FIXED POINT THEOREMS OF (α, ψ)-RATIONAL TYPE CONTRACTIVE MAPPINGS IN RECTANGULAR b-METRIC SPACE

LANGPOKLAKPAM PREMILA DEVI AND LAISHRAM SHAMBHU SINGH

ABSTRACT. In this paper, we use the concept of (α, ψ)-rational type contractive mappings to prove some unique fixed point theorems in the setting of rectangular b-metric space. We deduce some results as corollaries and examples are also given to verify the results obtained.

1. INTRODUCTION

Fixed point theory is one of the most important topics in Mathematics, specially in analysis. Due to its application in various disciplines like engineering, computer science, biological sciences, economics etc., many researchers took interest in fixed point theory and its application. It is well known that Banach Contraction Principle is the most important result in fixed point theory. During the last many years this result is extended in different directions.

The notion of b-metric space was introduced by Bakhtin[1] and extensively used by Czerwik[2,3]. b-metric is itself a generalisation of metric space. It is observed that metric space becomes a special case of b-metric space and hence every metric space is a b-metric space but the converse is not true. Following is the definition of b-metric space.

Definition 1.1 [1] Let X be a nonempty set and $d : X \times X \to [0, \infty)$ satisfies:
 (i) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$.
 (ii) $d(x, y) = d(y, x)$ for all $x, y \in X$.
 (iii) there exists a real number $s \geq 1$ such that
    $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$

Then the map d is called a b - metric on X and (X, d) is called a b - metric space with coefficient s. Another generalisation of metric space is also introduced by Branciari [4]. He replaces the triangle inequality of metric space by another more generalized inequality involving four elements against three elements in case

---

of triangular inequality. It can be noted that every metric space is a rectangular metric space but the converse is not true. Branciari's definition of rectangular metric space is given below.

Definition 1.2 [4] Let X be a nonempty set and the mapping $d : X \times X \to [0, \infty)$ satisfies:

(i) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$.
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$.
(iii) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)]$ for all $x, y \in X$ and all distinct points $u, v \in X/x, y$

Then the map d is called a rectangular metric on X and (X, d) is called a rectangular metric space (in short RMS).
Motivated by the notion of b-metric space and rectangular metric space, George et. al.[5] led to the introduction of a new generalization of metric space called rectangular b-metric space. This new notion comprises the properties of both b-metric and rectangular metric. The definition of rectangular b-metric space is as follows:

Definition 1.3 [5] Let X be a nonempty set and the mapping $d : X \times X \to [0, \infty)$ satisfies:

(i) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$.
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$.
(iii) there exists a real number $s \geq 1$ such that
$d(x, y) \leq s[d(x, u) + d(u, v) + d(v, y)]$ for all $x, y \in X$ and all distinct points $u, v \in X/x, y$.

Then the map d is called a rectangular b - metric on X and (X, d) is called a rectangular b - metric space (in short $RbMS$)with coefficient s.

It can be noted that every metric space is a rectangular metric space and every rectangular metric space is a rectangular b-metric space (with coefficient $s = 1$). However the converse of the above implication is not necessarily true.
Also every metric space is a b-metric space and every metric space is also a rectangular metric space. But the converse of the above statement may not be true.

Further, it can be noted that every b-metric space is a rectangular b-metric space and every rectangular metric space is a rectangular b-metric space but the converse of the above statements may not be true at all. Verification of these statements can be found in the examples given in George et. al.[5]
For more results on rectangular b-metric space and other generalized forms of metric space one can see research paper s in [5-22] and references therein.
Following definitions will be needed in the sequel:

Definition 1.4 [5] Let (X, d) be a rectangular b-metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. Then

(i) The sequence $\{x_n\}$ is said to be convergent in (X, d) and converges to x, if for every $\varepsilon > 0$ there exists $n_0 \in N$ such that $d(x_n, x) < \varepsilon$ for all $n > n_0$ and this fact is represented by $\lim_{n \to \infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(ii) $\{x_n\}$ is said to be a Cauchy sequence in (X, d) if for every $\varepsilon > 0$ there exists $n_0 \in N$ such that $d(x_n, x_{n+p}) < \varepsilon$ for all $n > n_0, p > 0$ or equivalently, if $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$ for all $p > 0$.

