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# COMPLETE HOMOGENEOUS SYMMETRIC FUNCTIONS OF THIRD AND SECOND-ORDER LINEAR RECURRENCE SEQUENCES

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Abstract. In this paper, we introduce an operator in order to derive a new symmetric functions of third and second-order linear recurrence sequences.

## 1. **Introduction and preliminaries**

In [20], the Gaussian generalized Tribonacci numbers $\{GV_n\}_{n\geq 0} = \{GV_n\,(GV_0, GV_1, GV_2)\}_{n\geq 0}$ is defined by

$$\left\{ \begin{array}{c} GV_n = GV_{n-1} + GV_{n-2} + GV_{n-3},\ n \geq 3 \\ GV_0 = c_0 + i\,(c_2 - c_1 - c_0)\,,\ GV_1 = c_1 + ic_0,\ GV_2 = c_2 + ic_1 \end{array} \right. .$$

Special cases of Gaussian generalized Tribonacci numbers $GV_n$ are Gaussian Tribonacci numbers $GV_n\,(0,\ 1,\ 1+i) = GT_n$ and Gaussian Tribonacci-Lucas numbers $GV_n\,(3-i,\ 1+3i,\ 3+i) = GK_n$. We formally define them as follows:

Gaussian Tribonacci numbers is defined by

$$GT_n = GT_{n-1} + GT_{n-2} + GT_{n-3},\ n \geq 3,$$

with initial conditions $GT_0 = 0$, $GT_1 = 1$, $GT_2 = 1+i$ and Gaussian Tribonacci-Lucas numbers is defined by

$$GK_n = GK_{n-1} + GK_{n-2} + GK_{n-3},\ n \geq 3,$$

with initial conditions $GK_0 = 3-i$, $GK_1 = 1+3i$ and $GK_2 = 3+i$ .

The authors in [10] defined and studied the trivariate Fibonacci and Lucas polynomials $H_n\,(x,y,t)$ and $K_n\,(x,y,t)$. They gave Binet's formulas, explicit formulas and some properties of these trivariate polynomials.

**Definition 1** For any integer $n \geq 3$, the trivariate Fibonacci polynomials, denoted by $(H_n\,(x,y,t))_{n\geq 3}$ is defined recursively by

$$H_n\,(x,y,t) = xH_{n-1}\,(x,y,t) + yH_{n-2}\,(x,y,t) + tH_{n-3}\,(x,y,t)\,,$$

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with the initials

$$H_0(x,y,t)=0,\ H_1(x,y,t)=1 \text{ and } H_2(x,y,t)=x.$$

**Definition 2** For any integer $n\geq 3$, the trivariate Lucas polynomials, denoted by $(K_n(x,y,t))_{n\geq 3}$ is defined recursively by

$$K_n(x,y,t)=xK_{n-1}(x,y,t)+yK_{n-2}(x,y,t)+tK_{n-3}(x,y,t),$$

with the initials

$$K_0(x,y,t)=3,\ K_1(x,y,t)=x \text{ and } K_2(x,y,t)=x^2+2y.$$

The Binet's formulas of trivariate Fibonacci and Lucas polynomials are

$$H_n(x,y,t)=\frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)},$$

and

$$K_n(x,y,t)=\alpha^n+\beta^n+\gamma^n,$$

respectively, where $\alpha$, $\beta$ and $\gamma$ are the roots of the characteristic equation $z^3-xz^2-yz-t=0$.

In [9], Kocer consider the bivariate Vieta-Fibonacci and bivariate Vieta-Lucas polynomials which are defined by the following recurrence relations, for $n\geq 2$

$$V_n(x,y)=xV_{n-1}(x,y)-yV_{n-2}(x,y) \text{ with } V_0(x,y)=0,\ V_1(x,y)=1,$$

and

$$v_n(x,y)=xv_{n-1}(x,y)-yv_{n-2}(x,y) \text{ with } v_0(x,y)=2,\ v_1(x,y)=x.$$

In 2018, Catarino introduced the $k$-Pell and $k$-Pell Lucas polynomials which are defined recursively by

$$P_{k,n+2}(x)=2xP_{k,n+1}(x)+kP_{k,n}(x) \text{ with } P_{k,0}(x)=0,\ P_{k,1}(x)=1,$$

and

$$Q_{k,n+2}(x)=2xQ_{k,n+1}(x)+kQ_{k,n}(x) \text{ with } Q_{k,0}(x)=2,\ Q_{k,1}(x)=2x,$$

respectively, for more information see the paper [17].

In [15], N. Karaaslan and T. Yagmur defined the $(p,q)$-modified Pell numbers by

$$MP_{p,q,n}=2pMP_{p,q,n-1}+qMP_{p,q,n-2},\ n\geq 2,$$

with $MP_{p,q,0}=1$ and $MP_{p,q,1}=p$.

We define some Gaussian numbers (see [8, 13, 14]).

| Gaussian numbers | Linear recurrence sequences | Initial conditions |
|---|---|---|
| Gaussian Perrin numbers | $Gr_n = Gr_{n-2} + Gr_{n-3},\ n \geq 3$ | $\begin{cases} Gr_0 = -1+3i \\ Gr_1 = 3 \\ Gr_2 = 2i \end{cases}$ |
| Gaussian Padovan numbers | $GP_n = GP_{n-2} + GP_{n-3},\ n \geq 3$ | $\begin{cases} GP_0 = 1 \\ GP_1 = 1+i \\ GP_2 = 1+i \end{cases}$ |
| Gaussian Pell Padovan numbers | $GR_n = 2GR_{n-2} + GR_{n-3},\ n \geq 3$ | $\begin{cases} GR_0 = 1-i \\ GR_1 = 1+i \\ GR_2 = 1+i \end{cases}$ |
| Gaussian $(p,q)$-Pell numbers | $GP_{p,q,n} = 2pGP_{p,q,n-1} + qGP_{p,q,n-2},\ n \geq 2$ | $\begin{cases} GP_{p,q,0} = i \\ GP_{p,q,1} = 1 \end{cases}$ |
| Gaussian $(p,q)$-Pell Lucas numbers | $GQ_{p,q,n} = 2pGQ_{p,q,n-1} + qGQ_{p,q,n-2},\ n \geq 2$ | $\begin{cases} GQ_{p,q,0} = 2-2ip \\ GQ_{p,q,1} = 2p+2iq \end{cases}$ |
| Gaussian $(p,q)$-Fibonacci numbers | $GF_{p,q,n} = pGF_{p,q,n-1} + qGF_{p,q,n-2},\ n \geq 2$ | $\begin{cases} GF_{p,q,0} = i \\ GF_{p,q,1} = 1 \end{cases}$ |
| Gaussian $(p,q)$-Lucas numbers | $GL_{p,q,n} = pGL_{p,q,n-1} + qGL_{p,q,n-2},\ n \geq 2$ | $\begin{cases} GL_{p,q,0} = 2-ip \\ GL_{p,q,1} = p+2iq \end{cases}$ |

**Table 1.** Gaussian numbers.

In this part we define some Gaussian polynomials.

