EXISTENCE OF FIXED POINT OF MEIR KEELER TYPE CONTRACTIVE CONDITION IN FUZZY METRIC SPACES

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ABSTRACT. In this paper, we prove a general common fixed point theorem for two pairs of weakly compatible self-mappings in Fuzzy metric space satisfying a generalized Meir-Keeler type contractive condition.

1. INTRODUCTION

It proved a turning point within the development of fuzzy mathematics while the perception of fuzzy set become brought by Zadeh [21]. Fuzzy set theory has many programs in carried out technology such as neural network principle, stability principle, mathematical programming, modelling theory, engineering sciences, clinical sciences (clinical genetics, apprehensive device), image processing, manage principle and so on. There are many view points of the notion of the metric area in fuzzy topology, see, e.g., Erceg [2], Deng [1], Kaleva and Seikkala [9], Kramosil and Michalek [10], George and Veermani [3]. In this paper, we are considering the Fuzzy metric space within the sense of Kramosil and Michalek [10].

Definition 1.1. A binary operation Δ on $[0,1]$ is a t-norm if it satisfies the following conditions:

    (i) Δ is associative and commutative,
    (ii) $\Delta(a,1) = a$ for every $a \in [0,1]$,
    (iii) $\Delta(a,b) \leq \Delta(c,d)$, whenever $a \leq c$ and $b \leq d$.

Basics examples of t-norm are Δ_L, $\Delta_L(a,b) = \max(a+b-1,0)$, t-norm Δ_P, $\Delta_P(a,b) = ab$ and t-norm Δ_M, $\Delta_M(a,b) = \min\{a,b\}$.

Definition 1.2 ([10])**.** The triplet $(\mathcal{K}, \mathcal{M}, \Delta)$ be a fuzzy metric space if Δ be a continuous t-norm, $\mathcal{K}$ be a arbitrary set and $\mathcal{M}$ is a fuzzy set in $\mathcal{K}^2[0,\infty)$ satisfying

    (i) $\mathcal{M}(x_1, y_1, 0) = 0$;
    (ii) $\mathcal{M}(x_1, y_1, t_1) = 1$ for all $t_1 > 0$ iff $x_1 = y_1$;
    (iii) $\mathcal{M}(x_1, y_1, t_1) = \mathcal{M}(y_1, x_1, t_1)$;
    (iv) $\Delta(\mathcal{M}(x_1, y_1, t_1), \mathcal{M}(y_1, z_1, s_1)) \leq \mathcal{M}(x_1, z_1, t_1 + s_1)$;

(v) $\mathcal{M}(x_1, y_1, \cdot) : [0, \infty) \to [0, 1]$ be a left continuous;

(vi) $\lim_{t_1 \to \infty} \mathcal{M}(x_1, y_1, t_1) = 1$ for all $x_1, y_1, t_1 \in \mathcal{K}$ and $s_1, t_1 > 0$;

$\mathcal{M}(x_1, y_1, t_1)$ can be thought as the degree of nearness between x_1 and y_1 with respect to t_1.

Definition 1.3 ([4])**.** Let $(\mathcal{K}, \mathcal{M}, \Delta)$ is a fuzzy metric space. A sequence $\{x_n\}$ in $\mathcal{K}$ is said to be

(i) Cauchy sequence if $\lim_{n \to \infty} \mathcal{M}(x_{n+p}, x_n, t_1) = 1$.

(ii) Converge to $x_1 \in \mathcal{K}$ if $\lim_{n \to \infty} \mathcal{M}(x_n, x_1, t_1) = 1$, $\forall\, t_1 > 0$.

(iii) Complete if every Cauchy sequence in $\mathcal{K}$ is convergent in $\mathcal{K}$.

In 1996, Jungck [8] brought the belief of weakly compatible as follows:

Definition 1.4 ([8])**.** Two maps $\mathcal{S}$ and $\mathcal{T}$ are said to be weakly compatible if they commute at their coincidence points.

In 1999, Vasuki [20] introduced the notion of weakly commuting as follows:

Definition 1.5 ([20])**.** Let $\mathcal{S}$ and $\mathcal{T}$ are two self-maps on a fuzzy metric space $(\mathcal{K}, \mathcal{M}, \Delta)$. Then $\mathcal{S}$ and $\mathcal{T}$ are said to be weakly commuting if

$$\mathcal{M}(\mathcal{S}\mathcal{T}x_1, \mathcal{T}\mathcal{S}x_1, t_1) \geq \mathcal{M}(\mathcal{S}x_1, \mathcal{T}x_1, t_1), \text{ for all } x_1 \text{ in } \mathcal{K}, \ t_1 > 0.$$

In 1994, Mishra [11] generalised the notion of weakly commuting to like minded mappings in fuzzy metric space akin to the concept of well suited mapping in metric space.

Definition 1.6 ([11])**.** Let $\mathcal{S}$ and $\mathcal{T}$ be self maps on a fuzzy metric space $(\mathcal{K}, \mathcal{M}, \Delta)$. Then $\mathcal{S}$ and $\mathcal{T}$ are said to be compatible if

$$\lim_{n \to \infty} \mathcal{M}(\mathcal{S}\mathcal{T}x_n, \mathcal{T}\mathcal{S}x_n, t_1) = 1, \ \forall\, t_1 > 0,$$

whenever a sequence $\{x_n\}$ in $\mathcal{K}$ satisfying $\lim_{n \to \infty} \mathcal{S}x_n = \lim_{n \to \infty} \mathcal{T}x_n = u_1$, where u_1 in $\mathcal{K}$.

