varphi(t) & \text{if } t \in J_0, \\ C(t)\varphi(0) & \text{if } t \in J, \end{cases}$$

then $\widetilde{\varphi} \in \mathcal{B}_T$.

Set $x(t) = z(t) + \widetilde{\varphi}(t)$, $t \in J_T$. It is easy to see that x satisfies (5) if and only if z satisfies $z_0 = 0$, $x'(0) = x_1 = z'(0) = z_1$ and

$$\begin{aligned} z(t) =& S(t)[z_1 - g(0,\widetilde{\varphi}_0,0)] + \int_0^t C(t-s)g(s,z_s+\widetilde{\varphi}_s,\int_0^s \sigma_1(s,\tau,z_\tau+\widetilde{\varphi}_\tau)d\tau)ds \\ & + \int_0^t S(t-s)f(s,z_s+\widetilde{\varphi}_s,\int_0^s \sigma_2(s,\tau,z_\tau+\widetilde{\varphi}_\tau)d\tau)ds \\ & + \int_0^t S(t-s)Bu_{z+\widetilde{\varphi}}^T(s)ds + \int_0^t S(t-s)\sigma(s)dB^H(s) \\ & + \sum_{0<t_k<t} C(t-t_k)I_k^1(z_{t_k}+\widetilde{\varphi}_{t_k}) + \sum_{0<t_k<t} S(t-t_k)I_k^2(z_{t_k}+\widetilde{\varphi}_{t_k}), \quad t \in J, \end{aligned}$$

where $u_{z+\widetilde{\varphi}}^T(t)$ is obtained from (6) by replacing $x_t = z_t + \widetilde{\varphi}_t$.

Let $\mathcal{B}_T^0 = \{z \in \mathcal{B}_T : z_0 = 0 \in \mathcal{B}\}$. For any $y \in \mathcal{B}_T^0$, we have

$$\|z\|_T = \|z_0\|_{\mathcal{B}} + \sup_{s\in J}(\mathbf{E}\|z(s)\|^2)^{\frac{1}{2}} = \sup_{s\in J}(\mathbf{E}\|z(s)\|^2)^{\frac{1}{2}},$$

and thus $(\mathcal{B}_T^0, \|\cdot\|_T)$ is a Banach space. Set

$$B_r = \{z \in \mathcal{B}_T^0 : \|z\|_T^2 \le r\} \quad \text{for some } r \ge 0,$$

then $B_r \subseteq \mathcal{B}_T^0$ is a bounded closed convex set, and for $u \in B_r$, by Lemma 2.2, we have

$$\begin{aligned}
\|z_t + \widetilde{\varphi}_t\|_{\mathcal{B}}^2 &\le 2(\|z_t\|_{\mathcal{B}}^2 + \|\widetilde{\varphi}_t\|_{\mathcal{B}}^2) \\
&\le 4\big(l_0^2 \sup_{s\in[0,t]}(\mathbf{E}\|z(s)\|^2 + \|z_0\|_{\mathcal{B}}^2 + l_0^2 \sup_{s\in[0,t]}(\mathbf{E}\|\widetilde{\varphi}(s)\|^2 + \|\widetilde{\varphi}_0\|_{\mathcal{B}}^2\big) \\
&\le 4l_0^2\Big(r + M_C\mathbf{E}\|\varphi(0)\|^2\Big) + 4\|\widetilde{\varphi}\|_{\mathcal{B}}^2 \\
&=: k. \qquad (7)
\end{aligned}$$

