

A NOTE ON UNIQUENESS OF TRANSCENDENTAL ENTIRE FUNCTIONS CONCERNING DIFFERENTIAL-DIFFERENCE POLYNOMIALS OF FINITE ORDER

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ABSTRACT. In this paper, using the notions of weakly weighted sharing and relaxed weighted sharing we study the uniqueness problems of differential-difference polynomials of transcendental entire functions that share a small function. The results of the paper extends the theorems given by Pulak Sahoo and Gurudas Biswas.

1. INTRODUCTION

Throughout this paper, we assume that the readers are familiar with the standard notations of Nevanlinna theory ([9]). Let f and g be two meromorphic functions in \mathbb{C} and $a \in \mathbb{C} \cup \{\infty\}$. We say that f and g share a -CM if $f - a$ and $g - a$ have the same zeros with multiplicities. Furthermore, if $f - a$ and $g - a$ have the same zeros without counting multiplicities, then we say that f and g share a -IM.

Let k and p be positive integers. We denote by $\bar{N}_{(k)}(r, a; f)$ ($\bar{N}_{(k)}(r, \frac{1}{f-a})$) the reduced counting function of a -point of f whose multiplicities are not less than k , and $\bar{N}_{(k)}(r, a; f)$ ($\bar{N}_{(k)}(r, \frac{1}{f-a})$) the reduced counting function of a -points of f whose multiplicities are at most k .

$$N_p(r, \frac{1}{f-a}) = \bar{N}(r, \frac{1}{f-a}) + \bar{N}_{(2)}(r, \frac{1}{f-a}) + \dots + \bar{N}_{(p)}(r, \frac{1}{f-a}), a \in \mathbb{C}$$

$$N_2(r, f) = \bar{N}(r, f) + \bar{N}_{(2)}(r, f).$$

We also recall that if $a \in \mathbb{C} \cup \{\infty\}$, the quantity

$$\delta(a; f) = 1 - \limsup_{r \rightarrow \infty} \frac{N(r, a; f)}{T(r, f)}$$

is called Nevanlinna deficiency of the value a and by ramification index we mean

$$\Theta(a; f) = 1 - \liminf_{r \rightarrow \infty} \frac{\bar{N}(r, a; f)}{T(r, f)}.$$

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Definition 1. ([10]) Let $a \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f| = 1)$ the counting function of simple a -points of f . For a positive integer k we denote by $N(r, a; f| \leq k)$ the counting function of those a -points of f (counted with proper multiplicities) whose multiplicities are not greater than k . By $\overline{N}(r, a; f| \leq k)$ we denote the corresponding reduced counting function. Analogously we can define $N(r, a; f| \geq k)$ and $\overline{N}(r, a; f| \geq k)$.

Definition 2. Let $a \in \mathbb{C} \cup \{\infty\}$. We denote $N_E(r, a; f, g)$ ($\overline{N}_E(r, a; f, g)$) by the counting function (reduced counting function) of all common zeros of $f - a$ and $g - a$ with the same multiplicities and by $N_0(r, a; f, g)$ ($\overline{N}_0(r, a; f, g)$) the counting function (reduced counting function) of all common zeros of $f - a$ and $g - a$ ignoring multiplicities. If

$$\overline{N}(r, a; f) + \overline{N}(r, a; g) - 2\overline{N}_E(r, a; f, g) = S(r, f) + S(r, g),$$

then we say that f and g share the value a “CM”, If

$$\overline{N}(r, a; f) + \overline{N}(r, a; g) - 2\overline{N}_0(r, a; f, g) = S(r, f) + S(r, g),$$

then we say that f and g share the value a “IM”.

Definition 3. ([14]) Let f and g share the value a “IM” and k be a positive integer or infinity. Then $\overline{N}_k^E(r, a; f, g)$ denotes the reduced counting function of those a -points of f whose multiplicities are equal to the corresponding a -points of g , and both of their multiplicities are not greater than k . $\overline{N}_{(k)}^0(r, a; f, g)$ denotes the reduced counting function of those a -points of f which are a -points of g , and both of their multiplicities are not less than k .

We now introduce the following definition of weakly weighted sharing which is scaling between IM and sharing CM.

Definition 4. ([14]) Let $a \in \mathbb{C} \cup \{\infty\}$ and k be a positive integer or infinity. If

$$\begin{aligned} \overline{N}(r, a; f| \leq k) - \overline{N}_k^E(r, a; f, g) &= S(r, f), \\ \overline{N}(r, a; g| \leq k) - \overline{N}_k^E(r, a; f, g) &= S(r, g), \\ \overline{N}(r, a; f| \geq k+1) - \overline{N}_{(k+1)}^0(r, a; f, g) &= S(r, f), \\ \overline{N}(r, a; g| \geq k) - \overline{N}_{(k+1)}^0(r, a; f, g) &= S(r, g), \end{aligned}$$

or if $k = 0$ and

$$\begin{aligned} \overline{N}(r, a; f) - \overline{N}_0(r, a; f, g) &= S(r, f), \\ \overline{N}(r, a; g) - \overline{N}_0(r, a; f, g) &= S(r, g), \end{aligned}$$

then we say that f and g share the value a weakly with weight k and we write f and g share “ (a, k) ”.