(iii) (X, d) is said to be complete rectangular b-metric space if every Cauchy sequence in X converges to some $x \in X$.

Proposition 1.5[24] Suppose that $\{x_n\}$ is a Cauchy sequence in a rectangular b-metric space (X, d) with $\lim_{n\to\infty} d(x_n, u) = 0$, where $u \in X$. Then $\lim_{n\to\infty} d(x_n, z) = d(u, z)$, for all $z \in X$. In particular, the sequence x_n does not converges to z if $z \neq u$.

Definition 1.6 [23] Let X be a nonempty set, $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be two mappings , we say that T is an α-admissible mapping if $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$ for all $x, y \in X$.

Definition 1.7 [11] Let (X, d) be a rectangular b-metric space and $\alpha : X \times X \to [0, \infty)$. Then X is called α-regular rectangular b-metric space if, for a sequence x_n in X such that $x_n \to x$ and $\alpha(x_n, x_{n+1}) \geq 1$ there exists a subsequence x_{n_k} of x_n such that $\alpha(x_{n_k}, x) \geq 1$ for all $k \in N$.

It can be noted that, limit of a sequence in a $RbMS$ is not necessarily unique and also every convergent sequence in a $RbMS$ is not necessarily a Cauchy sequence.

2. Main Results

In this paper we use (α, ψ)-rational type contraction mappings to prove some unique fixed point in the rectangular b-metric space. Throughout this paper we will consider the auxiliary function ψ defined by Alsulami et. al. [11].
Let Ψ be a family of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following properties:

(i) ψ is upper semi-continuous, strictly increasing;

(ii) $[\psi^n(t)]_{n\in N}$ converges to 0 as $n \to \infty$, for all $t > 0$

(iii) $\psi(t) < t$, for every $t > 0$.

Definition 2.1 Let X be a nonempty set and the mapping $\alpha : X \times X \to [0, \infty)$. A self function $\psi \in \Psi$, such that for all $x, y \in X$ the following conditions holds

$$\alpha(x, y)d(Tx, Ty) \leq \psi(M(x, y))$$

where

$$M(x, y) = \max\left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)}, \frac{d(x, Tx)d(y, Ty)}{1 + (Tx, Ty)} \right\} \tag{1}$$

Next, we state and prove an existence and uniqueness theorem for fixed point of (α, ψ)-rational type-I contractive mappings.

Theorem 2.2 Let (X, d) be a complete rectangular b-metric space and $s \geq 1$ is a real number, $T : X \to X$ be a self mapping and $\alpha : X \times X \to [0, \infty)$ a given function. Suppose that the following conditions are satisfied:

    (i) T is an α- admissible mapping.

    (ii) T is an (α, ψ)-rational type-I contractive mapping.

    (iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1, \alpha(x_0, T^2x_0) \geq 1$

    (iv) either T is continuous or X is α-regular.

Then T has a fixed point $x^* \in X$ and $\{T^n x_0\}$ converges to x^*. Further, if for all $x, y \in F(T)$, we have $\alpha(x, y) \geq 1$ then T has a unique fixed point in X.

Proof: Let $x_0 \in X$ satisfies $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$. We construct the sequence $\{x_n\}$ in X as $x_n = T^n x_0 = Tx_{n-1}$ for $n \in N$. It is obvious that if $x_{n_0} = x_{n_0+1}$, for some $n_0 \in N$, then x_{n_0} is a fixed point of T. Consequently, we suppose that $x_n \neq x_n + 1$ for all $n \in N$. Since T is α-admissible, therefore

$$\alpha(x_0, Tx_0) = \alpha(x_0, x_1) \geq 1 \Rightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1$$

and thus $\alpha(Tx_1, Tx_2) = \alpha(x_2, x_3) \geq 1$,...and hence by induction, we get $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \geq 0$.