Define the map $\overline{\Pi} : \mathcal{B}_T^0 \to \mathcal{B}_T^0$ defined by $\overline{\Pi}z(t) = 0$, for $t \in J_0$ and

$$\begin{aligned}
\overline{\Pi}z(t) =& S(t)[z_1 - g(0, \widetilde{\varphi}_0, 0)] + \int_0^t C(t-s)g(s, z_s + \widetilde{\varphi}_s, \int_0^s \sigma_1(s,\tau, z_\tau + \widetilde{\varphi}_\tau)d\tau)ds \\
&+ \int_0^t S(t-s)f(s, z_s + \widetilde{\varphi}_s, \int_0^s \sigma_2(s,\tau, z_\tau + \widetilde{\varphi}_\tau)d\tau)ds \\
&+ \int_0^t S(t-s)Bu_{z+\widetilde{\varphi}}^T(s)ds + \int_0^t S(t-s)\sigma(s)dB^H(s) \\
&+ \sum_{0<t_k<t} C(t-t_k)I_k^1(z_{t_k} + \widetilde{\varphi}_{t_k}) + \sum_{0<t_k<t} S(t-t_k)I_k^2(z_{t_k} + \widetilde{\varphi}_{t_k}), \quad t \in J.
\end{aligned}$$

Obviously, the operator Π has a fixed point which is equivalent to prove that $\overline{\Pi}$ has a fixed point. Note that, by our assumptions, we infer that all the functions involved in the operator are continuous therefore $\overline{\Pi}$ is continuous.

Let $z, \overline{z} \in \mathcal{B}_T^0$. From (6), by our assumptions, Hölder's inequality, Lemma 2.1, Lemma 2.2, in view of (7) and using the fact that $\Big(\sum_{i=1}^n a_i\Big)^2 \le n\sum_{i=1}^n a_i^2$, for any positive real numbers a_i, $i = 1, 2, ..., n$, we obtain the following estimates:

$$\begin{aligned}
&\mathbf{E}\|u_{z+\widetilde{\varphi}}^T(t)\|^2 \\
&\le 8M_W\Bigg\{\mathbf{E}\|y_1\|^2 + M_C\mathbf{E}\|\varphi(0)\|^2 + 2M_S\big[\mathbf{E}\|x_1\|^2 + 2(M_g\|\widetilde{\varphi}\|_{\mathcal{B}}^2 + C_2)\big] \\
&\quad + 2T^2M_C\Big[M_g\big([1+2M_{\sigma_1}]k + 2C_1\big) + C_2\Big] + 2T^2M_S\Big[M_f\big([1+2M_{\sigma_2}]k \\
&\quad + 2C_3\big) + C_4\Big] + 2M_SHT^{2H}\int_0^T \|\sigma(s)\|_{\mathcal{L}_2^0}^2 d(s) \\
&\quad + mM_C\sum_{k=1}^m M_{I_k^1} + mM_S\sum_{k=1}^m M_{I_k^2}\Bigg\} =: \Xi, \quad t \in J,
\end{aligned}$$

and

$$\begin{aligned}&\mathbf{E}\|u^T_{z+\widetilde{\varphi}}(t)-u^T_{\overline{z}+\widetilde{\varphi}}(t)\|^2\\&\le 8l_0^2M_W\Big\{T^2M_CM_g\big(1+M_{\sigma_1}\big)+T^2M_SM_f\big(1+M_{\sigma_2}\big)\\&+mM_C\sum_{k=1}^m\overline{M}_{I_k^1}+mM_S\sum_{k=1}^m\overline{M}_{I_k^2}\Big\}\sup_{s\in J}\mathbf{E}\|z(t)-\overline{z}(t)\|^2.\end{aligned}$$

Under the assumptions of Theorem 3.1, then there exists $r>0$ such that $\overline{\Pi}(B_r)\subseteq B_r$.

Proof. If this property is false, then for each $r>0$, there exists a function $z^r(\cdot)\in B_r$, but $\overline{\Pi}(z^r)\notin B_r$, i.e. $\|\overline{\Pi}(z^r)(t)\|^2>r$ for some $t\in J$. However, by our assumptions and Lemma 2.1, we have