In 2007, A. Banerjee and S. Mukherjee [1] introduced a new type of sharing known as relaxed weighted sharing, weaker than weakly weighted sharing and is defined as follows.

Definition 5. ([1]) We denote by $\overline{N}(r, a; f| = p; g| = q)$ the reduced counting function of common a -points of f and g with multiplicities p and q respectively.

Definition 6. ([1]) Let $a \in \mathbb{C} \cup \{\infty\}$ and k be a positive integer of infinity. Suppose f and g share the value a “IM”. If $p \neq q$,

$$\sum_{p, q \leq k} \bar{N}(r, a; f| = p; g| = q) = S(r),$$

then we say that f and g share the value a with weight k in a relaxed manner and in that case we write f and g share $(a, k)^*$.

In 2007, I. Laine and C. C. Yang [13] proved the following result.

Theorem A. Let $f(z)$ be a transcendental entire function of finite order and η be a nonzero complex constant. Then for $n \geq 2$, $f^n(z)f(z + \eta)$ assumes every non-zero value $a \in \mathbb{C}$ infinitely often.

In 2010, X. G. Qi, L. Z. Yang and K. Liu [19] proved the following uniqueness result corresponding to Theorem A.

Theorem B. Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and η be a non-zero complex constant, and let $n \geq 6$ be an integer. If $f^n(z)(z + \eta)$ and $g^n(z)(z + \eta)$ share 1 CM, then either $fg = t_1$ or $f = t_2g$ for some constants t_1 and t_2 satisfying $t_1^{n+1} = t_2^{n+1} = 1$.

In the same year J. L. Zhang [25] considered the zeros of one certain type of difference polynomial and proved the following result.

Theorem C. Let $f(z)$ be a transcendental entire function of finite order, $\alpha(z) (\neq 0)$ be a small function with respect to $f(z)$ and η be a non-zero complex constant. If $n \geq 2$ is an integer then $f^n(z)(f(z) - 1)f(z + \eta) - \alpha(z)$ has infinitely many zeros.

Theorem D. Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z) (\neq 0)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that η is a nonzero complex constant and $n \geq 7$ is an integer. If $f^n(z)(f(z) - 1)f(z + \eta)$ and $g^n(z)(g(z) - 1)g(z + \eta)$ share $\alpha(z)$ CM, then $f(z) \equiv g(z)$.

In 2014, using the idea of weakly weighted sharing and relaxed weighted sharing, C. Meng [18] proved the following results which improve and supplement Theorem D in different directions.

Theorem E. Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z) (\neq 0, \infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that η is a non-zero complex constant and $n \geq 7$ is an integer. If $f^n(z)(f(z) - 1)f(z + \eta)$ and $g^n(z)(g(z) - 1)g(z + \eta)$ share “ $(\alpha, 2)$ ”, then $f(z) \equiv g(z)$.

Theorem F. Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z) (\neq 0, \infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that η is a non-zero complex constant and $n \geq 10$ is an integer. If $f^n(z)(f(z) - 1)f(z + \eta)$ and $g^n(z)(g(z) - 1)g(z + \eta)$ share $(\alpha, 2)^*$, then $f(z) \equiv g(z)$.

Theorem G. Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z) (\neq 0, \infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that η is a non-zero complex constant and $n \geq 16$ is an integer. If $\overline{E}_2(\alpha(z), f^n(z)(f(z)-1)f(z+\eta)) = \overline{E}_2(\alpha(z), g^n(z)(g(z)-1)g(z+\eta))$, then $f(z) \equiv g(z)$.

In 2015, P. Sahoo[21], proved the following results.

Theorem H. Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z) (\neq 0, \infty)$ be a small function of both $f(z)$ and $g(z)$. Suppose that η is a non-zero complex constant, n and $m (\geq 1)$ are integers such that $n \geq m + 6$. If $f^n(z)(f^m(z) - 1)f(z + \eta)$ and $g^n(z)(g^m(z) - 1)g(z + \eta)$ share " $(\alpha(z), 2)^n$ ", then $f(z) \equiv tg(z)$ where $t^m = 1$.

Theorem I. Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z) (\neq 0, \infty)$ be a small function of both $f(z)$ and $g(z)$. Suppose that η is a non-zero complex constant, n and $m (\geq 1)$ are integers such that $n \geq 2m + 8$. If $f^n(z)(f^m(z) - 1)f(z + \eta)$ and $g^n(z)(g^m(z) - 1)g(z + \eta)$ share $(\alpha(z), 2)^*$, then $f(z) \equiv tg(z)$ where $t^m = 1$.

Theorem J. Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z) (\neq 0, \infty)$ be a small function of both $f(z)$ and $g(z)$. Suppose that η is a non-zero complex constant, n and $m (\geq 1)$ are integers such that $n \geq 4m + 12$. If $\overline{E}_2(\alpha(z), f^n(z)(f(z) - 1)f(z + \eta)) = \overline{E}_2(\alpha(z), g^n(z)(g^m(z) - 1)g(z + \eta))$ then $f(z) \equiv tg(z)$ where $t^m = 1$.

