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ON THE INFINITE DECOMPOSABILITY OF THE GEOMETRIC DISTRIBUTION

CHRISTOPHE CHESNEAU AND JOSÉ LUIS PALACIOS

Abstract. In this note, we prove a new result on the infinite decomposability of the geometric distribution. More precisely, we show that a random variable following the geometric distribution can be written as any sum of random variables following a modified geometric distribution at 0 plus one random variable following the geometric distribution, all of them independent.
Similar results are obtained for other distributions defined by a sum of random variables involving the geometric distribution.

1. Introduction

A random variable X is called decomposable if there exist two independent random variables X_1 and X_2 such that $X = X_1 + X_2$. More generally, the variable X is said to be infinitely decomposable if for every n it is possible to write $X = X_1 + X_2 + \cdots + X_n$, where $X_1, \ldots, X_n$ are all independent. Starting in the nineteen thirties, a considerable amount of research was focused on decomposable variables, culminating in the monograph [1], with a special emphasis on the Gaussian and the Poisson distributions. Nowadays the interest in decomposable variables has waned, though there are sporadic publications on this theme (see [2] and [3]), or they refer to this notion in passing, as in [4].
Recently, it was proved in [5] that a random variable following the exponential distribution can be infinitely decomposed into a sum of independent variables, all of which except one - which also has a an exponential distribution - are discontinuous with a well-identified distribution.
A similar result can be proved for the gamma distribution (see [6]). Since the exponential and gamma distributions yield appropriate continuous models for waiting times, it is somehow natural to expect that some similar decomposition should be applicable to the discrete models for waiting times, namely the geometric and the negative binomial distributions.
Indeed, in this article we show that, for any integer m, a random variable following the geometric distribution can be written as the sum of m random variables following the modified geometric distribution at 0 and another one following the

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geometric distribution, all of them independent.
The remainder of the paper is organized as follows. The essential definitions are put in Section 2. The main result is presented and proved in Section 3. Some consequences of this result are discussed in Section 4.

2. Definitions

Now, let $p \in (0,1)$. We say that a random variable X follows the geometric distribution with parameter p, denoted by $\mathcal{G}(p)$, if it has the following probability mass function (pmf):

$$P(X = k) = (1-p)p^k, \quad k = 0, 1, \ldots$$

Also, we say that a random variable X follows the modified geometric distribution in 0 with parameter p, denoted by $\mathcal{MG}_0(p)$, if it can be written as

$$X = Y_N, \tag{1}$$

where N denotes a random variable following the Bernoulli distribution with parameter p, denoted by $\mathcal{B}(p)$, i.e., with the pmf given by

$$P(N = 0) = 1 - p, \quad P(N = 1) = p,$$

and Y_0 and Y_1 denote 2 random variables independent of N such that $P(Y_0 = 0) = 1$ and Y_1 follows the distribution $\mathcal{G}(p)$. Then, for $k = 0, 1, \ldots$, we have

$$P(X = k) = P(Y_0 = k)P(N = 0) + P(Y_1 = k)P(N = 1),$$

which give the following pmf for X:

$$P(X = 0) = 1 - p^2, \quad P(X = k) = (1-p)p^{k+1}, \quad k = 1, 2, \ldots$$

In particular, if X and Y follow the distributions $\mathcal{G}(p)$ and $\mathcal{MG}_0(p)$, respectively, then the following basic relations hold, for any $r \geq 0$ and any integer m,

$$\frac{P(Y = 0)}{P(X = 0)} = 1 + p, \quad \frac{E(Y^r)}{E(X^r)} = p, \quad \frac{P(Y > m)}{P(X > m)} = p.$$

Other properties will be presented later.

3. Main results

The result below shows that the modified geometric distribution becomes the link between two geometric distributions with different parameters.
Proposition 1 Let X be a random variable following the distribution $\mathcal{MG}_0(p)$ and Y a random variable following the distribution $\mathcal{G}(p/(1+p))$, with X and Y independent. Then, $Z = X + Y$ follows the distribution $\mathcal{G}(p)$.
Proof. First of all, since X follows the distribution $\mathcal{MG}_0(p)$, by using (1), the moment generating function of X is given by, for $t < -\log(p)$,

$$\begin{aligned} M_X(t) = E(e^{tX}) &= E(e^{tY_0})P(N = 0) + E(e^{tY_1})P(N = 1) \\ &= (1-p) + \frac{1-p}{1-pe^t}p = (1-p)\left(\frac{1+p-pe^t}{1-pe^t}\right) \\ &= (1-p)(1+p)\left(\frac{1-[p/(1+p)]e^t}{1-pe^t}\right). \end{aligned}$$

On the other hand, since Y follows the distribution $\mathcal{G}(p/(1+p))$, we have, for $t < -\log(p) < \log(1+1/p)$,

$$M_Y(t) = E(e^{tY}) = \frac{1 - p/(1+p)}{1 - [p/(1+p)]e^t} = \frac{1}{1+p}\left(\frac{1}{1 - [p/(1+p)]e^t}\right).$$