$$\begin{aligned}r&<\mathbf{E}\|\overline{\Pi}(z^r)(t)\|^2\\&\le 7\Bigg[2M_S\big[\mathbf{E}\|x_1\|^2+2(M_g\|\widetilde{\varphi}\|^2_{\mathcal{B}}+C_2)\big]\\&\quad+2T^2M_C\Big[M_g\big([1+2M_{\sigma_1}]k+2C_1\big)+C_2\Big]\\&\quad+2T^2M_S\Big[M_f\big([1+2M_{\sigma_2}]k+2C_3\big)+C_4\Big]\\&\quad+T^2M_SM_B\Xi+2M_SHT^{2H}\int_0^T\|\sigma(s)\|^2_{\mathcal{L}_2^0}d(s)\\&\quad+mM_C\sum_{k=1}^mM_{I_k^1}+mM_S\sum_{k=1}^mM_{I_k^2}\Bigg],\\&\le\Theta+14T^2(1+8T^2M_BM_SM_W)\\&\quad\times\Big(M_CM_g(1+2M_{\sigma_1})+M_SM_f(1+2M_{\sigma_2})\Big)k,\end{aligned}\tag{8}$$

where

$$\begin{aligned}\Theta:=&\,56(1+8T^2M_BM_SM_W)\\&\times\Big[\mathbf{E}\|y_1\|^2+M_C(\mathbf{E}\|\varphi(0)\|^2)\Big]+7(1+8T^2M_BM_SM_W)\\&\times\Bigg[2M_S\big[\mathbf{E}\|x_1\|^2+2(M_g\|\widetilde{\varphi}\|^2_{\mathcal{B}}+C_2)\big]+2T^2M_C(2M_gC_1+C_2)\\&+2T^2M_S(2M_fC_3+C_4)+2M_SHT^{2H}\int_0^T\|\sigma(s)\|^2_{\mathcal{L}_2^0}d(s)\\&+mM_C\sum_{k=1}^mM_{I_k^1}+mM_S\sum_{k=1}^mM_{I_k^2}\Bigg].\end{aligned}$$

Dividing both sides of (8) by r and noting that

$$k=4l_0^2\Big(r+M_C\mathbf{E}\|\varphi(0)\|^2\Big)+4\|\widetilde{\varphi}\|^2_{\mathcal{B}}\xrightarrow{r\to\infty}\infty$$

then taking the limit as $r\to\infty$, we obtain $1\le\Sigma$, which contradicts our assumption. Thus, for some positive number r, $\overline{\Pi}(B_r)\subseteq B_r$. This completes the proof of Lemma 3.1. □

Under the assumptions of Theorem 3.1, then $\overline{\Pi} : \mathcal{B}_T^0 \to \mathcal{B}_T^0$ is a contraction mapping.

Proof. Let $z, \overline{z} \in \mathcal{B}_T^0$. Then, by our assumptions, Hölder's inequality, Lemma 2.1, Lemma 2.2 and since $\|z_0\|_{\mathcal{B}}^2 = 0$ and $\|\overline{z}_0\|_{\mathcal{B}}^2 = 0$, for each $t \in J$, we see that

$$\begin{aligned}
&\mathbf{E}\|(\overline{\Pi}z)(t) - (\overline{\Pi}\overline{z})(t)\|^2 \\
&\leq 10 l_0^2 \Big\{ T^2 M_C M_g \big(1 + M_{\sigma_1}\big) + T^2 M_S M_f \big(1 + M_{\sigma_2}\big) \\
&\quad + m M_C \sum_{k=1}^{m} \overline{M}_{I_k^1} + m M_S \sum_{k=1}^{m} \overline{M}_{I_k^2} \Big\} \sup_{s \in J} \mathbf{E}\|z(t) - \overline{z}(t)\|^2 \\
&\quad + 5 T^2 M_S M_B \mathbf{E}\|u^T_{z+\widetilde{\varphi}}(t) - u^T_{\overline{z}+\widetilde{\varphi}}(t)\|^2 \\
&\leq \Bigg\{ 10 l_0^2 \big(1 + 4 T^2 M_B M_S M_W\big) \\
&\quad \times \Big[T^2 M_C M_g \big(1 + M_{\sigma_1}\big) + T^2 M_S M_f \big(1 + M_{\sigma_2}\big) \\
&\quad + m M_C \sum_{k=1}^{m} \overline{M}_{I_k^1} + m M_S \sum_{k=1}^{m} \overline{M}_{I_k^2} \Big] \Bigg\} \sup_{s \in J} \mathbf{E}\|z(t) - \overline{z}(t)\|^2.
\end{aligned}$$

Taking the supremum over t, we obtain $\|(\overline{\Pi}z) - (\overline{\Pi}\overline{z})\|_T^2 \leq \Delta \|z - \overline{z}\|_T^2$.