In 2018, P. Sahoo and G. Biswas[27] proved the following theorems.

Theorem K. Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z) (\neq 0, \infty)$ be a small function of both $f(z)$ and $g(z)$ with finitely many zeros. Suppose that η is a non-zero complex constant, $n, k (\geq 0)$ and $m (\geq 1)$ are integers such that $n \geq 2k + m + 6$. If $(f^n(z)(f^m(z) - 1)f(z + \eta))^{(k)}$ and $(g^n(z)(g^m(z) - 1)g(z + \eta))^{(k)}$ share " $(\alpha(z), 2)^n$ ", then $f(z) \equiv tg(z)$ where $t^m = 1$.

Theorem L. Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z) (\neq 0, \infty)$ be a small function of both $f(z)$ and $g(z)$ with finitely many zeros. Suppose that η is a non-zero complex constant, $n, k (\geq 0)$ and $m (\geq 1)$ are integers such that $n \geq 3k + 2m + 8$. If $(f^n(z)(f^m(z) - 1)f(z + \eta))^{(k)}$ and $(g^n(z)(g^m(z) - 1)g(z + \eta))^{(k)}$ share $(\alpha(z), 2)^*$, then $f(z) \equiv tg(z)$ where $t^m = 1$.

Theorem M. Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z) (\neq 0, \infty)$ be a small function of both $f(z)$ and $g(z)$ with finitely many zeros. Suppose that η is a non-zero complex constant, $n, k (\geq 0)$ and $m (\geq 1)$ are integers such that $n \geq 5k + 4m + 12$. If $\overline{E}_2(\alpha(z), (f^n(z)(f^m(z) - 1)f(z + \eta))^{(k)}) = \overline{E}_2(\alpha(z), (g^n(z)(g^m(z) - 1)g(z + \eta))^{(k)})$, then $f(z) \equiv tg(z)$ where $t^m = 1$.

Question 1. What can be said about the relationship between two transcendental entire functions $f(z)$ and $g(z)$, if $f^n(z)(f^m(z) - 1) \prod_{j=1}^d f(z + \eta_j)^{v_j}$ is the

difference polynomials, where $n(\geq 1)$, $m(\geq 1)$ and $d \geq 1$ are integers?

In this article, our main aim is to find the possible answer to above question. We assume $v_j \in \mathbb{C} \setminus \{0\}$ ($j = 1, 2, \dots, d$) are distinct constants, n, m, v_j ($j = 1, 2, \dots, d$) are positive integers and $\sigma = \sum_{j=1}^d v_j = v_1 + v_2 + \dots + v_d$.

We prove the following results which improve and extend Theorems K-M. The following theorems are the main results of the paper.

Theorem 1. Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order and $\alpha(z) (\neq 0, \infty)$ be a small function of both $f(z)$ and $g(z)$ with finitely many zeros. Suppose that η is a nonzero complex constants, $n, k(\geq 0)$ and $m(\geq 1)$ are integers satisfying $n \geq 2k + m + \sigma + 5$. If $(f^n(z)(f^m(z) - 1) \prod_{j=1}^d f(z + \eta_j)^{v_j})^{(k)}$ and $(g^n(z)P(g) \prod_{j=1}^d g(z + \eta_j)^{v_j})^{(k)}$ share " $(\alpha(z), 2)$ ", then $f(z) \equiv tg(z)$ where $t^m = 1$.

Theorem 2. Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order and $\alpha(z) (\neq 0, \infty)$ be a small function of both $f(z)$ and $g(z)$ with finitely many zeros. Suppose that η is a nonzero complex constants, $n, k(\geq 0)$ and $m(\geq 1)$ are integers satisfying $n \geq 3k + 2m + 2\sigma + 6$. If $(f^n(z)(f^m(z) - 1) \prod_{j=1}^d f(z + \eta_j)^{v_j})^{(k)}$ and $(g^n(z)(g^m(z) - 1) \prod_{j=1}^d g(z + \eta_j)^{v_j})^{(k)}$ share $(\alpha(z), 2)^*$, then $f(z) \equiv tg(z)$ where $t^m = 1$.

Theorem 3. Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order and $\alpha(z) (\neq 0, \infty)$ be a small function of both $f(z)$ and $g(z)$ with finitely many zeros. Suppose that η is a nonzero complex constants, $n, k(\geq 0)$ and $m(\geq 1)$ are integers satisfying $n \geq 5k + 4m + 4\sigma + 8$. If $\overline{E}_2(\alpha(z), (f^n(z)(f^m(z) - 1) \prod_{j=1}^d f(z + \eta_j)^{v_j})^{(k)}) = \overline{E}_2(\alpha(z), (g^n(z)(g^m(z) - 1) \prod_{j=1}^d g(z + \eta_j)^{v_j})^{(k)})$, then $f(z) \equiv tg(z)$ where $t^m = 1$.