In 1994, Pant [14] introduced the concept of R-weakly commuting maps in metric area. Later on, Vasuki [20] initiated the idea of non-compatible mapping in fuzzy metric space and delivered the belief of R-weakly commuting mapping in fuzzy metric space and proved a few commonplace constant point theorems for those mappings.

Definition 1.7 ([20])**.** Let $(\mathcal{K}, \mathcal{M}, \Delta)$ be a fuzzy metric space and $\mathcal{S}$ and $\mathcal{T}$ are two self maps on $(\mathcal{K}, \mathcal{M}, \Delta)$. Then $\mathcal{S}$ and $\mathcal{T}$ are said to be non-compatible if

$$\lim_{n \to \infty} \mathcal{M}(\mathcal{S}\mathcal{T}x_n, \mathcal{T}\mathcal{S}x_n, t_1) \neq 1 \text{ or non-existent},$$

whenever a sequence $\{x_n\}$ in $\mathcal{K}$ satisfying $\lim_{n \to \infty} \mathcal{S}x_n = \lim_{n \to \infty} \mathcal{T}x_n = u_1$, where $u_1 \in \mathcal{K}$ and for all $t_1 > 0$.

Definition 1.8. A pair of self-mappings $\mathcal{S}$ and $\mathcal{T}$ on a fuzzy metric space $(\mathcal{K}, \mathcal{M}, \Delta)$ are said to satisfy the property (E.A) if there exists a sequence $\{x_n\}$ in $\mathcal{K}$ such that

$$\lim_{n \to \infty} \mathcal{M}(\mathcal{S}x_n, u, t_1) = \lim_{n \to \infty} \mathcal{M}(\mathcal{T}x_n, u, t_1) = 1, \text{ for some } u \in \mathcal{K} \text{ and for all } t_1 > 0.$$

The class of E.A mappings contains the class of non compatible mappings.

Definition 1.9. Let $(\mathcal{K}, \mathcal{M}, \Delta)$ is a fuzzy metric space. Two pairs $(P, \mathcal{S})$ and $(R, \mathcal{T})$ holds the common (E.A) property if we can find two sequences $\{z_n\}$ and $\{w_n\}$ in $\mathcal{K}$ satisfying

$$\lim_{n \to \infty} Pz_n = \lim_{n \to \infty} \mathcal{S}z_n = \lim_{n \to \infty} Rw_n = \lim_{n \to \infty} \mathcal{T}w_n = u, \text{ for some } u \in \mathcal{K}.$$

Definition 1.10. Let $(\mathcal{K}, \mathcal{M}, \Delta)$ is a fuzzy metric space. Two pairs $(P, \mathcal{S})$ and $(R, \mathcal{T})$ holds the $(JCLR_{ST})$ property if we find two sequences $\{z_n\}$ and $\{w_n\}$ in $\mathcal{K}$ satisfying

$$\lim_{n \to \infty} Pz_n = \lim_{n \to \infty} \mathcal{S}z_n = \lim_{n \to \infty} Rw_n = \lim_{n \to \infty} \mathcal{T}w_n = \mathcal{S}w = \mathcal{T}w \text{ for some } w \in \mathcal{K}.$$

Example 1.1. Let $\mathcal{K} = [-1, 1]$ and $(\mathcal{K}, \mathcal{M}, \Delta)$ is a fuzzy metric space defined by

$$\mathcal{M}(x_1, y_1, t_1) = \begin{cases} \frac{t_1}{t_1 + |x_1 - y_1|} & \text{if } t_1 > 0 \\ 0 & \text{if } t_1 = 0, \end{cases} \quad \text{for all } x_1, y_1 \text{ in } \mathcal{K}.$$

Define P, R, $\mathcal{S}$ and $\mathcal{T}$ on $\mathcal{K}$ by $Px_1 = \frac{x_1}{5}$, $Rx_1 = \frac{-x_1}{5}$, $\mathcal{S}x_1 = x_1$, $\mathcal{T}x_1 = -x_1$, for all $x_1 \in \mathcal{K}$.
Then with sequences $\{z_n\} = \left\{\frac{1}{3n}\right\}$ and $\{w_n\} = \left\{\frac{-1}{3n}\right\}$ in $\mathcal{K}$, one can easily verify that

$$\lim_{n \to \infty} Pz_n = \lim_{n \to \infty} \mathcal{S}z_n = \lim_{n \to \infty} Rw_n = \lim_{n \to \infty} \mathcal{T}w_n = \mathcal{S}0 = \mathcal{T}0.$$

Hence $(JCLR_{ST})$ property are satisfied by the pairs $(P, \mathcal{S})$ and $(R, \mathcal{T})$.

Now we prove the in fuzzy metric spaces satisfying a generalized Meir-Keeler type contractive condition.