By similar arguments, since $\alpha(x_0, T^2x_0) \geq 1$, we have $\alpha(x_0, x_2) = \alpha(x_0, T^2x_0) \geq 1, \alpha(Tx_0, Tx_2) = \alpha(x_1, x_3) \geq 1$. By induction, we get $\alpha(x_n, x_{n+2}) \geq 1$ for all $n \geq 0$. Consider (1) with $x = x_n$ and $y = x_{n+1}$. Clearly, we have

$$d(x_{n+1}, x_{n+2}) = d(Tx_n, Tx_{n+1}) \leq \alpha(x_n, x_{n+1})d(Tx_n, Tx_{n+1}) = \psi(M(x_n, Tx_{n+1}))$$

where

$$
\begin{aligned}
M(x_n, Tx_{n+1}) &= \max\left\{ d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}), \right. \\
&\qquad \left. \frac{d(x_n, Tx_n)d(x_{n+1}, Tx_{n+1})}{1 + d(x_n, x_{n+1})}, \frac{d(x_n, Tx_n)d(x_{n+1}, Tx_{n+1})}{1 + d(Tx_n, Tx_{n+1})} \right\} \\
&= \max\left\{ d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \right. \\
&\qquad \left. \frac{d(x_n, x_{n+1})d(x_{n+1}, x_{n+2})}{1 + d(x_n, x_{n+1})}, \frac{d(x_n, x_{n+1})d(x_{n+1}, x_{n+2})}{1 + d(x_{n+1}, x_{n+2})} \right\} \\
&= \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} \qquad (2)
\end{aligned}
$$

If for some n, we have

$$
\begin{aligned}
M(x_n, x_{n+1}) &= d(x_{n+1}, x_{n+2}), then \\
d(x_{n+1}, x_{n+2}) &\leq \psi(M(x_n, x_{n+1})) \\
&= \psi(d(x_{n+1}, x_{n+2})) \\
&< d(x_{n+1}, x_{n+2}) \qquad (3)
\end{aligned}
$$

which is impossible. Hence, $M(x_n, x_{n+1}) = d(x_n, x_{n+1})$, for all $n \in N$.

$$
\begin{aligned}
d(x_{n+1}, x_{n+2}) &\leq \psi(M(x_n, x_{n+1})) \\
&= \psi(d(x_n, x_{n+1})) \qquad (4)
\end{aligned}
$$

From the property (iii) of ψ, we conclude that

$$d(x_{n+1}, x_{n+2}) \quad < \quad d(x_n, x_{n+1}) \tag{5}$$

for every $n \in N$. Combining (4) and (5), we deduce $d(x_{n+1}, x_{n+2}) \leq \psi^n d(x_0, x_1)$, for all $n \in N$. Using the property (ii) of ψ, it is clear that

$$\lim_{n \to \infty} d(x_{n+1}, x_{n+2}) = 0 \tag{6}$$

Consider now (1) with $x = x_{n-1}$ and $y = x_{n+1}$, we have

$$
\begin{aligned}
d(x_n, x_{n+2}) &= d(Tx_{n-1}, Tx_{n+1}) \\
&\leq \alpha(x_{n-1}, x_{n+1}) d(Tx_{n-1}, Tx_{n+1}) \\
&< \psi(M(x_{n-1}, x_{n+1}))
\end{aligned}
\tag{7}
$$

where

$$
\begin{aligned}
M(x_{n-1}, x_{n+1}) &= \max \Bigg\{ d(x_{n-1}, x_{n+1}), d(x_{n-1}, Tx_{n-1}), d(x_{n+1}, Tx_{n+1}), \\
&\qquad \frac{d(x_{n-1}, Tx_{n-1})d(x_{n+1}, Tx_{n+1})}{1 + d(x_{n-1}, x_{n+1})}, \frac{d(x_{n-1}, Tx_{n-1})d(x_{n+1}, Tx_{n+1})}{1 + d(Tx_{n-1}, Tx_{n+1})} \Bigg\} \\
&= \max \Bigg\{ d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n), d(x_{n+1}, x_{n+2}), \\
&\qquad \frac{d(x_{n-1}, x_n)d(x_{n+1}, x_{n+2})}{1 + d(x_{n-1}, x_{n+1})}, \frac{d(x_{n-1}, x_n)d(x_{n+1}, x_{n+2})}{1 + d(x_n, x_{n+2})} \Bigg\}
\end{aligned}
\tag{8}
$$