Remark 1. Since Theorems K-M can be obtained from Theorems 1-3 respectively by putting $l = m$ and $\sigma = 1$, Theorems 1-3 improve and extend Theorems K-M respectively.

2. LEMMAS

In this section, we present some lemmas which will be needed in the sequel. We will denote by H the following function:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1} \right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1} \right),$$

where F and G are nonconstant meromorphic functions defined in the complex plane \mathbb{C} .

Lemma 2.1.[26] Let f be a non-constant meromorphic function and p, k be positive integers. Then

$$N_p(r, 0; f^{(k)}) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k}(r, 0; f) + S(r, f) \quad (1)$$

$$N_p(r, 0; f^{(k)}) \leq k\overline{N}(r, \infty, f) + N_{p+k}(r, 0; f) + S(r, f). \quad (2)$$

Lemma 2.2.[1] Let F and G be two non-constant meromorphic functions that share “(1,2)” and $H \neq 0$. Then

$$T(r, F) \leq N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) - \sum_{p=3}^{\infty} N_2(r, 0; \frac{G'}{G} | \geq p) + S(r, F) + S(r, G),$$

and the same inequality holds for $T(r, G)$.

Lemma 2.3.[1] Let F and G be two non-constant meromorphic functions that share $(1, 2)^*$ and $H \neq 0$. Then

$$T(r, F) \leq N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + \overline{N}(r, 0; F) + \overline{N}(r, \infty; F) - m(r, 1; G) + S(r, F) + S(r, G),$$

and the same inequality is true for $T(r, G)$.

Lemma 2.4.[8] Let F and G be two non-constant entire functions and $p \geq 2$ be an integer. If $\overline{E}_p(1, F) = \overline{E}_p(1, G)$ and $H \neq 0$, then

$$T(r, F) \leq N_2(r, 0; F) + N_2(r, 0; G) + 2\overline{N}(r, 0; F) + \overline{N}(r, 0; G) + S(r, F) + S(r, G),$$

and the same inequality holds for $T(r, G)$.

Lemma 2.5.[4] Let $f(z)$ be a meromorphic function in the complex plane of finite order $\rho(f)$, and let η be a fixed non-zero complex number. Then for each $\epsilon > 0$ one has

$$T(r, f(z + \eta)) = T(r, f(z)) + O(r^{\rho(f)-1+\epsilon}) + O\{\log r\}.$$

Lemma 2.6.[4] Let $f(z)$ be a entire function of finite order $\rho(f)$, c a fixed non-zero complex number, and $P(z) = a_n f^n(z) + a_{n-1} f^{n-1}(z) + \dots + a_1 f(z) + a_0$ where $a_j (j = 0, 1, \dots, n)$ are constants. If $F(z) = P(z)f(z + c)$, then $T(r, F) = (n + 1)T(r, f) + O(r^{\rho(f)-1+\epsilon}) + O(\log r)$.

Lemma 2.7.[4] Let f be meromorphic function of finite order and c be a non-zero complex constant. Then,

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = O\{r^{\rho(f)-1+\epsilon}\}.$$

Lemma 2.8. Let f is an entire function having order ρ and $F_1 = f^n(z)(f^m(z) - 1) \prod_{j=1}^d f(z + \eta_j)^{v_j}$ where n is integer. Then

$$T(r, F_1) = (n + m + \sigma)T(r, f) + O\{r^{\rho(f)-1+\epsilon}\} + S(r, f),$$

for all r outside of a set of finite linear measure, where $\sigma = v_1 + v_2 + \dots + v_d = \sum_{j=1}^d v_j$.

Proof. Suppose $f(z)$ is an entire function of finite order ρ , by standard Valiron-Mohon'ko theorem and Lemma 2.8, we get

$$\begin{aligned}
 (n+m+\sigma)T(r, f(z)) &= T(r, f^{n+\sigma}(z)(f^m(z)-1)) + S(r, f) \\
 &= m(r, f^{n+\sigma}(z)(f^m(z)-1)) + S(r, f) \\
 &\leq m\left(r, \frac{f^{n+\sigma}(z)((f^m(z)-1))}{F(z)}\right) + m(r, F(z)) + S(r, f) \\
 &\leq m\left(r, \frac{f^\sigma(z)}{\prod_{j=1}^d f(z+\eta_j)^{v_j}}\right) + m(r, F(z)) + S(r, f) \\
 &\leq T(r, F(z)) + O\{r^{\rho(f)-1+\varepsilon}\} + S(r, f). \tag{3}
 \end{aligned}$$

On the other side, from Lemma 2.6, we have

$$\begin{aligned}
 T(r, F(z)) &\leq m(r, f^n(z)) + m(r, ((f^m(z)-1))) + m\left(r, f^\sigma(z) \cdot \prod_{j=1}^d \frac{f(z+\eta_j)^{v_j}}{f(z)^{v_j}}\right) + S(r, f) \\
 &\leq (n+m)m(r, f(z)) + \sigma m(r, f(z)) + \sum_{j=1}^d v_j \cdot m\left(r, \frac{f(z+\eta_j)}{f(z)}\right) + S(r, f) \\
 &\leq (n+m+\sigma)m(r, f(z)) + O\{r^{\rho(f)-1+\varepsilon}\} + S(r, f) \\
 &\leq (n+m+\sigma)T(r, f(z)) + O\{r^{\rho(f)-1+\varepsilon}\} + S(r, f). \tag{4}
 \end{aligned}$$

From (3) and (4), we can prove this lemma easily.