**Definition 3** [7] For $n \in \mathbb{N}$, the generalized Gaussian Jacobsthal polynomials $\{GJ_{k,n}(x)\}_{n\in\mathbb{N}}$ is defined recurrently by

$$GJ_{k,n+1}(x) = GJ_{k,n}(x) + 2^k x GJ_{k,n-1}(x),\ \text{for } n \geq 1,$$

with initial conditions $GJ_{k,0}(x) = \frac{i}{2}$, $GJ_{k,1}(x) = 1$.

**Definition 4** [7] For $n \in \mathbb{N}$, the generalized Gaussian Jacobsthal Lucas polynomials $\{Gj_{k,n}(x)\}_{n\in\mathbb{N}}$ is defined recursively by

$$Gj_{k,n+1}(x) = Gj_{k,n}(x) + 2^k x Gj_{k,n-1}(x),\ \text{for } n \geq 1,$$

with initial conditions $Gj_{k,0}(x) = 2 - \frac{i}{2}$, $Gj_{k,1}(x) = 1 + 2xi$.

**Definition 5** For $n \in \mathbb{N}$, the Gaussian Padovan polynomials, denoted by $\{GP_n(x)\}_{n\in\mathbb{N}}$ is defined recurrently by

$$\begin{cases} GP_n(x) = xGP_{n-2}(x) + GP_{n-3}(x),\ n \geq 3 \\ GP_0(x) = 1,\ GP_1(x) = GP_2(x) = 1+i \end{cases}.$$

**Definition 6** For $n \in \mathbb{N}$, the Gaussian Pell Padovan polynomials, denoted by $\{GR_n(x)\}_{n\in\mathbb{N}}$ is defined recursively by

$$\begin{cases} GR_n(x) = 2xGR_{n-2}(x) + GR_{n-3}(x),\ n \geq 3 \\ GR_0(x) = 1-i,\ GR_1(x) = GR_2(x) = 1+i \end{cases}.$$

Next, we recall some properties of the symmetric functions that we will need in the sequel.

**Definition 7** Let $k$ and $n$ be two positive integers and $\{p_1, p_2, ..., p_n\}$ are set of given variables the $k$-th complete homogeneous symmetric function $h_k(p_1, p_2, ..., p_n)$ is defined by

$$h_k(p_1, p_2, ..., p_n) = \sum_{i_1+i_2+...+i_n=k} p_1^{i_1} p_2^{i_2} ... p_n^{i_n} \qquad (0 \leq k \leq n),$$

with $i_1, i_2, ..., i_n \geq 0$.

**Remark 1** Set $h_0\left(p_1, p_2, ..., p_n\right) = 1$, by usual convention. For $k < 0$, we set $h_k\left(p_1, p_2, ..., p_n\right) = 0$.

**Definition 8** [1] Let $A$ and $P$ be any two alphabets. We define $S_n(A-P)$ by the following form:

$$\frac{\Pi_{p\epsilon P}(1-pz)}{\Pi_{a\epsilon A}(1-az)} = \sum_{n=0}^{\infty} S_n(A-P)z^n, \tag{1}$$

with the condition $S_n(A-P) = 0$ for $n < 0$.

Equation (1) can be rewritten in the following form

$$\sum_{n=0}^{\infty} S_n(A-P)z^n = \left(\sum_{n=0}^{\infty} S_n(A)z^n\right) \times \left(\sum_{n=0}^{\infty} S_n(-P)z^n\right), \tag{2}$$

where

$$S_n(A-P) = \sum_{j=0}^{n} S_{n-j}(-P)S_j(A).$$

**Remark 2** Taking $A = \{0\}$ in (1) gives

$$\sum_{n=0}^{\infty} S_n\left(-P\right)z^n = \prod_{p\in P}\left(1-pz\right).$$

**Definition 9** [2] Given a function $g$ on $\mathbb{R}^n$, the divided difference operator is defined as follows

$$\partial_{p_i p_{i+1}}(g) = \frac{g(p_1, \cdots, p_i, p_{i+1}, \cdots p_n) - g(p_1, \cdots p_{i-1}, p_{i+1}, p_i, p_{i+2} \cdots p_n)}{p_i - p_{i+1}}.$$

**Definition 10** Let $n$ be a positive integer and $P = \{p_1, p_2\}$ be set of given variables, then, the $n$-th symmetric function $S_n(p_1+p_2)$ is defined by

$$S_n(P) = S_n(p_1+p_2) = \frac{p_1^{n+1} - p_2^{n+1}}{p_1 - p_2},$$

with

$$\begin{aligned} S_0(P) &= S_0(p_1+p_2) = 1, \\ S_1(P) &= S_1(p_1+p_2) = p_1+p_2, \\ S_2(P) &= S_2(p_1+p_2) = p_1^2 + p_1p_2 + p_2^2, \\ &\vdots \end{aligned}$$

**Definition 11** [3] Given an alphabet $P = \{p_1, p_2\}$, the symmetrizing operator $\delta_{p_1p_2}^k$ is defined by

$$\delta_{p_1p_2}^k g(p_1) = \frac{p_1^k g\left(p_1\right) - p_2^k g\left(p_2\right)}{p_1 - p_2}, \text{ for all } k \in \mathbb{N}_0. \tag{3}$$

If $g\left(p_1\right) = p_1$, the operator (3) gives us

$$\delta_{p_1p_2}^k g(p_1) = \frac{p_1^{k+1} - p_2^{k+1}}{p_1 - p_2} = S_k\left(p_1+p_2\right).$$

## 2. **Theorem and proof**

The following theorem is one of the key tools of the proof of our main results. It has been proved in [4]. For the completeness of the paper we state its proof here.