From (5) we have $d(x_{n+1}, x_{n+2}) < d(x_{n-1}, x_n)$. Define $a_n = d(x_n, x_{n+2})$ and $b_n = d(x_n, x_{n+1})$. Then

$$M(x_{n-1}, x_{n+1}) = max \Bigg\{ a_{n-1}, b_{n-1}, \frac{b_{n-1}b_{n+1}}{1+a_{n-1}} \frac{b_{n-1}b_{n+1}}{1+a_n} \Bigg\}.$$

If $M(x_{n-1}, x_{n+1}) = b_{n-1}$ or $\frac{b_{n-1}b_{n+1}}{1+a_{n-1}}$ or $\frac{b_{n-1}b_{n+1}}{1+a_n}$ then taking lim sup as $n \to \infty$ in (7) and using (6) and upper semi-continuity of ψ we get

$$
\begin{aligned}
0 \leq \limsup_{n \to \infty} a_n &\leq \limsup_{n \to \infty} \psi(M(x_{n-1}, x_{n+1})) \\
&= \psi(\limsup_{n \to \infty} M(x_{n-1}, x_{n+1})) \\
&= \psi(0) = 0,
\end{aligned}
$$

and hence

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} d(x_n, x_{n+2}) = 0.$$

If $M(x_{n-1}, x_{n+1}) = a_{n-1}$, then (7) implies $a_n \leq \psi(a_{n-1}) < a_{n-1}$ due to the property (iii) of ψ. In orther words, the sequence $\{a_n\}$ is positive monotone decreasing and hence it converges to some $t \geq 0$. Assume that $t > 0$. Now by (7), we have

$$t = \limsup_{n \to \infty} a_n \quad = \quad \limsup_{n \to \infty} \psi(a_{n-1}) = \psi(\limsup_{n \to \infty}(a_{n-1})) = \psi(t) < t,$$

which is a contradiction.
Therefore

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} d(x_n, x_{n+2}) = 0. \tag{9}$$

Now, we shall prove that $x_n \neq x_m$ for all $n \neq m$. Assume on the contrary that $x_n = x_m$ for some $m, n \in \mathbb{N}$ with $n \neq m$. Since $d(x_p, x_{p+1}) > 0$, for each $p \in \mathbb{N}$. Without loss of generality, we may assume that $m > n + 1$. Substituting again $x = x_n = x_m$ and $y = x_{n+1} = x_{m+1}$ in (1), which yields

$$
\begin{aligned}
d(x_n, x_{n+1}) &= d(x_n, Tx_n) = d(x_m, Tx_m) \\
&= d(Tx_{m-1}, Tx_m) \\
&\leq \alpha(x_{m-1}, x_m) d(Tx_{m-1}, Tx_m) \\
&\leq \psi(M(x_{m-1}, x_m)) \tag{10}
\end{aligned}
$$

where

$$
\begin{aligned}
M(x_{m-1}, x_m) &= \max \Bigg\{ d(x_{m-1}, x_m), d(x_{m-1}, Tx_{m-1}), d(x_m, Tx_m), \\
&\qquad \frac{d(x_{m-1}, Tx_{m-1})d(x_m, Tx_m)}{1 + d(x_{m-1}, x_m)}, \frac{d(x_{m-1}, Tx_{m-1})d(x_m, Tx_m)}{1 + d(Tx_{m-1}, Tx_m)} \Bigg\} \\
&= \max \Bigg\{ d(x_{m-1}, x_m), d(x_{m-1}, x_m), d(x_m, x_{m+1}), \\
&\qquad \frac{d(x_{m-1}, x_m)d(x_m, x_{m+1})}{1 + d(x_{m-1}, x_m)}, \frac{d(x_{m-1}, x_m)d(x_m, x_{m+1})}{1 + d(x_m, x_{m+1})} \Bigg\} \\
&= \max\{d(x_{m-1}, x_m), d(x_m, x_{m+1})\} \tag{11}
\end{aligned}
$$

If $M(x_{m-1}, x_m) = d(x_{m-1}, x_m)$, then (10) implies

$$
\begin{aligned}
d(x_n, x_{n+1}) &\leq \psi(d(x_{m-1}, x_m)) \\
&\leq \psi^{m-n}(d(x_n, x_{n+1})). \tag{12}
\end{aligned}
$$

If on the other hand $M(x_{m-1}, x_m) = d(x_m, x_{m+1})$, then from (10) we have

$$
\begin{aligned}
d(x_n, x_{n+1}) &\leq \psi(d(x_m, x_{m+1})) \\
&\leq \psi^{m-n+1}(d(x_n, x_{n+1})). \tag{13}
\end{aligned}
$$

Using the property (iii) of ψ, the two inequalities (12) and (13) imply

$$d(x_n, x_{n+1}) < d(x_n, x_{n+1}),$$

which is impossible.