Therefore, by the independence of X and Y, we get $M_Z(t) = M_X(t)M_Y(t)$, implying that, for $t < -\log(p)$,

$$\begin{aligned} M_Z(t) &= (1-p)(1+p)\left(\frac{1 - [p/(1+p)]e^t}{1 - pe^t}\right) \times \frac{1}{1+p}\left(\frac{1}{1 - [p/(1+p)]e^t}\right) \\ &= \frac{1-p}{1-pe^t}. \end{aligned}$$

We recognize the moment generating function of the distribution $\mathcal{G}(p)$, implying the desired results

We are now in the position to state the main result of the paper, which mainly follows from Proposition 1.

Theorem 1 Let X be random variable following the distribution $\mathcal{G}(p)$. Then, for every integer m, we can write

$$X = \sum_{i=1}^{m+1} X_i,$$

where X_i follows the distribution $\mathcal{MG}_0(p/[(i-1)p+1])$ for $i = 1, \ldots, m$, and X_{m+1} follows the distribution $\mathcal{G}(p/(mp+1))$, with $X_1, \ldots, X_m, X_{m+1}$ independent.

Proof. The proof follows by induction and Proposition 1. Indeed, by using the generic notations for the distributions for the sake of simplicity, we have for $m = 2$,

$$\begin{aligned} \mathcal{G}(p) &= \mathcal{MG}_0(p) + \mathcal{G}(p/(p+1)) \\ &= \mathcal{MG}_0(p) + \mathcal{MG}_0(p/(p+1)) + \mathcal{G}(p_*/(p_*+1)), \end{aligned}$$

with $p_* = p/(p+1)$ and $p_*/(p_*+1) = p/(2p+1)$. This can be extended for any m by noticing that, for any integer a and $p_* = p/(ap+1)$, we have $p_*/(ap_*+1) = p/[(a+1)p+1]$.

4. Discussion

Some useful consequences of Theorem 1 are given below.

Proposition 2 Theorem 1 still holds by replacing the geometric distribution by the “second type of” geometric distribution $\mathcal{G}_*(p)$ with pmf:

$$P(X = k) = (1-p)p^{k-1}, \quad k = 1, 2, \ldots$$

Proof. By adopting the setting of Theorem 1, we have $X = \sum_{i=1}^{m+1} X_i$, implying that

$$X + 1 = \sum_{i=1}^{m} X_i + (X_{m+1} + 1).$$

We end the proof by noticing that $X+1$ follows the distribution $\mathcal{G}_*(p)$ and $X_{m+1}+1$ follows the distribution $\mathcal{G}_*(p/(mp+1))$.

We also can extend the infinite decomposability result of Theorem 1 to a random variable X following the binomial negative distribution with parameters n and p, denoted by $\mathcal{H}(n,p)$, i.e., with the following pdf:

$$P(X=k)=\binom{k+n-1}{k}(1-p)^n p^k, \quad k=0,1,\ldots$$

This is formulated in the result below.

Proposition 3 Let X be random variable following the distribution $\mathcal{H}(n,p)$. Then, for every integer m, we can write

$$X=\sum_{i=1}^{m+1} X_i,$$

where X_i is a sum of n independent random variable following the distribution $\mathcal{MG}_0(p/[(i-1)p+1])$ for $i=1,\ldots,m$, and X_{m+1} follows the distribution $\mathcal{H}(n,p/(mp+1))$, with $X_1,\ldots,X_m,X_{m+1}$ independent.

Proof. By using a well-known property of the distribution $\mathcal{H}(n,p)$, we can write $X=\sum_{i=1}^{n} U_i$, where $U_1,\ldots,U_n$ are n i.i.d. random variables following the distribution $\mathcal{G}(p)$.

Then, by applying Theorem 1, we can write

$$U_i=\sum_{j=1}^{m+1} V_{j,i},$$

where $V_{j,i}$ follows the distribution $\mathcal{MG}_0(p/[(j-1)p+1])$ for $j=1,\ldots,m$, and $V_{m+1,i}$ follows the distribution $\mathcal{G}(p/(mp+1))$, with $V_{1,i},\ldots,V_{m,i},V_{m+1,i}$ independent.

Hence, we can write

$$X=\sum_{j=1}^{m+1} W_j,$$

where $W_j=\sum_{i=1}^{n} V_{j,i}$, with $W_{m+1}=\sum_{i=1}^{n} V_{m+1,i}$ following the distribution $\mathcal{H}(n,p/(mp+1))$ and, for $j=1,\ldots,m$.

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Christophe Chesneau

LMNO, University of Caen-Normandie, Caen, 14032, France

E-mail address: christophe.chesneau@unicaen.fr

José Luis Palacios
Department of Electrical and Computing Engineering, University of New Mexico, Albuquerque, NM, 87131, USA
E-mail address: jpalacios@unm.edu