**Theorem 1** Given two alphabets $P=\{p_1,p_2\}$ and $A=\{a_1,a_2,a_3\}$, we have

$$ {}_{n=0}^{\infty} S_n(A)\partial_{p_1p_2}(p_1^{n+1})z^n=\frac{S_0(-A)-p_1p_2S_2(-A)z^2-p_1p_2S_3(-A)S_1(P)z^3}{\left(\sum\limits_{n=0}^{\infty}S_n(-A)p_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty}S_n(-A)p_2^nz^n\right)}, \quad (4)$$

with $S_0(-A)=1,\ S_2(-A)=a_1a_2+a_1a_3+a_2a_3,\ S_3(-A)=-a_1a_2a_3$.

**Proof.** Let $\sum\limits_{n=0}^{\infty}S_n(A)z^n$ and $\sum\limits_{n=0}^{\infty}S_n(-A)z^n$ be two sequences such that $\sum\limits_{n=0}^{\infty}S_n(A)z^n=\frac{1}{\sum\limits_{n=0}^{\infty}S_n(-A)z^n}$. On one hand, since $g(p_1)=\sum\limits_{n=0}^{\infty}S_n(A)p_1^nz^n$ and $g(p_2)=\sum\limits_{n=0}^{\infty}S_n(A)p_2^nz^n$, we have

$$\begin{aligned}
\delta_{p_1p_2}g(p_1) &= \delta_{p_1p_2}\left(\sum_{n=0}^{\infty}S_n(A)p_1^nz^n\right)\\
&= \frac{p_1\sum\limits_{n=0}^{\infty}S_n(A)p_1^nz^n-p_2\sum\limits_{n=0}^{\infty}S_n(A)p_2^nz^n}{p_1-p_2}\\
&= \sum_{n=0}^{\infty}S_n(A)\left(\frac{p_1^{n+1}-p_2^{n+1}}{p_1-p_2}\right)z^n\\
&= \sum_{n=0}^{\infty}S_n(A)\partial_{p_1p_2}(p_1^{n+1})z^n,
\end{aligned}$$

which is the right-hand side of (4). On the other part, since

$$g(p_1)=\frac{1}{\sum\limits_{n=0}^{\infty}S_n(-A)p_1^nz^n},$$

we have

$$\begin{aligned}
\delta_{p_1p_2}g(p_1) &= \frac{\frac{p_1}{\sum\limits_{n=0}^{\infty}S_n(-A)p_1^nz^n}-\frac{p_2}{\sum\limits_{n=0}^{\infty}S_n(-A)p_2^nz^n}}{p_1-p_2}\\
&= \frac{p_1\sum\limits_{n=0}^{\infty}S_n(-A)p_2^nz^n-p_2\sum\limits_{n=0}^{\infty}S_n(-A)p_1^nz^n}{(p_1-p_2)\left(\sum\limits_{n=0}^{\infty}S_n(-A)p_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty}S_n(-A)p_2^nz^n\right)}\\
&= \frac{\sum\limits_{n=0}^{\infty}S_n(-A)\frac{p_1p_2^n-p_2p_1^n}{p_1-p_2}z^n}{\left(\sum\limits_{n=0}^{\infty}S_n(-A)p_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty}S_n(-A)p_2^nz^n\right)}\\
&= \frac{S_0(-A)-p_1p_2S_2(-A)z^2-p_1p_2S_3(-A)S_1\left(P\right)z^3}{\left(\sum\limits_{n=0}^{\infty}S_n(-A)p_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty}S_n(-A)p_2^nz^n\right)}.
\end{aligned}$$

This completes the proof.

## 3. Applications on third-order linear recurrence sequences

In this part, we now derive the generating functions of Gaussian generalized Tribonacci numbers, Gaussian Padovan numbers and Gaussian Perrin numbers, Gaussian Pell Padovan numbers, trivariate Fibonacci polynomials and trivariate Lucas polynomials, Gaussian Padovan polynomials and Gaussian Pell Padovan polynomials. The technique used is based on the theory of the so called symmetric functions.

- For the case $A = \{a_1, a_2, a_3\}$ and $P = \{1, 0\}$ in theorem 1 we deduce the following lemma.

**Lemma 1** Given an alphabet $A = \{a_1, a_2, a_3\}$, we have

$$\sum_{n=0}^{\infty} S_n(A) z^n = \frac{1}{(1-a_1z)(1-a_2z)(1-a_3z)}. \tag{5}$$

Based on the relationship (5) we have

$$\sum_{n=0}^{\infty} S_{n-1}(A) z^n = \frac{z}{(1-a_1z)(1-a_2z)(1-a_3z)}, \tag{6}$$

and

$$\sum_{n=0}^{\infty} S_{n-2}(A) z^n = \frac{z^2}{(1-a_1z)(1-a_2z)(1-a_3z)}, \tag{7}$$

with $(1-a_1z)(1-a_2z)(1-a_3z) = 1 - (a_1+a_2+a_3)z + (a_1a_2+a_1a_3+a_2a_3)z^2 - a_1a_2a_3z^3$.

### 3.1. Construction of generating functions of some well-known numbers.

This part consists of three cases.

**Case 1.** The substitution of $\begin{cases} a_1+a_2+a_3 = 1 \\ a_1a_2+a_1a_3+a_2a_3 = -1 \\ a_1a_2a_3 = 1 \end{cases}$ in (5), (6) and (7), we obtain

$$\sum_{n=0}^{\infty} S_n(A) z^n = \frac{1}{1-z-z^2-z^3}, \tag{8}$$

$$\sum_{n=0}^{\infty} S_{n-1}(A) z^n = \frac{z}{1-z-z^2-z^3}, \tag{9}$$

$$\sum_{n=0}^{\infty} S_{n-2}(A) z^n = \frac{z^2}{1-z-z^2-z^3}, \tag{10}$$

respectively.