Now we shall prove that x_n is a cauchy sequence that is $\lim_{n \to \infty} d(x_n, x_{n+k}) = 0$, for all $k \in N$. We shall already proved the cases for $k = 1$ and $k = 2$ in (6) and (9), respectively. Take arbitrary $k \geq 3$. We discuss two cases.

Case I
Suppose that $k = 2m+1$, where $m \geq 1$. Using the b-rectangular inequality, we have

$$
\begin{aligned}
d(x_n, x_{n+k}) &= d(x_n, x_{n+2m+1}) \\
&\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m+1})]
\end{aligned}
$$

$$\begin{aligned}
&\leq \quad s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + \\
&\qquad s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2m+1})] \\
&\leq \quad s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + \\
&\qquad s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4})] + \\
&\qquad s^3[d(x_{n+4}, x_{n+5}) + d(x_{n+5}, x_{n+6}) + d(x_{n+6}, x_{n+2m+1})] \\
&\leq \quad s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + \\
&\qquad s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4})] + \\
&\qquad s^3[d(x_{n+4}, x_{n+5}) + d(x_{n+5}, x_{n+6})] + \cdots + s^m d(x_{n+2m}, x_{n+2m+1}) \\
&\leq \quad [s(\psi^n + \psi^{n+1}) + s^2(\psi^{n+2} + \psi^{n+3}) + s^3(\psi^{n+4} + \psi^{n+5}) + \cdots + \\
&\qquad s^m \psi^{n+2m+1}]d(x_0, x_1)]
\end{aligned}$$

$$= s\psi^n[1 + s\psi^2 + s^2\psi^4 + \ldots]d(x_0, x_1) + s\psi^{n+1}[1 + s\psi^2 + s^2\psi^4 + \ldots]d(x_0, x_1)$$

$$= s\psi^n(1 + \psi)[1 + s\psi^2 + s^2\psi^4 + \ldots]d(x_0, x_1)$$

$$= s\psi^n(1 + \psi)\frac{1}{1 - s\psi^2}d(x_0, x_1) \to 0 \; as \; n \to \infty. \tag{14}$$

Case II
Suppose that $k = 2m$, where $m \geq 2$. Again, by applying the rectangular inequality, we have

$$\begin{aligned}
d(x_n, x_{n+k}) &= \quad d(x_n, x_{n+2m}) \\
&\leq \quad s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m})] \\
&\leq \quad s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + \\
&\qquad s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2m})] \\
&\leq \quad s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4})] + \\
&\qquad s^3[d(x_{n+4}, x_{n+5}) + d(x_{n+5}, x_{n+6})] + \cdots + \\
&\qquad s^{m-1}[d(x_{n+2m-3}, x_{n+2m-2}) + d(x_{n+2m-2}, x_{n+2m-1})] + s^m d(x_{n+2m-1}, x_{n+2m}) \\
&\leq \quad [s(\psi^n + \psi^{n+1}) + s^2(\psi^{n+2} + \psi^{n+3}) + s^3(\psi^{n+4} + \psi^{n+5}) + \cdots + \\
&\qquad s^{m-1}(\psi^{n+2m-3} + \psi^{n+2m-2})]d(x_0, x_1) + s^m \psi^{n+2m-1}d(x_0, x_1)
\end{aligned}$$

$$= s\psi^n[1 + s\psi^2 + s^2\psi^4 + \ldots]d(x_0, x_1) + s\psi^{n+1}[1 + s\psi^2 + s^2\psi^4 + \ldots]d(x_0, x_1) + s^m \psi^{n+2m-1}d(x_0, x_1)$$

$$\leq \frac{1 + \psi}{1 - s\psi^2}s\psi^n d(x_0, x_1) + (s\psi^2)^m \psi^{n-2}d(x_0, x_1)$$

$$= s\psi^n(1 + \psi)\frac{1}{1 - s\psi^2}d(x_0, x_1) \to 0 \; as \; n \to \infty. \tag{15}$$