2. Main Results

Theorem 2.1. *Let P, R, S and T be four self mapping on a fuzzy metric space $(\mathcal{M}, \mathcal{K}, \Delta)$ with minimum t-norm such that*

(2.1) *$P(\mathcal{K}) \subseteq T(\mathcal{K})$, $R(\mathcal{K}) \subseteq S(\mathcal{K})$;*

(2.2) *given an $\epsilon > 0$ and for all $z, w \in \mathcal{K}$, we can find a $\delta \in (0, \epsilon)$*

$$\text{satisfying } \epsilon - \delta < m(z, w, t_1) \leq \epsilon \Longrightarrow \mathcal{M}(Pz, Rw, t_1) > \epsilon,$$

where $m(z, w, t_1) = \min\{\mathcal{M}(Sz, Tw, t_1), \mathcal{M}(Pz, Sz, t_1), \mathcal{M}(Rw, Tw, t_1)\}$;

(2.3) *one of $P\mathcal{K}$, $R\mathcal{K}$, $S\mathcal{K}$ or $T\mathcal{K}$ be a complete subspace of $\mathcal{K}$;*

(2.4) *the pair (P, S) and (R, T) are weakly compatible.*

Then the pairs (P, S) and (R, T) have coincidence points and $Pu_1 = Ru_1 = Tu_1 = Su_1 = u_1$, where u_1 is unique in $\mathcal{K}$.

Proof. Since $P(\mathcal{K}) \subseteq T(\mathcal{K})$. Now consider a point $z_0 \in \mathcal{K}$, there exists a point $z_1 \in \mathcal{K}$ satisfying $Pz_0 = Tz_1 = w_1$ and a given point z_1, we can find a point $z_2 \in \mathcal{K}$ such that $Rz_1 = Sz_2 = w_2$. Continuing in this way, we can define sequences $\{z_n\}$ and $\{w_n\}$ in $\mathcal{K}$ such that

$$w_{2n} = Sz_{2n} = Rz_{2n-1}; \quad w_{2n-1} = Tz_{2n-1} = Pz_{2n-2}.$$

We show that $\{w_n\}$ be a Cauchy sequence. Let us denote $\mathcal{M}_n = \mathcal{M}(w_n, w_{n+1}, t_1)$ and $\mathcal{G}_n = \mathcal{G}(w_n, w_{n+1}, t_1)$, where $t_1 > 0$.
Suppose that $\mathcal{M}_n = 1$ for some $n = 2k - 1$. Then $\mathcal{M}(w_{2k-1}, w_{2k}, t_1) = 1$, this gives $w_{2k-1} = w_{2k}$, which implies that $Tz_{2k-1} = Pz_{2k-2} = Sz_{2k} = Rz_{2k-1}$, so T and R have a coincidence point.

Further, if $\mathcal{F}_n = 1$ for some $n = 2k$, then $\mathcal{M}(w_{2k}, w_{2k+1}, t_1) = 1$, this gives $w_{2k} = w_{2k+1}$, which implies that $Tz_{2k+1} = Pz_{2k} = Sz_{2k} = Rz_{2k-1}$, so P and S have a coincidence point.

Now suppose that $\mathcal{M}_n \neq 1$, for all n.

If for some $z, w \in \mathcal{K}$, $m(z, w, t_1) = 1$, then we get $Pz = Sz$ and $Tw = Rw$. Hence the result.

If $m(z, w, t_1) < 1$, for all $z, w \in \mathcal{K}$, then, by (2.2), we have

$$\mathcal{M}(Pz, Rw, t_1) > m(z, w, t_1). \tag{1}$$

Hence, we have

$$
\begin{aligned}
\mathcal{M}_{2n-1} = \mathcal{M}(w_{2n-1}, w_{2n}, t_1) &= \mathcal{M}(Pz_{2n-2}, Rz_{2n-1}, t_1) \\
&> m(z_{2n-2}, z_{2n-1}, t_1) \\
&= \min\{\mathcal{M}(Sz_{2n-2}, Tz_{2n-1}, t_1), \mathcal{M}(Pz_{2n-2}, Sz_{2n-2}, t_1), \mathcal{M}(Rz_{2n-1}, Tz_{2n-1}, t_1)\} \\
&= \min\{\mathcal{M}(w_{2n-2}, w_{2n-1}, t_1), \mathcal{M}(w_{2n-1}, w_{2n-2}, t_1), \mathcal{M}(w_{2n}, w_{2n-1}, t_1)\} \\
&= \min\{\mathcal{M}_{2n-2}, \mathcal{M}_{2n-1}\} = \mathcal{M}_{2n-2}.
\end{aligned}
$$

Therefore,

$$\mathcal{M}_{2n-1} > \mathcal{M}_{2n-2}. \tag{2}$$

Similarly, $\mathcal{M}_{2n} > \mathcal{M}_{2n-1}$.

Hence we deduce that $\mathcal{M}_n > \mathcal{M}_{n-1}$, for all n.