By our assumption, we conclude that $\overline{\Pi}$ is a contraction on $\mathcal{B}_T^0$. Thus we have completed the proof of Lemma 3.2. □

On the other hand, by Banach fixed point theorem there exists a unique fixed point $x(\cdot) \in \mathcal{B}_T^0$ such that $(\Pi x)(t) = x(t)$. This fixed point is then the mild solution of the system (1). Clearly, $x(T) = (\Pi x)(T) = y_1$. Thus, the system (1) is controllable on J_T. The proof for Theorem 3.1 is thus complete. □

4. Application

In this section, we apply the results established in the previous section to discuss the controllability of the second-order stochastic nonlinear wave equation with infinite delay and fBm. Now, we consider only a simple type of stochastic wave equation driven by fBm in the following form:

$$\begin{cases} \dfrac{\partial}{\partial t}\Big[\dfrac{\partial}{\partial t}y(t,\xi) - \displaystyle\int_{-\infty}^{t} \delta_1(t,\xi,s-t,)P_1(y(s,\xi))ds \\ \quad - \displaystyle\int_0^t \int_{-\infty}^{s} b_1(s-\tau)P_2(y(\tau,\xi))d\tau ds\Big] \\ = \Big[\dfrac{\partial^2}{\partial \xi^2}y(t,\xi) + \displaystyle\int_{-\infty}^{t} \delta_2(t,\xi,s-t,)G_1(y(s,\xi))ds \\ \quad + \displaystyle\int_0^t \int_{-\infty}^{s} b_2(s-\tau)G_2(y(\tau,\xi))d\tau ds \\ \quad + b(\xi)u(t)\Big]dt + \sigma(t)dB^H(t), \quad t_k \neq t \in J, \quad \xi \in [0,\pi], \\ \Delta y(t_k)(\xi) = \int_{-\infty}^{t_k} \eta_k(t_k - s)y(s,\xi)ds, \quad k = 1,2,..,m, \quad \xi \in [0,\pi], \\ \Delta y'(t_k)(\xi) = \int_{-\infty}^{t_k} \rho_k(t_k - s)y(s,\xi)ds, \quad k = 1,2,..,m, \quad \xi \in [0,\pi], \\ y(t,0) = y(t,\pi) = 0, \quad t \in J, \\ \frac{\partial}{\partial t}y(0,\xi) = x_1(\xi), \quad \xi \in [0,\pi], \\ y(t,\xi) = \varphi(t,\xi), \quad t \in J_0, \quad \xi \in [0,\pi], \end{cases} \tag{9}$$

where B^H is is a fractional Brownian motion ; $0 < t_1 < t_2 < \cdots < t_n < T$, $n \in \mathbb{N}$; $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} < T$ are prefixed numbers, and $\varphi \in \mathcal{B}$.

To rewrite (9) into the abstract from of (1) we consider the space $\mathbb{H} = \mathbb{K} = \mathbb{U} = \mathcal{L}^2([0,\pi])$ with the norm $\|\cdot\|$.. Let $e_n(\xi) := \sqrt{\frac{2}{\pi}}\sin n\xi$, $n = 1,2,3,...$ denote the completed orthogonal basics in $\mathbb{H}$.