Multiplying the equation (8) by $(GV_0)$ and adding it to the equation obtained by (9) multiplying by $(GV_1 - GV_0)$ and adding it to the equation obtained by (10) multiplying by $(GV_2 - GV_1 - GV_0)$, then we obtain the following proposition.

**Proposition 1** For $n \in \mathbb{N}$, the generating function of Gaussian generalized Tribonacci numbers is given by

$$\sum_{n=0}^{\infty} GV_n z^n = \frac{GV_0 + (GV_1 - GV_0)z + (GV_2 - GV_1 - GV_0)z^2}{1-z-z^2-z^3}. \tag{11}$$

We can state the following corollary.

**Corollary 1** The following identity holds true:

$$GV_n = GV_0 S_n(A) + (GV_1 - GV_0) S_{n-1}(A) + (GV_2 - GV_1 - GV_0) S_{n-2}(A).$$

- Put $GV_0 = 0$, $GV_1 = 1$ and $GV_2 = 1 + i$ in the relationship (11), we can state the following corollary.

**Corollary 2** For $n \in \mathbb{N}$, the generating function of Gaussian Tribonacci numbers is given by

$$\sum_{n=0}^{\infty} GT_n z^n = \frac{z + iz^2}{1 - z - z^2 - z^3}, \text{ with } GT_n = S_{n-1}(A) + iS_{n-2}(A). \tag{12}$$

- Put $GV_0 = 3 - i$, $GV_1 = 1 + 3i$ and $GV_2 = 3 + i$ in the relationship (11), we can state the following corollary.

**Corollary 3** For $n \in \mathbb{N}$, the generating function of Gaussian Tribonacci-Lucas numbers is given by

$$\sum_{n=0}^{\infty} GK_n z^n = \frac{3 - i - (2 - 4i) z - (1 + i) z^2}{1 - z - z^2 - z^3}, \tag{13}$$

with $GK_n = (3 - i) S_n(A) - (2 - 4i) S_{n-1}(A) - (1 + i) S_{n-2}(A)$.

**Case 2.** The substitution $\begin{cases} a_1 + a_2 + a_3 = 0 \\ a_1a_2 + a_1a_3 + a_2a_3 = -1 \\ a_1a_2a_3 = 1 \end{cases}$ in (5), (6) and (7), we obtain

$$\sum_{n=0}^{\infty} S_n(A) z^n = \frac{1}{1 - z^2 - z^3}, \tag{14}$$

$$\sum_{n=0}^{\infty} S_{n-1}(A) z^n = \frac{z}{1 - z^2 - z^3}, \tag{15}$$

$$\sum_{n=0}^{\infty} S_{n-2}(A) z^n = \frac{z^2}{1 - z^2 - z^3}, \tag{16}$$

respectively.

Multiplying the equation (16) by $(i)$ and adding it to the equation obtained by (15) multiplying by $(1 + i)$ and adding it to the equation (14), then we have the following proposition.

**Proposition 2** For $n \in \mathbb{N}$, the generating function of Gaussian Padovan numbers is given by

$$\sum_{n=0}^{\infty} GP_n z^n = \frac{1 + (1 + i) z + iz^2}{1 - z^2 - z^3}. \tag{17}$$

We have the following corollary.

**Corollary 4** The following identity holds true:

$$GP_n = S_n(A) + (1 + i) S_{n-1}(A) + iS_{n-2}(A).$$

Multiplying the equation (14) by $(-1 + 3i)$ and adding it to the equation obtained by (15) multiplying by $(3)$ and adding it to the equation obtained by (16) multiplying by $(1 - i)$, then we obtain

$$\sum_{n=0}^{\infty} \left((-1 + 3i) S_n(A) + 3S_{n-1}(A) + (1 - i) S_{n-2}(A)\right) z^n = \frac{-1 + 3i + 3z + (1 - i) z^2}{1 - z^2 - z^3},$$

and we have the following proposition.

**Proposition 3** For $n \in \mathbb{N}$, the generating function of Gaussian Perrin numbers is given by

$$\sum_{n=0}^{\infty} Gr_n z^n = \frac{-1+3i+3z+(1-i)\,z^2}{1-z^2-z^3}, \tag{18}$$

with $Gr_n = (-1+3i)\,S_n(A) + 3S_{n-1}(A) + (1-i)\,S_{n-2}\,(A)$.

**Case 3.** The setting of $\begin{cases} a_1+a_2+a_3=0 \\ a_1a_2+a_1a_3+a_2a_3=-2 \\ a_1a_2a_3=1 \end{cases}$ in (5), (6) and (7), we obtain

$$\sum_{n=0}^{\infty} S_n\,(A)\,z^n = \frac{1}{1-2z^2-z^3}, \tag{19}$$

$$\sum_{n=0}^{\infty} S_{n-1}\,(A)\,z^n = \frac{z}{1-2z^2-z^3}, \tag{20}$$

$$\sum_{n=0}^{\infty} S_{n-2}\,(A)\,z^n = \frac{z^2}{1-2z^2-z^3}, \tag{21}$$

respectively.

Multiplying the equation (19) by $(1-i)$ and adding it to the equation obtained by (20) multiplying by $(1+i)$ and adding it to the equation obtained by (21) multiplying by $(-1+3i)$, then we deduce the following proposition and corollary.

**Proposition 4** For $n \in \mathbb{N}$, the generating function of Gaussian Pell-Padovan numbers is given by

$$\sum_{n=0}^{\infty} GR_n z^n = \frac{1-i+(1+i)\,z+(-1+3i)z^2}{1-2z^2-z^3}. \tag{22}$$

**Corollary 5** The following identity holds true:

$$GR_n = (1-i)\,S_n(A) + (1+i)\,S_{n-1}(A) + (-1+3i)S_{n-2}\,(A)\,.$$

### 3.2. **Construction of generating functions of some well-known polynomials.** This part consists of three cases.

**Case 1.** The Setting of $\begin{cases} a_1+a_2+a_3=x \\ a_1a_2+a_1a_3+a_2a_3=-y \\ a_1a_2a_3=t \end{cases}$ in (5), (6) and (7), we obtain

$$\sum_{n=0}^{\infty} S_n\,(A)\,z^n = \frac{1}{1-xz-yz^2-tz^3}, \tag{23}$$

$$\sum_{n=0}^{\infty} S_{n-1}\,(A)\,z^n = \frac{z}{1-xz-yz^2-tz^3}, \tag{24}$$

$$\sum_{n=0}^{\infty} S_{n-2}\,(A)\,z^n = \frac{z^2}{1-xz-yz^2-tz^3}, \tag{25}$$

respectively, and we have the following corollary.