Thus $\{\mathcal{M}_n\}$ is a strictly increasing sequence of positive real numbers in $[0, 1]$.

$$\text{Hence } \{\mathcal{M}_n\} \text{ converges to some limit say } s. \tag{3}$$

Next we claim that $s = 1$. If $s \neq 1$, then by (3), there exists a $\delta > 0$ and a natural number m such that for each $n \geq m$,

$$s - \delta < \mathcal{M}(w_n, w_{n+1}, t_1) = \mathcal{M}_n \leq s. \tag{4}$$

In particular, $m(z_{2n}, z_{2n-1}, t_1) = \min\{\mathcal{M}_{2n}, \mathcal{M}_{2n-1}\} = \mathcal{M}_{2n-1}$, then we get

$$s - \delta < \mathcal{M}_{2n-1} \leq s.$$

Therefore, by using (2.2), we have

$$
\begin{aligned}
\mathcal{M}(Pz_{2n}, &Rz_{2n-1}, t_1) \\
&> m(z_{2n}, z_{2n-1}, t_1) \\
&= \min\{\mathcal{M}(Sz_{2n}, Tz_{2n-1}, t_1), \mathcal{M}(Pz_{2n}, Sz_{2n}, t_1), \mathcal{M}(Rz_{2n-1}, Tz_{2n-1}, t_1)\} \\
&= \min\{\mathcal{M}(w_{2n}, w_{2n-1}, t_1), \mathcal{M}(w_{2n+1}, w_{2n}, t_1), \mathcal{M}(w_{2n}w_{2n-1}, t_1)\} \\
&= \min\{\mathcal{M}(w_{2n}, w_{2n-1}, t_1), \mathcal{M}(w_{2n+1}, w_{2n}, t_1)\} \\
&= \min\{\mathcal{M}_{2n-1}, \mathcal{M}_{2n}\} = \mathcal{M}_{2n}.
\end{aligned}
$$

Thus, we have $\mathcal{M}(z_{2n+1}, z_{2n}, t_1) = \mathcal{M}_{2n} > s$, a contradiction. Hence $s = 1$ i.e., $\lim\limits_{n\to\infty} \mathcal{M}_n = \lim\limits_{n\to\infty} \mathcal{M}(w_n, w_{n+1}, t_1) = 1$.

Now, for any positive integer k,

$$
\begin{aligned}
\mathcal{M}(w_n, &w_{n+k}, t_1) \\
&\geq \mathcal{M}\left(w_n, w_{n+1}, \frac{t_1}{k}\right) \Delta \mathcal{M}\left(w_{n+1}, w_{n+2}, \frac{t_1}{k^2}\right) \Delta \ldots \Delta \mathcal{M}\left(w_{n+k-1}, w_{n+k}, \frac{t_1}{k^n}\right).
\end{aligned}
$$

Since $\lim\limits_{n \to \infty} \mathcal{M}(w_n, w_{n+1}, t_1) = 1$, for $t_1 > 0$, then

$$\lim_{n \to \infty} \mathcal{M}(w_n, w_{n+k}, t_1) \geq 1,$$

which shows that $\{w_n\}$ be a Cauchy sequence in $\mathcal{K}$.

Suppose $S\mathcal{K}$ be a complete subspace of $\mathcal{K}$. Then the subsequence $w_{2n} = Sz_{2n}$ must have a limit in $S\mathcal{K}$, say u_1 and $v_1 \in S^{-1}(u_1)$, so that $Sv_1 = u_1$. Since $\{w_n\}$ is a Cauchy sequence containing a convergent subsequence $\{w_{2n}\}$, the sequence $\{w_n\}$ also converges to u_1.

First we claim that $Pv_1 = u_1$. If $Pv_1 \neq u_1$. On putting $z = v_1$ and $w = z_{2n-1}$ in (2.2), we have for $t_1 > 0$,

$$\mathcal{M}(Pv_1, Rz_{2n-1}, t_1) > m(v_1, z_{2n-1}, t_1)$$
$$= \min\{\mathcal{M}(Sv_1, Tz_{2n-1}, t_1), \mathcal{M}(Pv_1, Sv_1, t_1), \mathcal{M}(Rz_{2n-1}, Tz_{2n-1}, t_1)\}$$

Putting $n \to \infty$, we have

$$\mathcal{M}(Pv_1, u_1, t_1) > \min\{\mathcal{M}(u_1, u_1, t_1), \mathcal{M}(u_1, Pv_1, t_1), \mathcal{M}(u_1, u_1, t_1)\}$$
$$= \mathcal{M}(Pv_1, u_1, t_1)$$

a contradiction.

Thus $Pv_1 = u_1 = Sv_1$.

Hence the pair (P, S) has a point of coincidence.

As $P(\mathcal{K}) \subseteq T(\mathcal{K})$, $Pv_1 = u_1$ gives $u_1 \in T\mathcal{K}$.

Let $x_1 \in T^{-1}(u_1)$, then $Tx_1 = u_1$.

Next we claim that $Rx_1 = u_1$. If $Rx_1 \neq u_1$.

On putting $z = w_{2n}$ and $w = x_1$ in (2.2), we get $t_1 > 0$,

$$\mathcal{M}(Pw_{2n}, Rx_1, t_1) > m(w_{2n}, x_1, t_1)$$
$$= \min\{\mathcal{M}(Sw_{2n}, Tx_1, t_1), \mathcal{M}(Pw_{2n}, Sw_{2n}, t_1), \mathcal{M}(Rx_1, Tx_1, t_1)\}.$$

Putting $n \to \infty$, we have

$$\mathcal{M}(u_1, Rx_1, t_1) > \min\{\mathcal{M}(u_1, u_1, t_1), \mathcal{M}(u_1, u_1, t_1), \mathcal{M}(Rx_1, u_1, t_1)\},$$
$$= \mathcal{M}(Rx_1, u_1, t_1),$$

a contradiction. Therefore, $Rx_1 = u_1 = Tx_1$.