Since $\lim_{n \to \infty} d(x_n, x_{n+2}) = 0$ because of (9). In both of the above cases, we have $\lim_{n \to \infty} d(x_n, x_{n+k}) = 0$, for all $k \geq 3$. Hence, we conclude that $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is complete, there exist $x^* \in X$ such that

$$\lim_{n \to \infty} d(x_n, x^*) = 0. \tag{16}$$

Now we will show that the limit x^* of the sequence $\{x_n\}$ is a fixed point of T. First, we suppose that T is continuous. Then from (16), we have

$$\lim_{n \to \infty} d(Tx_n, Tx^*) = \lim_{n \to \infty} d(x_{n+1}, Tx^*) = 0.$$

Due to Proposition 1.5, we conclude that $x^* = Tx^*$, that is , x^* is a fixed point of T.

Now, we suppose that X is α-regular. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k-1}, x^*) \geq 1$ for all $k \in \mathbb{N}$. Now, from inequality (1) with $x = x_{n_k}$ and $y = x^*$, we obtain

$$\begin{aligned}
d(x_{n_k+1}, Tx^*) &= d(Tx_{n_k}, Tx^*) \\
&\leq \alpha(x_{n_k}, x^*)d(Tx_{n_k}, Tx^*) \\
&\leq \psi(M(x_{n_k}, x^*))
\end{aligned} \tag{17}$$

where

$$\begin{aligned}
M(x_{n_k}, x^*) &= \max\left\{ d(x_{n_k}, x^*), d(x_{n_k}, Tx_{n_k}), d(x^*, Tx^*), \right. \\
&\qquad \left. \frac{d(x_{n_k}, Tx_{n_k})d(x^*, Tx^*)}{1 + d(x_{n_k}, x^*)}, \frac{d(x_{n_k}, Tx_{n_k})d(x^*, Tx^*)}{1 + d(Tx_{n_k}, Tx^*)} \right\} \\
&= \max\left\{ d(x_{n_k}, x^*), d(x_{n_k}, x_{n_k+1}), d(x^*, Tx^*) \right. \\
&\qquad \left. \frac{d(x_{n_k}, x_{n_k+1})d(x^*, Tx^*)}{1 + d(x_{n_k}, x^*)}, \frac{d(x_{n_k}, x_{n_k+1})d(x^*, Tx^*)}{1 + d(x_{n_k+1}, Tx^*)} \right\} \tag{18}
\end{aligned}$$

Letting $k \to \infty$ in (18), we obtain $M(x_{n_k}, x^*) = d(x^*, Tx^*)$. Therefore, upon taking the limit as $k \to \infty$ in inequality (17), we have

$$d(x^*, Tx^*) \leq \psi(d(x^*, Tx^*)) < d(x^*, Tx^*),$$

which implies $x^* = Tx^*$, that is, x^* is a fixed point of T.

Finally, suppose that x^* and y^* are two fixed points of T such that $x^* \neq y^*$. Then by the hypothesis, $\alpha(x^*, y^*) \geq 1$. Hence, from (1) with $x = x^*$ and $y = y^*$ we have

$$\begin{aligned}
d(x^*, y^*) &= d(Tx^*, Ty^*) \\
&\leq \alpha(x^*, y^*)d(Tx^*, Ty^*) \\
&\leq \psi(M(x^*, y^*))
\end{aligned}$$

where

$$\begin{aligned}
M(x^*, y^*) &= \max\left\{ d(x^*, y^*), d(x^*, Tx^*), d(y^*, Ty^*), \right. \\
&\qquad \left. \frac{d(x^*, Tx^*)d(y^*, Ty^*)}{1 + d(x^*, y^*)}, \frac{d(x^*, Tx^*)d(y^*, Ty^*)}{1 + d(Tx^*, Ty^*)} \right\} \\
&= d(x^*, y^*) \tag{19}
\end{aligned}$$

Hence, we get $d(x^*, y^*) \leq \psi(d(x^*, y^*)) < d(x^*, y^*)$, which is possible only if $d(x^*, y^*) = 0$, that is, $x^* = y^*$. Hence T has a unique fixed point.