Lemma 2.9. Suppose that $f(z)$ and $g(z)$ be two entire functions, $n(\geq 1), m(\geq 1), k(\geq 0)$ be integers, and let $F = (f^n(z)(f^m(z)-1) \prod_{j=1}^d f(z+\eta_j)^{v_j})^{(k)}$, $G = (g^n(z)(g^m(z)-1) \prod_{j=1}^d g(z+\eta_j)^{v_j})^{(k)}$. If there exists constants $c_1 \neq 0$ and $c_2 \neq 0$ such that $\bar{N}(r, c_1; F) = \bar{N}(r, 0; G)$ and $\bar{N}(r, c_2; G) = \bar{N}(r, 0; F)$, then $n_0 \leq 2k+m+2+\sigma$.

Proof. We take $F_1 = f^n(z)(f^m(z)-1) \prod_{j=1}^d f(z+\eta_j)^{v_j}$ and $G_1 = g^n(z)(g^m(z)-1) \prod_{j=1}^d g(z+\eta_j)^{v_j}$.

1) $\prod_{j=1}^d g(z+\eta_j)^{v_j}$.

By the Nevanlinna second fundamental theorem, we have

$$T(r, F) \leq \bar{N}(r, 0; F) + \bar{N}(r, c_1; F) + S(r, F) \leq \bar{N}(r, 0; F) + \bar{N}(r, 0; G) + S(r, F) \tag{5}$$

Using (1),(2),(5) and Lemmas 2.6 and 2.8 we obtain

$$\begin{aligned}
 (n+m+\sigma)T(r, f) &\leq T(r, F) - \bar{N}(r, 0; F) + N_{k+1}(r, 0; F_1) + S(r, f) \\
 &\leq \bar{N}(r, 0; G) + N_{k+1}(r, 0; F_1) + S(r, f) \\
 &\leq N_{k+1}(r, 0; F_1) + N_{k+1}(r, 0; G_1) + S(r, f) + S(r, g) \\
 &\leq (k+1)(\bar{N}(r, 0; f) + \bar{N}(r, 0; g)) + N(r, 1; (f^m(z) - 1)) + N(r, 1; (g^m(z) - 1)) \\
 &\quad + N(r, 0; \prod_{j=1}^d f(z + \eta_j)^{v_j}) + N(r, 0; \prod_{j=1}^d g(z + \eta_j)^{v_j}) + S(r, f) + S(r, g) \\
 &\leq (k+1+m+\sigma)(T(r, f) + T(r, g)) + O\{r^{\rho(f)-1+\epsilon}\} + O\{r^{\rho(g)-1+\epsilon}\} \\
 &\quad + S(r, f) + S(r, g).
 \end{aligned} \tag{6}$$

Similarly,

$$\begin{aligned}
 (n+m+\sigma)T(r, g) &\leq (k+m+1+\sigma)(T(r, f) + T(r, g)) + O(r^{\rho(f)-1+\epsilon}) + O(r^{\rho(g)-1+\epsilon}) \\
 &\quad + S(r, f) + S(r, g).
 \end{aligned} \tag{7}$$

Combining (6) and (7) we obtain

$$(n-2k-m-\sigma)(T(r, f) + T(r, g)) \leq O(r^{\rho(f)-1+\epsilon}) + O(r^{\rho(g)-1+\epsilon}) + S(r, f) + S(r, g) \tag{8}$$

which gives $n \leq 2k + m + 2 + \sigma$. This proves the Lemma.

Lemma 2.10.[2] Suppose that f and g are transcendental meromorphic functions of finite order, $\eta_j (j = 1, 2, \dots, d)$ are distinct finite complex numbers, and $n, m, d, v_j (j = 1, 2, \dots, d)$ are integers. If $n > m + 5\sigma$, we have

$$f^n(z)(f^m(z) - 1) \prod_{j=1}^d f(z + \eta_j)^{v_j} = g^n(z)(g^m(z) - 1) \prod_{j=1}^d g(z + \eta_j)^{v_j}.$$

Then, $f = tg$, where $t^m = t^{n+\sigma} = 1$.