**Corollary 6** For $n \in \mathbb{N}$, the generating function of trivariate Fibonacci polynomials is given by

$$\sum_{n=0}^{\infty} H_n(x,y,t)\, z^n = \frac{z}{1-xz-yz^2-tz^3}, \text{ with } H_n(x,y,t) = S_{n-1}(A). \tag{26}$$

Multiplying the equation (23) by (3) and adding it to the equation obtained by (24) multiplying by $(-2x)$ and adding it to the equation obtained by (25) multiplying by $(-y)$, then we deduce the following proposition and corollary.

**Proposition 5** For $n \in \mathbb{N}$, the generating function of trivariate Lucas polynomials is given by

$$\sum_{n=0}^{\infty} K_n(x,y,t)\, z^n = \frac{3-2xz-yz^2}{1-xz-yz^2-tz^3}. \tag{27}$$

**Corollary 7** The following identity holds true:

$$K_n(x,y,t) = 3S_n(A) - 2xS_{n-1}(A) - yS_{n-2}(A).$$

- Writing $x^2$ instead of $x$, $x$ instead of $y$ and taking $t=1$ in (26) and (27), we have the following corollaries.

**Corollary 8** For $n \in \mathbb{N}$, the generating function of Tribonacci polynomials is given by

$$\sum_{n=0}^{\infty} T_n(x)\, z^n = \frac{z}{1-x^2z-xz^2-z^3}, \text{ with } T_n(x) = S_{n-1}(A).$$

**Corollary 9** For $n \in \mathbb{N}$, the generating function of Tribonacci Lucas polynomials is given by

$$\sum_{n=0}^{\infty} K_n(x)\, z^n = \frac{3-2x^2z-xz^2}{1-x^2z-xz^2-z^3}, \text{ with } K_n(x) = 3S_n(A) - 2x^2S_{n-1}(A) - xS_{n-2}(A).$$

- Based on the relationships (26) and (27) and with $x=y=t=1$, we obtain the following corollaries.

**Corollary 10** For $n \in \mathbb{N}$, the generating function of Tribonacci numbers is given by

$$\sum_{n=0}^{\infty} T_n z^n = \frac{z}{1-z-z^2-z^3}, \text{ with } T_n = S_{n-1}(A).$$

**Corollary 11** [12] For $n \in \mathbb{N}$, the generating function of Tribonacci Lucas numbers is given by

$$\sum_{n=0}^{\infty} K_n z^n = \frac{3-2z-z^2}{1-z-z^2-z^3}, \text{ with } K_n = 3S_n(A) - 2S_{n-1}(A) - S_{n-2}(A).$$

**Case 2.** The substitution of $\begin{cases} a_1+a_2+a_3=0 \\ a_1a_2+a_1a_3+a_2a_3=-x \\ a_1a_2a_3=1 \end{cases}$ in (5), (6) and (7), we obtain

$$\sum_{n=0}^{\infty} S_n(A)\, z^n = \frac{1}{1-xz^2-z^3}, \tag{28}$$

$$\sum_{n=0}^{\infty} S_{n-1}(A) z^n = \frac{z}{1 - xz^2 - z^3}, \tag{29}$$

$$\sum_{n=0}^{\infty} S_{n-2}(A) z^n = \frac{z^2}{1 - xz^2 - z^3}, \tag{30}$$

respectively.

Multiplying the equation (30) by $(1 - x + i)$ and adding it to the equation obtained by (29) multiplying by $(1 + i)$ and adding it to the equation (28), then we obtain

$$\sum_{n=0}^{\infty} (S_n(A) + (1+i) S_{n-1}(A) + (1 - x + i) S_{n-2}(A)) z^n = \frac{1 + (1+i) z + (1 - x + i) z^2}{1 - xz^2 - z^3},$$

and we have the following proposition.

**Proposition 6** For $n \in \mathbb{N}$, the generating function of Gaussian Padovan polynomials is given by

$$\sum_{n=0}^{\infty} GP_n(x) z^n = \frac{1 + (1+i) z + (1 - x + i) z^2}{1 - xz^2 - z^3}, \tag{31}$$

with $GP_n(x) = S_n(A) + (1+i) S_{n-1}(A) + (1 - x + i) S_{n-2}(A)$.

**Case 3.** Taking $\begin{cases} a_1 + a_2 + a_3 = 0 \\ a_1a_2 + a_1a_3 + a_2a_3 = -2x \\ a_1a_2a_3 = 1 \end{cases}$ in (5), (6) and (7), we obtain

$$\sum_{n=0}^{\infty} S_n(A) z^n = \frac{1}{1 - 2xz^2 - z^3}, \tag{32}$$

$$\sum_{n=0}^{\infty} S_{n-1}(A) z^n = \frac{z}{1 - 2xz^2 - z^3}, \tag{33}$$

$$\sum_{n=0}^{\infty} S_{n-2}(A) z^n = \frac{z^2}{1 - 2xz^2 - z^3}, \tag{34}$$

respectively.

Multiplying the equation (32) by $(1 - i)$ and adding it to the equation obtained by (33) multiplying by $(1 + i)$ and adding it to the equation obtained by (34) multiplying by $(1 - 2x + i(1 + 2x))$, then we have the following proposition and corollary.