Hence we have shown that $u_1 = Sv_1 = Pv_1 = Rx_1 = Tx_1$.

The same result is obtained if we assume $T\mathcal{K}$ to be complete. Indeed if $P\mathcal{K}$ is complete, then $u_1 \in P\mathcal{K} \subseteq T\mathcal{K}$ and if $R\mathcal{K}$ is complete, then $u_1 \in R\mathcal{K} \subseteq S\mathcal{K}$. Now weakly compatible of the pairs (P, S) and (R, T) implies, $Pu_1 = PSv_1 = SPv_1 = Su_1$ and $Ru_1 = RTx_1 = TRx_1 = Tu_1$.

Finally we claim that $Pu_1 = u_1$.

If $Pu_1 \neq u_1$.

Putting $z = u_1$ and $w = x_1$ in (2.2), we get $t_1 > 0$,

$$\mathcal{M}(Pu_1, Rx_1, t_1) = \mathcal{M}(Pu_1, u_1, t_1)$$
$$> m(u_1, x_1, t_1)$$
$$= \min\{\mathcal{M}(Su_1, Tx_1, t_1), \mathcal{M}(Pu_1, Su_1, t_1), \mathcal{M}(Rx_1, Tx_1, t_1)\}$$
$$= \min\{\mathcal{M}(Pu_1, u_1, t_1), \mathcal{M}(Pu_1, Pu_1, t_1), \mathcal{M}(u_1, u_1, t_1)\}$$
$$= \mathcal{M}(Pu_1, u_1, t_1),$$

a contradiction. Therefore, $Pu_1 = u_1$.

We can easily show that $Ru_1 = u_1$.

Hence $Pu_1 = Ru_1 = Tu_1 = Su_1 = u_1$.

Uniqueness. Suppose $w_1(u_1 \neq w_1)$ be other point in $\mathcal{K}$ such that

$$Pw_1 = Rw_1 = Tw_1 = Sw_1 = w_1.$$

On setting $z = u_1$, $w = w_1$ in (2.2), we get $t_1 > 0$,

$$\begin{aligned}
\mathcal{M}(Pu_1, Rw_1, t_1) &= \mathcal{M}(u_1, w_1, t_1) \\
&> \min\{\mathcal{M}(Su_1, Tw_1, t_1), \mathcal{M}(Pu_1, Su_1, t_1), \mathcal{M}(Rw_1, Tw_1, t_1)\} \\
&= \min\{\mathcal{M}(u_1, w_1, t_1), \mathcal{M}(u_1, u_1, t_1), \mathcal{M}(w_1, w_1, t_1)\} \\
&= \mathcal{M}(u_1, w_1, t_1), \\
\mathcal{M}(u_1, w_1, t_1) &> \mathcal{M}(u_1, w_1, t_1)
\end{aligned}$$

a contradiction.

Hence $Pu_1 = Ru_1 = Tu_1 = Su_1 = u_1$, where u_1 is unique in $\mathcal{K}$. $\qquad\square$

Example 2.1. Let $\mathcal{K} = [2, 20]$ and $(\mathcal{K}, \mathcal{M}, \Delta)$ is a fuzzy metric space defined same as in Example 1.10.

Define self maps P, R, S and T on $\mathcal{K}$ as follows:

$$Px_1 = \begin{cases} 2 & \text{if } x_1 = 2 \text{ or } x_1 > 5 \\ x_1 + 1 & \text{if } 2 < x_1 \leq 5 \end{cases} \qquad Rx_1 = \begin{cases} 2 & \text{if } x_1 = 2 \text{ or } x_1 > 5 \\ x_1 + 2 & \text{if } 2 < x_1 \leq 5 \end{cases}$$

$$Sx_1 = \begin{cases} 2 & \text{if } x_1 = 2 \\ 8 & \text{if } 2 < x_1 = 5 \\ \frac{x_1 + 1}{3} & \text{if } x_1 > 5, \end{cases} \qquad Tx_1 = \begin{cases} 2 & \text{if } x_1 = 2 \text{ or } x_1 > 5 \\ x_1 + 1 & \text{if } 2 < x_1 = 5 \end{cases}$$

Then the self maps $P2 = S2 = T2 = R2 = 2$ and satisfy all the conditions of Theorem 2.1 and moreover the maps satisfy neither the φ-contractive condition nor the Banach type contractive condition.

Now we shall improve the above theorem the usage of commonplace property (E.A), because it relaxes containment of the variety of 1 map into the variety of other, that's utilized to assemble the collection of joint iterates in not unusual constant point issues.