3. Proof of the Theorems

Proof of Theorem 1. Let $F(z) = \frac{F_1^{(k)}}{\alpha(z)}$, $G(z) = \frac{G_1^{(k)}}{\alpha(z)}$. Where $F_1 = [f(z)^n (f^m(z) - 1) \prod_{j=1}^d f(z + c_j)^{v_j}]$ and $G_1 = [g(z)^n (g^m(z) - 1) \prod_{j=1}^d g(z + c_j)^{v_j}]$. Then $F(z)$ and $G(z)$ are transcendental meromorphic functions that share “(1, 2)” except the zeros or poles of $\alpha(z)$. By Lemma 2.8, we have

$$T(r, F_1) = (n+m+\sigma)T(r, f) + O\{r^{\rho(f)-1+\epsilon}\} + S(r, f) \tag{9}$$

$$T(r, G_1) = (n+m+\sigma)T(r, g) + O\{r^{\rho(g)-1+\epsilon}\} + S(r, g) \tag{10}$$

If hopeful, we may assume that $H \not\equiv 0$. Using (1),(9) and Lemma 2.8 we get Also, we have

$$\begin{aligned}
 N_2(r, 0, F) &= N_2(r, 0; F_1^{(k)}) + S(r, f) \\
 &\leq T(r, F_1^{(k)}) - (n+m+\sigma)T(r, f) + N_{k+2}(r, 0; F_1) + S(r, f) \\
 &= T(r, F) - (n+m+\sigma)T(r, f) + N_{k+2}(r, 0; F_1) + S(r, f)
 \end{aligned}$$

we get from this

$$(n + m + \sigma)T(r, f) \leq T(r, F) - N_2(r, 0; F) + N_{k+2}(r, 0; F_1) + S(r, f). \quad (11)$$

Using (2) we get

$$N_2(r, 0; F) \leq N_2(r, 0; F_1^{(k)}) + S(r, f) \leq N_{k+2}(r, 0; F_1) + S(r, f). \quad (12)$$

Correspondingly,

$$N_2(r, 0; G) \leq N_{k+2}(r, 0; G_1) + S(r, g). \quad (13)$$

By (12),(13) and Lemmas 2.2, 2.8 we get from (11)

$$\begin{aligned} (n + m + \sigma)T(r, f) &\leq N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + N_{k+2}(r, 0; F_1) \\ &\quad + S(r, f) + S(r, g) \\ &\leq N_{k+2}(r, 0; F_1) + N_{k+1}(r, 0; G_1) + S(r, f) + S(r, g) \\ &\leq N_{k+2}(r, 0; f^n(f^m(z) - 1) \prod_{j=1}^d f(z + \eta_j)^{v_j}) \\ &\quad + N_{k+2}(r, 0; g^n(g^m(z) - 1) \prod_{j=1}^d g(z + \eta_j)^{v_j}) + S(r, f) + S(r, g) \\ &\leq (k + 2 + m + \sigma)(T(r, f) + T(r, g)) + O\{r^{\rho(f)-1+\epsilon}\} + O\{r^{\rho(g)-1+\epsilon}\} \\ &\quad + S(r, f) + S(r, g). \end{aligned} \quad (14)$$

Correspondingly we get

$$\begin{aligned} (n + m + \sigma)T(r, g) &\leq (k + 2 + m + \sigma)(T(r, f) + T(r, g)) + O\{r^{\rho(f)-1+\epsilon}\} + O\{r^{\rho(g)-1+\epsilon}\} \\ &\quad + S(r, f) + S(r, g). \end{aligned} \quad (15)$$

Adding (14) and (15) we get

$$(n - 2k - m - \sigma - 4)\{T(r, f) + T(r, g)\} \leq O\{r^{\rho(f)-1+\epsilon}\} + O\{r^{\rho(g)-1+\epsilon}\} + S(r, f) + S(r, g) \quad (16)$$

leads to a contradiction to the hypothesis that $n \geq 2k + m + \sigma + 5$. Therefore we have $H = 0$. Then

$$\left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right) = 0.$$

On twice integration the above equation becomes

$$\frac{1}{F-1} = \frac{C}{G-1} + D$$

here $A(\neq 0)$ and B are constants. By (16) we can say that F, G share the value 1 CM and they share the value “(1,2)”. Therefore $n \geq 2k + m + 5 + \sigma$. We now consider three cases as follows.

Case 1. Suppose that $C \neq 0, C = D$. Then by (16) we get

$$\frac{1}{F-1} = \frac{DG}{G-1}. \quad (17)$$

If $D = -1$, then by (17) we get $FG = 1$. Then

$$(f^n(z)(f^m(z) - 1) \prod_{j=1}^d f(z + \eta_j)^{v_j})^{(k)} (g^n(z)(g^m(z) - 1) \prod_{j=1}^d g(z + \eta_j)^{v_j})^{(k)} = \alpha^2.$$