**Proposition 7** For $n \in \mathbb{N}$, the generating function of Gaussian Pell-Padovan polynomials $GR_n(x)$ is given by

$$\sum_{n=0}^{\infty} GR_n(x) z^n = \frac{1 - i + (1+i) z + (1 - 2x + i(1+2x)) z^2}{1 - 2xz^2 - z^3}. \tag{35}$$

**Corollary 12** The following identity holds true:

$$GR_n(x) = (1 - i) S_n(A) + (1+i) S_{n-1}(A) + (1 - 2x + i(1+2x)) S_{n-2}(A).$$

## 4. Applications on second-order linear recurrence sequences

In this part, we now derive the generating functions of Gaussian $(p,q)$ numbers, $(p,q)$-modified Pell numbers and bivariate Vieta polynomials, Gaussian generalized polynomials, $k$-Pell polynomials and $k$-Pell Lucas polynomials.

- For the case $A=\{a_1,-a_2,0\}$ and $P=\{1,0\}$ in theorem 1 we deduce the following lemma.

**Lemma 2** Given an alphabet $A=\{a_1,-a_2\}$, we have

$$\sum_{n=0}^{\infty} S_n\left(a_1+[-a_2]\right)z^n=\frac{1}{1-\left(a_1-a_2\right)z-a_1a_2z^2}. \tag{36}$$

Based on the relationship (36) we have

$$\sum_{n=0}^{\infty} S_{n-1}\left(a_1+[-a_2]\right)z^n=\frac{z}{1-\left(a_1-a_2\right)z-a_1a_2z^2}. \tag{37}$$

### 4.1. Construction of generating functions for Gaussian $(p,q)$-numbers and $(p,q)$-modified Pell numbers.

This part consists of two cases.

**Case 1.** The substitution of $\begin{cases} a_1-a_2=p \\ a_1a_2=q \end{cases}$ in (36) and (37), we obtain

$$\sum_{n=0}^{\infty} S_n\left(a_1+[-a_2]\right)z^n=\frac{1}{1-pz-qz^2}, \tag{38}$$

$$\sum_{n=0}^{\infty} S_{n-1}\left(a_1+[-a_2]\right)z^n=\frac{z}{1-pz-qz^2}, \tag{39}$$

respectively.

Multiplying the equation (38) by $(i)$ and adding it to the equation obtained by (39) multiplying by $(1-pi)$, then we obtain the following proposition.

**Proposition 8** For $n\in\mathbb{N}$, the generating function of Gaussian $(p,q)$-Fibonacci numbers $GF_{p,q,n}$ is given by

$$\sum_{n=0}^{\infty} GF_{p,q,n}z^n=\frac{i+(1-pi)\,z}{1-pz-qz^2}, \tag{40}$$

with $GF_{p,q,n}=iS_n(a_1+[-a_2])+(1-pi)\,S_{n-1}(a_1+[-a_2])$.

Multiplying the equation (38) by $(2-pi)$ and adding it to the equation obtained by (39) multiplying by $\left(i\left(p^2+2q\right)-p\right)$, then we have the following proposition.

**Proposition 9** For $n\in\mathbb{N}$, the generating function of Gaussian $(p,q)$-Lucas numbers $GL_{p,q,n}$ is given by

$$\sum_{n=0}^{\infty} GL_{p,q,n}z^n=\frac{(2-pi)+\left(i\left(p^2+2q\right)-p\right)z}{1-pz-qz^2}, \tag{41}$$

with $GL_{p,q,n}=(2-pi)\,S_n(a_1+[-a_2])+\left(i\left(p^2+2q\right)-p\right)S_{n-1}(a_1+[-a_2])$.

- Based on the relationships (40) and (41) and with $p=q=1$, we obtain the following corollaries.

**Corollary 13** [18] For $n \in \mathbb{N}$, the generating function of Gaussian Fibonacci numbers $GF_n$ is given by

$$\sum_{n=0}^{\infty} GF_n z^n = \frac{i + (1-i)\,z}{1 - z - z^2}, \text{ with } GF_n = iS_n(a_1 + [-a_2]) + (1-i)\,S_{n-1}(a_1 + [-a_2]).$$

**Corollary 14** [18] For $n \in \mathbb{N}$, the generating function of Gaussian Lucas numbers $GL_n$ is given by

$$\sum_{n=0}^{\infty} GL_n z^n = \frac{(2-i) + (3i-1)\,z}{1 - z - z^2}, \text{ with } GL_n = (2-i)\,S_n(a_1+[-a_2]) + (3i-1)\,S_{n-1}(a_1+[-a_2]).$$

**Case 2.** Assuming that $\left\{ \begin{array}{l} a_1 - a_2 = 2p \\ a_1 a_2 = q \end{array} \right.$ in (36) and (37), we get

$$\sum_{n=0}^{\infty} S_n\,(a_1 + [-a_2])\,z^n = \frac{1}{1 - 2pz - qz^2}, \tag{42}$$

$$\sum_{n=0}^{\infty} S_{n-1}\,(a_1 + [-a_2])\,z^n = \frac{z}{1 - 2pz - qz^2}, \tag{43}$$

respectively.

Multiplying the equation (42) by $(i)$ and adding it to the equation obtained by (43) multiplying by $(1-2pi)$, then we have the following proposition and corollary.

**Proposition 10** For $n \in \mathbb{N}$, the generating function of Gaussian $(p,q)$-Pell numbers $GP_{p,q,n}$ is given by

$$\sum_{n=0}^{\infty} GP_{p,q,n} z^n = \frac{i + (1-2pi)\,z}{1 - 2pz - qz^2}. \tag{44}$$

**Corollary 15** The following identity holds true:

$$GP_{p,q,n} = iS_n(a_1 + [-a_2]) + (1-2pi)\,S_{n-1}(a_1 + [-a_2]).$$

Multiplying the equation (42) by $(2-2pi)$ and adding it to the equation obtained by (43) multiplying by $\left(i\left(4p^2 + 2q\right) - 2p\right)$, then we obtain the following proposition.

**Proposition 11** For $n \in \mathbb{N}$, the generating function of Gaussian $(p,q)$-Pell Lucas numbers $GQ_{p,q,n}$ is given by

$$\sum_{n=0}^{\infty} GQ_{p,q,n} z^n = \frac{(2-2pi) + \left(i\left(4p^2 + 2q\right) - 2p\right)z}{1 - 2pz - qz^2}, \tag{45}$$

with $GQ_{p,q,n} = (2-2pi)\,S_n(a_1 + [-a_2]) + \left(i\left(4p^2 + 2q\right) - 2p\right)S_{n-1}(a_1 + [-a_2])$.