Theorem 2.2. *Let P, R, S and T be four self mappings on a fuzzy metric space $(\mathcal{K}, \mathcal{M}, \Delta)$ with minimum t-norm satisfying (2.2), (2.4) and the following:*

$$\text{the pairs } (P, S) \text{ and } (R, T) \text{ holds common (E.A) property;} \qquad (5)$$

$$S\mathcal{K} \text{ and } T\mathcal{K} \text{ are closed subsets of } \mathcal{K}. \qquad (6)$$

Then the pairs (P, S) and (R, T) have coincidence points and $Pu_1 = Ru_1 = Tu_1 = Su_1 = u_1$, where u_1 is unique in $\mathcal{K}$.

Proof. In view of (5), there exists two sequences $\{z_n\}$ and $\{w_n\}$ in $\mathcal{K}$ satisfying

$$\begin{aligned}
\lim_{n \to \infty} Pz_n &= \lim_{n \to \infty} Sz_n \\
&= \lim_{n \to \infty} Rw_n = \lim_{n \to \infty} Tw_n = u_1, \text{ for some } u_1 \in \mathcal{K}.
\end{aligned}$$

Since $S\mathcal{K}$ be a closed subset of $\mathcal{K}$, we can find a point $v_1 \in \mathcal{K}$ satisfying $u_1 = Sv_1$.

First we claim that $Pv_1 = u_1$. If $Pv_1 \neq u_1$.

Putting $z = v_1$ and $w = w_n$ in (2.2), we have for $t_1 > 0$,

$$\mathcal{M}(Pv_1, Rw_n, t_1) > m(v_1, w_n, t_1)$$
$$= \min\{\mathcal{M}(Sv_1, Tw_n, t_1), \mathcal{M}(Pv_1, Sv_1, t_1), \mathcal{M}(Rw_n, Tw_n, t_1)\}.$$

Letting the limit as $n \to \infty$, we obtain

$$\mathcal{M}(Pv_1, u_1, t_1) > \min\{\mathcal{M}(u_1, u_1, t_1), \mathcal{M}(Pv_1, u_1, t_1), \mathcal{M}(u_1, u_1, t_1)\}$$
$$= \mathcal{M}(Pv_1, u_1, t_1),$$

a contradiction. Therefore, $Pv_1 = u_1 = Sv_1$.
Since $T\mathcal{K}$ is also closed subset of $\mathcal{K}$, $\lim\limits_{n \to \infty} Tw_n = u_1 \in T\mathcal{K}$ and hence we can find a point p_1 in $\mathcal{K}$ satisfying $Tp_1 = u_1 = Pv_1 = Sv_1$.
Now we show that $Rp_1 = u_1$. If $Rp_1 \neq u_1$.
On setting $z = v_1$ and $w = p_1$ in (2.2), we get for $t_1 > 0$,

$$\mathcal{M}(Pv_1, Rp_1, t_1) > m(v_1, p_1, t_1)$$
$$= \min\{\mathcal{M}(Sv_1, Tp_1, t_1), \mathcal{M}(Pv_1, Sv_1, t_1), \mathcal{M}(Rp_1, Tp_1, t_1)\},$$
$$\mathcal{M}(u_1, Rp_1, t_1) > \min\{\mathcal{M}(u_1, u_1, t_1), \mathcal{M}(u_1, u_1, t_1), \mathcal{M}(Rp_1, u_1, t_1)\}$$
$$= \mathcal{M}(Rp_1, u_1, t_1),$$

a contradiction. Therefore, $Rp_1 = u_1 = Tp_1$, which shows the pair (R, T) has a coincidence point p_1. Now weakly compatible of the pairs (P, S) and (R, T) implies, $Pv_1 = Sv_1, Rp_1 = Tp_1, Pu_1 = PSv_1 = SPv_1 = Su_1$ and $Ru_1 = RTp_1 = TRp_1 = Tu_1$.
Now we claim that $Pu_1 = u_1$. If $Pu_1 \neq u_1$.
Putting $z = u_1$ and $w = p_1$ in (2.2), for $t_1 > 0$,

$$\mathcal{M}(Pu_1, Rp_1, t_1) > m(u_1, p_1, t_1)$$
$$= \min\{\mathcal{M}(Su_1, Tp_1, t_1), \mathcal{M}(Pu_1, Su_1, t_1), \mathcal{M}(Rp_1, Tp_1, t_1)\},$$

i.e.,

$$\mathcal{M}(Pu_1, u_1, t_1) > \min\{\mathcal{M}(Pu_1, u_1, t_1), \mathcal{M}(Pu_1, Pu_1, t_1), \mathcal{M}(Rp_1, Rp_1, t_1)\}$$
$$= \mathcal{M}(Pu_1, u_1, t_1)$$

a contradiction. Therefore, $Pu_1 = u_1 = Su_1$. We can easily show that $Ru_1 = u_1 = Tu_1$.
Hence $Pu_1 = Ru_1 = Tu_1 = Su_1 = u_1$.
Uniqueness. Suppose $w_1(u_1 \neq w_1)$ be other point in $\mathcal{K}$ such that

$$Pw_1 = Rw_1 = Tw_1 = Sw_1 = w_1.$$

On setting $z = u_1$, $w = w_1$ in (2.2), we get $t_1 > 0$,

$$\mathcal{M}(Pu_1, Rw_1, t_1) = \mathcal{M}(u_1, w_1, t_1)$$
$$> \min\{\mathcal{M}(Su_1, Tw_1, t_1), \mathcal{M}(Pu_1, Su_1, t_1), \mathcal{M}(Rw_1, Tw_1, t_1)\}$$
$$= \min\{\mathcal{M}(u_1, w_1, t_1), \mathcal{M}(u_1, u_1, t_1), \mathcal{M}(w_1, w_1, t_1)\}$$
$$= \mathcal{M}(u_1, w_1, t_1),$$
$$\mathcal{M}(u_1, w_1, t_1) > \mathcal{M}(u_1, w_1, t_1)$$

a contradiction.
Hence $Pu_1 = Ru_1 = Tu_1 = Su_1 = u_1$, where u_1 is unique in $\mathcal{K}$. $\square$

We now give an example to illustrate the above theorem.