Defined $A : \mathbb{H} \to \mathbb{H}$ by $A = \frac{\partial^2}{\partial \xi^2}$, with domain $D(A) = \mathbb{H}^2([0,\pi]) \cap \mathbb{H}_0^1([0,\pi])$, where

$$\mathbb{H}_0^1([0,\pi]) = \{w \in L^2([0,\pi]) : \frac{\partial w}{\partial z} \in L^2([0,\pi]), w(0) = w(\pi) = 0\}$$

and

$$\mathbb{H}^2([0,\pi]) = \{w \in L^2([0,\pi]) : \frac{\partial w}{\partial z}, \frac{\partial^2 w}{\partial z^2} \in L^2([0,\pi])\}.$$

Then

$$Ax = -\sum_{n=1}^{\infty} n^2 \langle x, e_n\rangle e_n, \quad x \in D(A), \tag{10}$$

(see Ref. [29]). Using (10), one can easily verify that the operators $C(t)$ defined by

$$C(t)x = \sum_{n=1}^{\infty} cos(nt)\langle x, e_n\rangle e_n, \quad t \in \mathbb{R},$$

form a cosine function on $\mathbb{H}$, with associated sine function

$$S(t)x = \sum_{n=1}^{\infty} \frac{sin(nt)}{n}\langle x, e_n\rangle e_n, \quad t \in \mathbb{R}.$$

It is clear that (see Ref. [28]), for all $x \in \mathbb{H}$, $t \in \mathbb{R}$, $C(\cdot)x$ and $S(\cdot)x$ are periodic functions with $\|C(t)\| \leq 1$ and $\|S(t)\| \leq 1$. Thus, (**H**1) is true.

Define the fractional Brownian motion in $\mathbb{K}$ by

$$B^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\beta_n^H(t)e_n,$$

where $H \in (\frac{1}{2}, 1)$ and $\{\beta_n^H\}_{n\in N}$ is a sequence of one-dimensional fractional Brownian motions mutually independent.

Next, we give a special $\mathcal{B}$-space. Let $l(s) = e^{2s}$, $s \leq 0$, then $l_0 = \int_{J_0} l(s)ds = \frac{1}{2}$ and define

$$\|\psi\|_{\mathcal{B}} = \int_{J_0} l(s) \sup_{\theta\in[s,0]} (\mathbf{E}\|\psi(\theta)\|^2)^{\frac{1}{2}} ds, \quad \forall \psi \in \mathcal{B}.$$

It follows from [15], that $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a Banach space. Hence for $(t, \psi) \in J \times \mathcal{B}$, where $\psi(\theta)x = \psi(\theta, x)$, $(\theta, x) \in J_0 \times [0, \pi]$. Let $y(t)(\xi) = y(t, \xi)$.

To study the system (9), we assume the following conditions hold:

(i) Let $B \in \mathcal{L}(\mathbb{R}, \mathbb{H})$ be defined as

$$Bu(\xi) = b(\xi)u, \quad 0 \leq \xi \leq \pi, \quad u \in \mathbb{R}, \quad b(\xi) \in \mathcal{L}_2([0, \pi]).$$

(ii) The linear operator $W : \mathcal{L}_2(J, U) \to \mathbb{H}$ defined by

$$Wu = \int_J S(T - s)b(\xi)u(s)ds$$

is a bounded linear operator but not necessarily one-to-one. Let $KerW = \{u \in \mathcal{L}_2(J, U) : Wu = 0\}$ be null space of W and $[KerW]^\perp$ be its orthogonal complement in $\mathcal{L}_2(J, U)$. Let $W^* : [KerW]^\perp \to Range(W)$ be the restriction of W to $[KerW]^\perp$, W^* is necessarily one-to-one operator. The inverse mapping theorem says that $(W^*)^{-1}$ is bounded since $[KerW]^\perp$ and $Range(W)$ are Banach spaces. So that inverse operator W^{-1} is bounded and takes values in $\mathcal{L}_2(J, U)/KerW$, the assumption (**H10**) is satisfied.