Since the number of zeros of $\alpha(z)$ is finite that f as well as g has finitely many zeros. We put $f(z) = h(z)e^{\beta(z)}$, where $h(z)$ is a non-zero polynomial and $\beta(z)$ is a non-constant polynomial. By replacing $\beta(z + \eta_j)$ by $\gamma(z)$ and $h(z + \eta_j)$ by $\mu(z)$ we get

$$\begin{aligned} (f^n(z)(f^m(z) - 1) \prod_{j=1}^d f(z + \eta_j)^{v_j})^{(k)} &= (h^n(z)e^{n\beta(z)}(h^m(z)e^{m\beta(z)} - 1) \prod_{j=1}^d h(z + c_j)^{v_j} e^{\beta(z+\eta_j)v_j})^{(k)} \\ &= (h^n(z) \prod_{j=1}^d (\mu^{s_j}(z)e^{\gamma(z)v_j}) \cdot e^{n\beta(z)}(h^m(z)e^{m\beta(z)} - 1))^{(k)} \\ &= (h^{n+m}(z) \prod_{j=1}^d \mu^{v_j}(z)e^{(n+m)\beta(z)+\gamma(z)v_j} - h^n(z) \prod_{j=1}^d \mu^{v_j}(z)e^{n\beta(z)+\gamma(z)v_j})^{(k)} \\ &= e^{(n+m)\beta(z)+\gamma(z)(v_1+v_2+\dots+v_d)} P_1(\beta(z), \gamma(z), h(z), \mu(z), \dots, \beta^{(k)}(z), \gamma^{(k)}(z), h^{(k)}(z), \mu^{(k)}(z)) \\ &\quad - e^{n\beta(z)+\gamma(z)(s_1+s_2+\dots+s_d)} P_2(\beta(z), \gamma(z), h(z), \mu(z), \dots, \beta^{(k)}(z), \gamma^{(k)}(z), h^{(k)}(z), \mu^{(k)}(z)) \\ &= e^{n\beta(z)+\gamma(z)(v_1+v_2+\dots+v_d)} (P_1 e^{m\beta(z)} - P_2). \end{aligned}$$

Obviously $P_1 e^{m\beta(z)} - P_2$ has infinite number of zeros, which contradicts with the fact that g is an entire function.

If $D \neq -1$, from (17), we have $\frac{1}{F} = \frac{DG}{(1+D)G-1}$ and so $\overline{N}(r, \frac{1}{1+D}; G) = \overline{N}(r, 0; F)$. Using (1),(2),(10) and Nevanlinna’s second fundamental theorem, we get that

$$\begin{aligned} T(r, G) &\leq \overline{N}(r, 0; G) + \overline{N}(r, 0; G - 1 + D) + \overline{N}(r, G) + S(r, G) \\ &\leq \overline{N}(r, 0; F) + \overline{N}(r, 0; G) + \overline{N}(r, G) + S(r, G) \\ &\leq N_{k+1}(r, 0; F_1) + T(r, G) + N_{k+1}(r, 0; G_1) - (n + m + \sigma)T(r, g) + S(r, g). \end{aligned}$$

This yields

$$\begin{aligned} (n + m + \sigma)T(r, g) &\leq (k + 1 + m + \sigma)(T(r, f) + T(r, g)) + O\{r^{\rho(f)-1+\epsilon}\} \\ &\quad + O\{r^{\rho(g)-1+\epsilon}\} + S(r, g). \end{aligned}$$

Thus we get

$$\begin{aligned} (n - 2k - m - \sigma - 2)\{T(r, f) + T(r, g)\} &\leq O\{r^{\rho(f)-1+\epsilon}\} + O\{r^{\rho(g)-1+\epsilon}\} \\ &\quad + S(r, f) + S(r, g) \end{aligned}$$

its contradiction since $n \geq 3 + m + \sigma + 2k$.

Case 2. Let $A \neq B$. and $B \neq 0$. Then by (16) we have $F = \frac{(B+1)G-(B-A+1)}{BG+(A-B)}$ and so $\overline{N}(r, \frac{B-A+1}{B+1}; G) = \overline{N}(r, 0; F)$. Arguing similarly as in case 1 we arrive at a contradiction.

Case 3. Let $A \neq 0$ and $B = 0$. By (16) we get $F = \frac{G+A-1}{A}$ and $G = AF - (A - 1)$. If $A \neq 1$, it follows that $\overline{N}(r, \frac{A-1}{A}; F) = \overline{N}(r, 0; G)$ and $\overline{N}(r, 1 - A; G)$. Now Lemma

2.10 it can be shown that $n \leq 2k + m + 2 + \sigma$, which is a contradiction. Thus $A = 1$ and the $F = G$. Then

$$(f^n(z)(f^m(z) - 1) \prod_{j=1}^d f(z + \eta_j)^{v_j})^{(k)} = (g^n(z)(g^m(z) - 1) \prod_{j=1}^d g(z + \eta_j)^{v_j})^{(k)}.$$

Integrating once we obtain

$$(f^n(z)(f^m(z) - 1) \prod_{j=1}^d f(z + \eta_j)^{v_j})^{(k)} = (g^n(z)(g^m(z) - 1) \prod_{j=1}^d g(z + \eta_j)^{v_j})^{(k)} + c_{k-1},$$

where c_{k-1} is a constant. If using Lemma 2.9 it follows that $n \leq 2k + m + \sigma$ a contradiction. Hence $c_{k-1} = 0$. Repeating the process k -times, we deduce that

$$f^n(z)(f^m(z) - 1) \prod_{j=1}^d f(z + \eta_j)^{v_j} = g^n(z)(g^m(z) - 1) \prod_{j=1}^d g(z + \eta_j)^{v_j},$$

which by Lemma 2.10 gives $f = tg$, where t is a constant satisfying $t = 1$. Hence the proof of Theorem 1.