Example 2.2. Let $\mathcal{K} = [2, 20]$ and $(\mathcal{K}, \mathcal{M}, \Delta)$ be a fuzzy metric space defined same as in Example 1.10.
Define P, R, S and T on $\mathcal{K}$ as:

$$Px_1 = \begin{cases} 2 & \text{if } x_1 = 2 \text{ or } x_1 > 5 \\ x_1 + 1 & \text{if } 2 < x_1 \leq 5 \end{cases} \qquad Rx_1 = \begin{cases} 2 & \text{if } x_1 = 2 \text{ or } x_1 > 5 \\ x_1 + 2 & \text{if } 2 < x_1 \leq 5 \end{cases}$$

$$Sx_1 = \begin{cases} 2 & \text{if } x_1 = 2 \text{ or } x_1 > 5 \\ 8 & \text{if } 2 < x_1 \leq 5 \end{cases} \qquad Tx_1 = \begin{cases} 2 & \text{if } x_1 = 2 \text{ or } x_1 > 5 \\ 9 & \text{if } 2 < x_1 \leq 5 \end{cases}$$

Take $\left\{z_n = 5 + \frac{1}{n}\right\}$ and $\left\{w_n = 5 + \frac{1}{n}\right\}$. Then

$$\lim_{n \to \infty} Pz_n = \lim_{n \to \infty} Sz_n = \lim_{n \to \infty} Rw_n = \lim_{n \to \infty} Tw_n = 2 \in \mathcal{K}.$$

Thus pairs (P, S) and (R, T) satisfy common (E.A) property, $P2 = S2 = T2 = R2 = 2$ and holds all conditions of Theorem 2.2. Here $S\mathcal{K}$ and $T\mathcal{K}$ are closed subspaces of $\mathcal{K}$ whereas neither $P\mathcal{K}$ nor $R\mathcal{K}$ is closed subspace of $\mathcal{K}$. Moreover the maps satisfy neither the φ-contractive condition nor the Banach type contractive condition. Also, one may notice that neither $R\mathcal{K} \not\subseteq S\mathcal{K}$ nor $P\mathcal{K} \not\subseteq T\mathcal{K}$.

Finally, it is found that common (E.A) property calls for the completeness or closeness of the subspaces. So an attempt has been made to drop the closeness of the subspaces from Theorem 2.2 with the aid of the use of the $(JCLR_{ST})$ property.

Theorem 2.3. *Let P, R, S and T be four self maps on a fuzzy metric space $(\mathcal{K}, \mathcal{M}, \Delta)$ with minimum t-norm satisfying condition (2.2), (2.3) and the following:*

$$\text{the pairs } (P, S) \text{ and } (R, T) \text{ holds } (JCLR_{ST}) \text{ property.} \tag{7}$$

Then the pairs (P, S) and R, T have a coincidence point and $Pv_1 = Rv_1 = Tv_1 = Sv_1 = v_1$, where v_1 is unique in $\mathcal{K}$.

Proof. As the pairs (P, S) and (R, T) holds the $(JCLR_{ST})$ property, there exists two sequences $\{z_n\}$ and $\{w_n\}$ in $\mathcal{K}$ satisfying

$$\lim_{n \to \infty} Pz_n = \lim_{n \to \infty} Sz_n = \lim_{n \to \infty} Rw_n = \lim_{n \to \infty} Tw_n = Su_1 = Tu_1,$$

where $u_1 \in \mathcal{K}$.
First we claim that $Pu_1 = Su_1$. If $Pu_1 \neq Su_1$.
Setting $z = u_1$ and $w = w_n$ in (2.2), we have $t_1 > 0$,

$$\mathcal{M}(Pu_1, Rw_n, t_1) > m(u_1, w_n, t_1)$$
$$= \min\{\mathcal{M}(Su_1, Tw_n, t_1), \mathcal{M}(Pu_1, Su_1, t_1), \mathcal{M}(Rw_n, Tw_n, t_1)\}$$