(iii) The functions $\eta_k, \rho_k \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ such that for $k = 1, 2, .., m$,

$$\overline{M}_{I_k^1} = \int_{J_0} l(s)\eta_k^2(s)ds < \infty, \quad \overline{M}_{I_k^2} = \int_{J_0} l(s)\rho_k^2(s)ds < \infty.$$

We define the functions $g, f : J \times \mathcal{B} \times \mathbb{H} \to \mathbb{H}$, $\sigma : J \to \mathcal{L}_2^0$, and $I_k^1, I_k^2 : \mathcal{B} \to \mathbb{H}$, $k = 1, 2, .., m$, by

$$\begin{aligned}
g(t, \psi, V_1\psi)(\xi) &= \int_{J_0} \delta_1(t, \xi, \theta)P_1(\psi(\theta)(\xi))d\theta + V_1\psi(\xi), \\
f(t, \psi, V_2\psi)(\xi) &= \int_{J_0} \delta_2(t, \xi, \theta)G_1(\psi(\theta)(\xi))d\theta + V_2\psi(\xi), \\
I_k^1(t, \psi)(\xi) &= \int_{J_0} \eta_k(-s)\psi(\theta)(\xi)ds, \quad k = 1, 2, ..., m, \\
I_k^2(t, \psi)(\xi) &= \int_{J_0} \rho_k(-s)\psi(\theta)(\xi)ds, \quad k = 1, 2, ..., m,
\end{aligned}$$

where

$$\begin{aligned}
V_1\psi(\xi) &= \int_0^t \int_{J_0} b_1(s - \theta)P_2(\psi(\theta)(\xi))d\theta ds, \\
V_2\psi(\xi) &= \int_0^t \int_{J_0} b_2(s - \theta)G_2(\psi(\theta)(\xi))d\theta ds.
\end{aligned}$$

Then, the system (9) can be written in the abstract form as the system (1):

$$\begin{cases} d\big[x'(t) - g\big(t, x_t, \int_0^t \sigma_1(t,s,x_s)ds\big)\big] = \big[Ax(t) + f\big(t, x_t, \int_0^t \sigma_2(t,s,x_s)ds\big)\big]dt \\ \qquad\qquad\qquad\qquad\qquad\qquad +Bu(t)dt + \sigma(t)dB^H(t), \quad t \in J := [0,T], \\ \Delta x(t_k) = I_k^1(x_{t_k}), \quad k = 1,2..,m, \\ \Delta x'(t_k) = I_k^2(x_{t_k}), \quad k = 1,2..,m, \\ x'(0) = x_1 \in \mathbb{H}, \\ x_0 = \varphi \in \mathcal{B}, \quad \text{for a.e. } s \in J_0 := (-\infty, 0], \end{cases}$$

Furthermore, we can impose some suitable conditions on the above defined functions as those in the assumptions $(\mathbf{H1}) - (\mathbf{H9})$. Therefore, by Theorem 3.1, we can conclude that the system (9) is controllable on J_T.