Multiplying the equation (43) by $(-p)$ and adding it to the equation (42), then we have the following proposition and corollary.

**Proposition 12** For $n \in \mathbb{N}$, the generating function of $(p,q)$-modified Pell numbers $MP_{p,q,n}$ is given by

$$\sum_{n=0}^{\infty} MP_{p,q,n} z^n = \frac{1 - pz}{1 - 2pz - qz^2}. \tag{46}$$

**Corollary 16** The following identity holds true:

$$MP_{p,q,n} = S_n(a_1 + [-a_2]) - pS_{n-1}(a_1 + [-a_2]).$$

- Based on the relationships (44), (45) and (46) and with $p = q = 1$, we obtain the following corollaries.

**Corollary 17** [18] For $n \in \mathbb{N}$, the generating function of Gaussian Pell numbers $GP_n$ is given by

$$\sum_{n=0}^{\infty} GP_n z^n = \frac{i + (1-2i)\,z}{1-2z-z^2},\ \text{with } GP_n = iS_n(a_1+[-a_2]) + (1-2i)\,S_{n-1}(a_1+[-a_2]).$$

**Corollary 18** [18] For $n \in \mathbb{N}$, the generating function of Gaussian Pell Lucas numbers $GQ_n$ is given by

$$\sum_{n=0}^{\infty} GQ_n z^n = \frac{(2-2i) + (6i-2)\,z}{1-2z-z^2},$$

with $GQ_n = (2-2i)\,S_n(a_1+[-a_2]) + (6i-2)\,S_{n-1}(a_1+[-a_2])$.

**Corollary 19** [19] For $n \in \mathbb{N}$, the generating function of modified Pell numbers $q_n$ is given by

$$\sum_{n=0}^{\infty} q_n z^n = \frac{1-z}{1-2z-z^2},\ \text{with } q_n = S_n(a_1+[-a_2]) - S_{n-1}(a_1+[-a_2]).$$

4.2. **Construction of generating functions of bivariate Vieta-Fibonacci and Lucas polynomials.** This part consists of three cases.

**Case 1.** The substitution of $\begin{cases} a_1 - a_2 = x \\ a_1 a_2 = -y \end{cases}$ in (36) and (37), we obtain

$$\sum_{n=0}^{\infty} S_n\left(a_1+[-a_2]\right) z^n = \frac{1}{1-xz+yz^2}, \tag{47}$$

$$\sum_{n=0}^{\infty} S_{n-1}\left(a_1+[-a_2]\right) z^n = \frac{z}{1-xz+yz^2}, \tag{48}$$

respectively, and we have the following corollary.

**Corollary 20** For $n \in \mathbb{N}$, the generating function of bivariate Vieta-Fibonacci polynomials $V_n(x,y)$ is given by

$$\sum_{n=0}^{\infty} V_n(x,y)\, z^n = \frac{z}{1-xz+yz^2},\ \text{with } V_n(x,y) = S_{n-1}\left(a_1+[-a_2]\right). \tag{49}$$

Multiplying the equation (47) by (2) and adding it to the equation obtained by (48) multiplying by $(-x)$, then we have the following proposition.

**Proposition 13** For $n \in \mathbb{N}$, the generating function of bivariate Vieta-Lucas polynomials $v_n(x,y)$ is given by

$$\sum_{n=0}^{\infty} v_n(x,y)\, z^n = \frac{2-xz}{1-xz+yz^2}, \tag{50}$$

with $v_n(x,y) = 2S_n(a_1+[-a_2]) - xS_{n-1}(a_1+[-a_2])$.

- Based on the relationships (49) and (50) and with $y = 1$, we obtain the following corollaries.

**Corollary 21** For $n \in \mathbb{N}$, the generating function of Vieta-Fibonacci polynomials $V_n(x)$ is given by

$$\sum_{n=0}^{\infty} V_n(x) z^n = \frac{z}{1 - xz + z^2}, \text{ with } V_n(x) = S_{n-1}(a_1 + [-a_2]).$$

**Corollary 22** For $n \in \mathbb{N}$, the generating function of Vieta-Lucas polynomials $v_n(x)$ is given by

$$\sum_{n=0}^{\infty} v_n(x) z^n = \frac{2 - xz}{1 - xz + z^2}, \text{ with } v_n(x) = 2S_n(a_1 + [-a_2]) - xS_{n-1}(a_1 + [-a_2]).$$

**Case 2.** Assuming that $\begin{cases} a_1 - a_2 = 2x \\ a_1 a_2 = k \end{cases}$ in (36) and (37), we obtain

$$\sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) z^n = \frac{1}{1 - 2xz - kz^2}, \tag{51}$$

$$\sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2]) z^n = \frac{z}{1 - 2xz - kz^2}, \tag{52}$$

respectively, and we have the following corollary.

**Corollary 23** For $n \in \mathbb{N}$, the generating function of $k$-Pell polynomials $P_{k,n}(x)$ is given by

$$\sum_{n=0}^{\infty} P_{k,n}(x) z^n = \frac{z}{1 - 2xz - kz^2}, \text{ with } P_{k,n}(x) = S_{n-1}(a_1 + [-a_2]). \tag{53}$$

Multiplying the equation (51) by (2) and adding it to the equation obtained by (52) multiplying by $(-2x)$, then we have the following proposition and corollary.

**Proposition 14** For $n \in \mathbb{N}$, the generating function of $k$-Pell Lucas polynomials $Q_{k,n}(x)$ is given by

$$\sum_{n=0}^{\infty} Q_{k,n}(x) z^n = \frac{2 - 2xz}{1 - 2xz - kz^2}. \tag{54}$$

**Corollary 24** The following identity holds true:

$$Q_{k,n}(x) = 2S_n(a_1 + [-a_2]) - 2xS_{n-1}(a_1 + [-a_2]).$$

- Based on the relationships (53) and (54) and with $x = 1$, we obtain the following corollaries.