Putting $n \to \infty$, we have

$$\mathcal{M}(Pu_1, Su_1, t_1) > \min\{\mathcal{M}(Su_1, Su_1, t_1), \mathcal{M}(Pu_1, Su_1, t_1), \mathcal{M}(Su_1, Su_1, t_1)\}$$
$$= \mathcal{M}(Pu_1, u_1, t_1),$$

a contradiction. Therefore, $Pu_1 = Su_1$.
Now we claim that $Ru_1 = Tu_1$.
If $Ru_1 \neq Tu_1$. Putting $z = u_1$ and $w = u_1$ in (2.2), we get $t_1 > 0$,

$$\mathcal{M}(Pu_1, Ru_1, t_1) > m(u_1, u_1, t_1)$$
$$= \min\{\mathcal{M}(Su_1, Tu_1, t_1), \mathcal{M}(Pu_1, Su_1, t_1), \mathcal{M}(Ru_1, Tu_1, t_1)\},$$

$$\mathcal{M}(Tu_1, Ru_1, t_1) > \min\{\mathcal{M}(Su_1, Su_1, t_1), \mathcal{M}(Tu_1, Tu_1, t_1), \mathcal{M}(Ru_1, Tu_1, t_1)\}$$
$$= \mathcal{M}(Ru_1, Tu_1, t_1),$$

a contradiction. Therefore, $Ru_1 = Tu_1$.

Thus we have $Tu_1 = Ru_1 = Pu_1 = Su_1$. Now we assume that $v_1 = Tu_1 = Ru_1 = Pu_1 = Su_1$. Now weakly compatible of the pairs (P, S) and (R, T) implies, $Pu_1 = Su_1$, $Ru_1 = Tu_1$, $Pv_1 = PSu_1 = SPu_1 = Sv_1$ and $Rv_1 = RTu_1 = TRu_1 = Tv_1$.

Now we claim that $Pv_1 = v_1$.

If $Pv_1 \neq v_1$. On putting $z = v_1$ and $w = u_1$ in (2.2), for $t_1 > 0$,

$$\mathcal{M}(Pv_1, Ru_1, t_1) > m(v_1, u_1, t_1)$$
$$= \min\{\mathcal{M}(Sv_1, Tu_1, t_1), \mathcal{M}(Pv_1, Sv_1, t_1), \mathcal{M}(Ru_1, Tu_1, t_1)\},$$

i.e.,

$$\mathcal{M}(Pv_1, v_1, t_1) > \min\{\mathcal{M}(Pv_1, u, t_1), \mathcal{M}(Pv_1, Pv_1, t_1), \mathcal{M}(v_1, v_1, t_1)\}$$
$$= \mathcal{M}(Pv, v_1, t_1)$$

a contradiction. Therefore, $Pv_1 = v_1 = Sv_1$. We can easily show that $Rv_1 = v_1 = Tv_1$.

Hence $Tv_1 = Rv_1 = Pv_1 = Sv_1 = v_1$.   □

We now give an example to illustrate the above theorem.

Example 2.3. Let $\mathcal{K} = [2, 20]$ and $(\mathcal{K}, \mathcal{M}, \Delta)$ is a fuzzy metric space defined same as in Example 1.10.

Define self maps P, R, S and T on $\mathcal{K}$ as follows:

$$Px_1 = \begin{cases} 2 & \text{if } x_1 = 2 \text{ or } x_1 > 5 \\ x_1 + 1 & \text{if } 2 < x_1 \leq 5 \end{cases} \qquad Rx_1 = \begin{cases} 2 & \text{if } x_1 = 2 \text{ or } x_1 > 5 \\ x_1 + 2 & \text{if } 2 < x_1 \leq 5 \end{cases}$$

$$Sx_1 = \begin{cases} 2 & \text{if } x_1 = 2 \text{ or } x_1 > 5 \\ x_1 + 1 & \text{if } 2 < x_1 \leq 5 \end{cases} \qquad Tx_1 = \begin{cases} 2 & \text{if } x_1 = 2 \text{ or } x_1 > 5 \\ x_1 + 9 & \text{if } 2 < x_1 \leq 5 \end{cases}$$

Take $\left\{z_n = 5 + \frac{1}{n}\right\}$ and $\left\{w_n = 5 + \frac{1}{n}\right\}$. Then

$$\lim_{n \to \infty} Pz_n = \lim_{n \to \infty} Sz_n = \lim_{n \to \infty} Rw = \lim_{n \to \infty} Tw_n = 2 = S2 = T2.$$

Thus pairs (P, S) and (R, T) satisfy $(JCLR_{ST})$ property, $P2 = S2 = T2 = R2 = 2$ and holds all the conditions of Theorem 2.3. Notice that none of $P\mathcal{K}$, $R\mathcal{K}$, $S\mathcal{K}$ and $T\mathcal{K}$ is a closed or complete subspace of $\mathcal{K}$. Moreover the maps satisfy neither the φ-contractive condition nor the Banach type contractive condition. Also $R\mathcal{K} \nsubseteq S\mathcal{K}$ and $P\mathcal{K} \nsubseteq T\mathcal{K}$.

3. Conclusions

Section 1 is essentially central to text. In this section, we give some basic definitions and results that we need in the sequel. It consists two sections. In first section, we deal with background of fuzzy fixed point theory. In second section, we give notations, preliminaries, basic definitions and basic results which are used throughout paper. Then in Section 1, first we prove a common fixed point theorem for four self maps in fuzzy metric spaces with minimum t-norm satisfying some Meir-keeler type contractive condition in which two pairs of mappings are weakly compatible and have coincidence point and in next (Theorem 2.2) we prove the same with E.A

property and Theorem 2.3 with $(JCLR_{ST})$ property. Some suitable examples are given to support our theorems.

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