References

[1] H. M. Ahmed, Controllability of impulsive neutral stochastic differential equations with fractional Brownian motion. IMA J. Math. Contr. Infor., doi:10.1093/imamci/dnu019, 1-14, 2014.
[2] D. Applebum, Lévy Processes and Stochastic Calculus. Cambridge University Press, 2004.
[3] G. Arthi and K. Balachandran, Controllability of Damped Second-Order Impulsive Neutral Functional Differential Systems with Infinite Delay. J. Optim. Theory Appl., 152, 799-813, 2012.
[4] P. Balasubramaniam and P. Muthukumar,Approximate controllability of second-order stochastic distributed implicit functional differential systems with infinite delay. J. Optim. Theory Appl., 143, 225-244,2009.
[5] P. Balasubramaniam and S. K. Ntouyas, Controllability for neutral stochastic functional differential inclusions with infinite delay in abstract space. J. Math. Anal. Appl., 324, 161-176, 2006.
[6] B. Boufoussi and S. Hajji, Neutral stochastic functional differential equations driven by a fractional Brownian motion in a Hilbert space. Statist. Probab. Lett., 82, 1549-1558, 2012.
[7] A. Boudaoui and E. Lakhel, Controllability of Stochastic Impulsive Neutral Functional Differential Equations Driven by Fractional Brownian Motion with Infinite Delay. Differ Equ Dyn Syst., DOI 10.1007/s12591-017-0401-7, 1-17, 2017.
[8] N. Carimichel and M. D. Quinn, Fixed Point Methods in Nonlinear Control, in: Lecture Notes in Control and Information Society. Vol. 75, Springer, Berlin, 1984.
[9] Y. K. Chang, Controllability of impulsive functional differential systems with infinite delay in Banach spaces. Chaos Solitons Fracals, 33, 1601-1609, 2001.
[10] M. Chen, Approximate controllability of stochastic equations in a Hilbert space with fractional Brownian motions. Stoch. Dyn., 15 (1): 1550005, 2015.
[11] J. Cui and L. T. Yan, Controllability for neutral stochastic evolution equations driven by fractional Brownian motion. Acta Math. Sci., 37B(1), 108-118,2017.
[12] G. Da Pratoand J. Zabczyk, Stochastic Equations in Infinite Dimensions. Vol. 44, Cambridge University Press, Cambridge, UK, 1992.
[13] W. E. Fitzgibbon Global existence and boundedness of solutions to the extensible beam equation. SIAM J. Math. Anal., 13 (5), 739745, 1982.
[14] H. O. Fattorini, Second Order Linear Differential Equations in Banach Spaces. Vol. 108 of North-Holland Mathematics Studies, North-Holland, Amsterdam, The Netherlands, 1985.
[15] Y. Hino and S. Murakami, T. Naito, Functional Differential Equations with Infinite Delay, in: Lecture Notes in Mathematics. Vol. 1473, Springer-Verlag, Berlin, 1991.
[16] D. D. Huan, Global attracting set and stability for stochastic integro-differential equations. Electr. J. Math. Anal. Appl., 7 (2), 267-283 2019.
[17] D. D. Huan, Approximate controllability of damped second-order impulsive neutral stochastic integro-differential system with state-dependent delay . Electr. J. Math. Anal. Appl., 5 (2), 221-238, 2017.
[18] D. D. Huan and H. J. Gao; Controllability of nonlocal second order impulsive neutral stochastic functional integro-differential equations with delay and Poisson jumps. Cogent Engineering, 2 (1) , 1-16, 2015.

[19] D. D. Huan, Ravi P. Agarwal and H. J. Gao, Approximate controllability for time-dependent impulsive neutral stochastic partial differential equations with memory. Filomat, 31 (11), 3433-3442,2017.
[20] X. Mao, Stochastic Differential Equations and Applications. Horwood Publishing Limited, Chichester, UK, 1997.
[21] Y. S. Mishura, Stochastic Calculus for Fractional Brownian Motion and Related Processes. Springer, 2008.
[22] P. Muthukumar and C. Rajivganthi, Approximate Controllability of Second-Order Neutral Stochastic Differential Equations with Infinite Delay and Poisson Jumps, J. Syst. Sci. Complex, 28, 10331048, 2015.
[23] B. Øksendal, Stochastic Differential Equations. Springer, New York, 6th ed, 2005.
[24] J. Y. Park, P. Balasubramaniam and N. Kumaresan, Controllability for neutral stochastic functional integrodifferential infinite delay systems in abstract space. Numer. Func. Anal. Optim., 28, 1369-1386, 2007.
[25] A. Pazy, Semigroup of Linear operators and Applications to Partial Differential Equations. Springer Verlag, New York, 1992.
[26] P. E. Protter, Stochastic integration and differential equations. Springer, New York, 2nd ed., 2004.
[27] R. Sakthivel, Y. Ren and N.I. Mahmudov, Approximate controllability of second-order stochastic differential equations with impulsive effects. Modern Phys. Lett. B, 24 (14), 1559-1572, 2010.
[28] C. C. Travis and G.F. Webb, Compactness, regularity, and uniform continuity properties of strongly continuous cosine families. Houston J. Math., 3(4), 555-567, 1977.
[29] C. C. Travis and G.F. Webb, Second oder differential equations in Banach spaces. In: Proceedinds International Symposium on Nonlinear Equations in Abstract Spaces, pp. 331-361. Academic Press, New York, 1987.
[30] C. C. Travis and G.F. Webb, Cosine families and abstract nonlinear second order differential equations. Acta Math. Acad. Sci. Hung., 32 , 76-96, 1978.

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