**Corollary 25** For $n \in \mathbb{N}$, the generating function of $k$-Pell numbers $P_{k,n}$ is given by

$$\sum_{n=0}^{\infty} P_{k,n} z^n = \frac{z}{1 - 2z - kz^2}, \text{ with } P_{k,n} = S_{n-1}(a_1 + [-a_2]).$$

**Corollary 26** For $n \in \mathbb{N}$, the generating function of $k$-Pell Lucas numbers $Q_{k,n}$ is given by

$$\sum_{n=0}^{\infty} Q_{k,n} z^n = \frac{2 - 2z}{1 - 2z - kz^2}, \text{ with } Q_{k,n} = 2S_n(a_1 + [-a_2]) - 2S_{n-1}(a_1 + [-a_2]).$$

- Put $k = 1$ in the relationships (53) and (54), we obtain the following corollaries.

**Corollary 27** For $n \in \mathbb{N}$, the generating function of Pell polynomials $P_n(x)$ is given by

$$\sum_{n=0}^{\infty} P_n(x) z^n = \frac{z}{1-2xz-z^2}, \text{ with } P_n(x) = S_{n-1}(a_1 + [-a_2]).$$

**Corollary 28** For $n \in \mathbb{N}$, the generating function of Pell Lucas polynomials $Q_n(x)$ is given by

$$\sum_{n=0}^{\infty} Q_n(x) z^n = \frac{2-2xz}{1-2xz-z^2}, \text{ with } Q_n(x) = 2S_n(a_1+[-a_2]) - 2xS_{n-1}(a_1+[-a_2]).$$

**Case 3.** By taking $\begin{cases} a_1 - a_2 = 1 \\ a_1 a_2 = 2^k x \end{cases}$ in (36) and (37), we obtain

$$\sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) z^n = \frac{1}{1-z-2^k xz^2}, \tag{55}$$

$$\sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2]) z^n = \frac{z}{1-z-2^k xz^2}, \tag{56}$$

respectively.

Multiplying the equation (55) by $\left(\frac{i}{2}\right)$ and adding it to the equation obtained by (56) multiplying by $(1 - \frac{i}{2})$, then we have the following proposition.

**Proposition 15** For $n \in \mathbb{N}$, the generating function of generalized Gaussian Jacobsthal polynomials $GJ_{k,n}(x)$ is given by

$$\sum_{n=0}^{\infty} GJ_{k,n}(x) z^n = \frac{i + (2-i) z}{2 - 2z - 2^{k+1} xz^2}, \tag{57}$$

with $GJ_{k,n}(x) = \frac{i}{2} S_n(a_1 + [-a_2]) + \left(1 - \frac{i}{2}\right) S_{n-1}(a_1 + [-a_2])$.

Multiplying the equation (55) by $\left(2 - \frac{i}{2}\right)$ and adding it to the equation obtained by (56) multiplying by $\left(i\left(2x + \frac{1}{2}\right) - 1\right)$, then we obtain the following proposition.

**Proposition 16** For $n \in \mathbb{N}$, the generating function of generalized Gaussian Jacobsthal Lucas polynomials $Gj_{k,n}(x)$ is given by

$$\sum_{n=0}^{\infty} Gj_{k,n}(x) z^n = \frac{4 - i + (i(4x+1) - 2) z}{2 - 2z - 2^{k+1} xz^2}, \tag{58}$$

with $Gj_{k,n}(x) = \left(2 - \frac{i}{2}\right) S_n(a_1 + [-a_2]) + \left(i\left(2x + \frac{1}{2}\right) - 1\right) S_{n-1}(a_1 + [-a_2])$.

- Based on the relationships (57) and (58) and with $x = 1$, we obtain the following corollaries.

**Corollary 29** For $n \in \mathbb{N}$, the generating function of generalized Gaussian Jacobsthal numbers $GJ_{k,n}$ is given by

$$\sum_{n=0}^{\infty} GJ_{k,n} z^n = \frac{i + (2-i) z}{2 - 2z - 2^{k+1} z^2}, \text{ with } GJ_{k,n} = \frac{i}{2} S_n(a_1+[-a_2]) + \left(1 - \frac{i}{2}\right) S_{n-1}(a_1+[-a_2]).$$

**Corollary 30** For $n \in \mathbb{N}$, the generating function of generalized Gaussian Jacobsthal Lucas numbers $Gj_{k,n}$ is given by

$$\sum_{n=0}^{\infty} Gj_{k,n} z^n = \frac{4 - i + (5i - 2) z}{2 - 2z - 2^{k+1} z^2},$$

with $Gj_{k,n} = \left(2 - \frac{i}{2}\right) S_n(a_1 + [-a_2]) + \left(\frac{5i}{2} - 1\right) S_{n-1}(a_1 + [-a_2])$.

- Put $k = 1$ in the relationships (57) and (58), we obtain the following corollaries.

**Corollary 31** [18] For $n \in \mathbb{N}$, the generating function of Gaussian Jacobsthal polynomials $GJ_n(x)$ is given by

$$\sum_{n=0}^{\infty} GJ_n(x)\, z^n = \frac{i + (2-i)\, z}{2 - 2z - 4xz^2},$$

with $GJ_n(x) = \frac{i}{2} S_n(a_1 + [-a_2]) + \left(1 - \frac{i}{2}\right) S_{n-1}(a_1 + [-a_2])$.

**Corollary 32** [18] For $n \in \mathbb{N}$, the generating function of Gaussian Jacobsthal Lucas polynomials $Gj_n(x)$ is given by

$$\sum_{n=0}^{\infty} Gj_n(x)\, z^n = \frac{4 - i + (i\,(4x+1) - 2)\, z}{2 - 2z - 4xz^2},$$

with $Gj_n(x) = \left(2 - \frac{i}{2}\right) S_n(a_1 + [-a_2]) + \left(i\left(2x + \frac{1}{2}\right) - 1\right) S_{n-1}(a_1 + [-a_2])$.

## 5. Conclusion

In this paper, we have derived theorem 1 by making use of symmetrizing operator given by definition 11. By making use of theorem 1, we have obtained propositions and corollaries which is led to generating function for a class of new family of complete functions.

In our forthcoming investigation, we plan to establish further results and properties associated with some generalized forms of the above mentioned families of new class of generating functions of binary products of some special numbers and polynomials.

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