Definition 2.3. Let (X, d) be a rectangular b-metric space and $\alpha : X \times X \to [0, \infty)$. A mapping $T : X \to X$ is said to be (α, ψ)-rational type-II

contractive mapping if there exists $\psi \in \Psi$, such that, for all $x, y \in X$ the following condition holds:

$$\alpha(x,y)d(Tx,Ty) \le \psi(M(x,y)) \qquad (20)$$

where

$$M(x,y) \;=\; \max \left\{ d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)+d(x,Ty)+d(y,Tx)}, \right.$$
$$\left. \frac{d(x,Ty)d(x,y)}{1+d(x,Tx)+d(y,Tx)+d(y,Ty)} \right\}$$

For this class of mappings we state a similar existence and uniqueness theorem.

Theorem 2.4. Let (X,d) be a complete rectangular b-metric space, $T : X \to X$ be a self mapping, and $\alpha : X \times X \to [0,\infty)$. Suppose that the following conditions are satisfied:

   (i) T is an α-admissible mapping;
   (ii) T is an (α, ψ)-rational type-II contractive mapping;
   (iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$ and $\alpha(x_0, T^2x_0) \ge 1$
   (iv) either T is continuous, or X is α-regular.

Then T has a fixed point $x^* \in X$ and $\{T^n x_0\}$ converges to x^*. Further, if for all $x, y \in F(T)$, we have $\alpha(x,y) \ge 1$, then T has a unique fixed point in X.

Proof: The proof can be done by following the lines of the proof of Theorem 2.2.

Example 2.5. Let X be a finite set defined as $X = \{1, 2, 3, 4\}$. Defined $d : X \times X \to R^+$ as
$d(1,1) = d(2,2) = d(3,3) = d(4,4) = 0$
$d(1,2) = d(2,1) = 20$
$d(2,3) = d(3,2) = d(1,3) = d(3,1) = 2$
$d(1,4) = d(4,1) = d(2,4) = d(4,2) = d(3,4) = d(4,3) = 4$
$20 = d(1,2) \ge d(1,3) + d(3,2) = 2 + 4 = 6$
$20 = d(1,2) \ge d(1,3) + d(3,2) + d(4,2) = 2 + 4 + 4 = 10$
that is, the triangle inequality is not satisfied and condition for rectangular metric is also not satisfied(i.e. rectangular inequality).However, d is a rectangular b-metric on X with $s = 2 > 1$, and hence (X,d) is a rectangular b-metric space. Define $T : X \to X$ as $T1 = T2 = T3 = 2, T4 = 3$, $\alpha(x,y) = 1$ and $\psi(t) = \frac{t}{2}$. Then, for $x = 1, 2, 3$ and $y = 1, 2, 3$, we have
$\alpha(x,y)d(Tx,Ty) = 0 \le \psi(M(x,y)) = 0$.
On the other hand, for $x = 1, 2, 3$ and $y = 4$ we obtain
$\alpha(x,y)d(Tx,T4) = d(2,3) = 1$ and

$$M(x,4) = max\left\{d(x,4), d(x,T4), d(y,T4), \frac{d(x,Tx)d(y,Ty)}{1+d(x,4)}, \frac{d(x,Tx)d(x,Ty)}{1+d(x,T4)}\right\}$$

$$= max\left\{0 \text{ or } 4, 0 \text{ or } 2 \text{ or } 4, 0 \text{ or } 2 \text{ or } 4, \frac{d(x,Tx)d(y,Ty)}{1+0 \text{ or } 4}, \frac{d(x,Tx)d(x,Ty)}{1+0 \text{ or } 2 \text{ or } 4}\right\}$$

$$M(1,2) = max\left\{d(1,2), d(1,T1), d(2,T2), \frac{d(1,T1)d(2,T2)}{1+d(1,2)}, \frac{d(1,T1)d(2,T2)}{1+d(T1,T2)}\right\}$$

$$= max\left\{20,, d(1,2), d(2,2), \frac{d(1,2)d(2,2)}{1+20}, \frac{d(1,2)d(2,2)}{1+d(2,2)}\right\}$$

$$= max\left\{20, 20, 0, \frac{20 \times 0}{1+20}, \frac{20 \times 0}{1+0}\right\}$$

$$= 20$$

$$M(2,2) = max\left\{d(2,2), d(2,T2), d(2,T2), \frac{d(2,T2)d(2,T2)}{1+d(2,2)}, \frac{d(2,T2)d(2,T2)}{1+d(T2,T2)}\right\}$$

$$= max\left\{0, d(2,2), d(2,2), \frac{d(2,2)d(2,2)}{1+0}, \frac{d(2,2)d(2,2)}{1+d(2,2)}\right\}$$

$$= 0.$$

3. Some Consequences

In this section we give some consequences of the main results presented above. Specially, we apply our results to rectangular b- metic spaces endowed with a partial order.