Proof of Theorem 2. Let F, G, F_1 and G_1 be defined as in Theorem 1. Then $F(z)$ and $G(z)$ are two transcendental meromorphic functions that share $(1, 2)^*$ except at zeros and poles of $\alpha(z)$. Let $\psi \neq 0$. Then by using (2) for $p = 1$, (13) and Lemmas 2.3 and 2.5 obtain from (11).

$$\begin{aligned} (n + m + \sigma)T(r, f) &\leq N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + \bar{N}(r, 0; F) \\ &\quad + \bar{N}(r, \infty; F) + N_{k+2}(r, 0; F_1) + S(r, f) + S(r, g) \\ &\leq N_{k+2}(r, 0; F_1) + N_{k+2}(r, 0; G_1) + N_{k+1}(r, 0; F_1) \\ &\quad + S(r, f) + S(r, g) \\ &\leq (2k + 2m + 2\sigma + 3)T(r, f) + (k + m + 2 + \sigma)T(r, g) \\ &\quad + O\{r^{\rho(f)-1+\epsilon}\} + O\{r^{\rho(g)-1+\epsilon}\} + S(r, f) + S(r, g). \end{aligned} \quad (18)$$

Correspondingly, we obtain

$$\begin{aligned} (n + m + \sigma)T(r, g) &\leq (2k + 2m + 2\sigma + 3)T(r, g) + (k + l + 2 + \sigma)T(r, f) \\ &\quad + O\{r^{\rho(f)-1+\epsilon}\} + O\{r^{\rho(g)-1+\epsilon}\} + S(r, f) + S(r, g). \end{aligned} \quad (19)$$

From (18) and (19) we get

$$\begin{aligned} (n - 2m - 3k - 2\sigma - 5)T(r, f) + T(r, g) &\leq O\{r^{\rho(f)-1+\epsilon}\} + O\{r^{\rho(g)-1+\epsilon}\} \\ &\quad + S(r, f) + S(r, g), \end{aligned}$$

contradicting with the fact that $n \geq 3k + 2m + 2\sigma + 6$. Thus we must have $H = 0$. Then the result follows from the proof of Theorem 1. This completes the proof of Theorem 2.

Proof of Theorem 3. Let F, G, F_1 and G_1 be similar as in Theorem 1. Then F and G are transcendental meromorphic functions such that $\bar{E}_2(1, F) = \bar{E}_2(1, G)$ except the zeros and poles of $\alpha(z)$. Let $H \neq 0$. Then by (2), (13) and Lemmas 2.4

and 2.5 and we obtain from (11),

$$\begin{aligned}
 (n+m+\sigma)T(r, f) &\leq N_2(r, 0; G) + 2\overline{N}(r, 0; F) + \overline{N}(r, 0; G) + N_{k+2}(r, 0; F_1) \\
 &\quad + S(r, f) + S(r, g) \\
 &\leq N_{k+2}(r, 0; F_1) + N_{k+2}(r, 0; G_1) + 2N_{k+1}(r, 0; F_1) \\
 &\quad + N_{k+1}(r, 0; G_1) + S(r, f) + S(r, g) \\
 &\leq (3k+3m+3\sigma+4)T(r, f) + (2k+2m+2\sigma+3)T(r, g) \\
 &\quad + O\{r^{\rho(f)-1+\epsilon}\} + O\{r^{\rho(g)-1+\epsilon}\} + S(r, f) + S(r, g).
 \end{aligned} \tag{20}$$

Similarly,

$$\begin{aligned}
 (n+m+\sigma)T(r, g) &\leq (3k+3m+3\sigma+4)T(r, g) + (2k+2m+2\sigma+3)T(r, f) \\
 &\quad + O\{r^{\rho(f)-1+\epsilon}\} + O\{r^{\rho(g)-1+\epsilon}\} + S(r, f) + S(r, g).
 \end{aligned} \tag{21}$$

Combining (20) and (21) we obtain

$$(n-5k-4m-4\sigma-7)T(r, f) + T(r, g) \leq O\{r^{\rho(f)-1+\epsilon}\} + O\{r^{\rho(g)-1+\epsilon}\} + S(r, f) + S(r, g),$$

a contradiction with the assumption that $n \geq 5k + 4m + 4\sigma + 8$. Thus $H = 0$ and the rest of the theorem follows from the proof of Theorem 1. Hence the proof of Theorem 3.

4. Open Questions

Question 4.1. Can the Theorems 1-3 be extend to meromorphic functions?

Question 4.2. Can the difference polynomials in Theorems 1-3 be replaced by the difference polynomials of form $f^n(z)(f^m(z)-1)\prod_{j=1}^d f(z+\eta_j)^{v_j}\prod_{j=1}^k f^{(i)}(z)$?

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