Definition 3.1. Let $(X, \preceq)$ be a partially ordered set. A mapping $T : X \to X$ is said to be nondecreasing with respect to $\preceq$ if for every $x, y \in X, x \preceq y$ implies $Tx \preceq Ty$.

Definition 3.2. Let $(X, d, \preceq)$ be a partially ordered rectangular b-metric space. X is called regular rectangular b-metric space if $\{x_n\}$ is a sequence in X such that $x_n \to x$ and $x_n \preceq x_{n+1}$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \preceq x$ for all $k \in N$.

Theorem 3.3. Let $(X, d, \preceq)$ be a partially ordered rectangular b-metric space and $T : X \to X$ be a nondecreasing self mapping. Suppose that the following conditions are satisfied:
(i) There exists a function $\psi \in \Psi$ for which, $d(Tx, Ty) \leq \psi(M(x,y))$ where

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}, \frac{d(x,Tx)d(x,Ty)}{1+(Tx,Ty)}\right\}$$

for all $x, y \in X$ with $x \preceq y$.
(ii) There exists $x_0 \in X$ such that $x_0 \preceq Tx_0$ and $x_0 \preceq T^2x_0$.
(iii) Either T is continuous or X is regular.
Then T has a fixed point $x^* \in X$ and $\{T^2x_0\}$ converges to x^*.

Proof: Define a mapping $\alpha : X \times X \to [0, \infty)$ defined as follows

$$\alpha(x, y) \;\; = \;\; \begin{cases} 1, & \text{if } x \preceq y \text{ or } y \preceq x; \\ 0, & \text{otherwise}. \end{cases}$$

Then the existence conditions of theorem **2.2** hold and hence T has a fixed point which is the limit of the sequence $\{T^n x_0\}$.

Theorem 3.4. Let $(X, d, \preceq)$ be a partially ordered rectangular b-metric space and $T : X \to X$ be a nondecreasing self mapping. Suppose that the following conditions are satisfied:
(i) There exists a function $\psi \in \Psi$ for which, $d(Tx, Ty) \leq \psi(M(x, y))$, where

$$M(x, y) \;\; = \;\; \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Tx)d(y, Ty)}{1 + d(x, y) + d(x, Ty) + d(y, Tx)}, \right.$$
$$\left. \frac{d(x, Ty)d(x, y)}{1 + d(x, Tx) + d(y, Tx) + d(y, Ty)} \right\}$$

for all $x, y \in X$ with $x \preceq y$.
(ii) There exists $x_0 \in X$ such that $x_0 \preceq Tx_0$ and $x_0 \preceq T^2 x_0$.
(iii) Either T is continuous or X is regular.
Then T has a fixed point $x^* \in X$ and $\{T^n x_0\}$ converges to x^*.

Proof: Employing again a mapping $\alpha : X \times X \to [0, \infty)$ defined as

$$\alpha(x, y) \;\; = \;\; \begin{cases} 1, & \text{if } x \preceq y \text{ or } y \preceq x; \\ 0, & \text{otherwise} \end{cases}$$

We observe that the existence conditions of Theorem **2.4** holds and hence, T has a fixed point which is the limit of the sequence $\{T^n x_0\}$. Several particular cases can be deduced from the above results.

Corollary 3.5. Let (X, d) be a complete rectangular b-metric space, $T : X \to X$ and $\alpha : X \times X \to R$. Suppose that the following conditions are satisfied: **(i)** T is an α-admissible mapping.
(ii) T satisfies $d(Tx, Ty) \leq kM(x, y)$ where

$$M(x, y) \;\; = \;\; \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Tx)d(y, Ty)}{1 + d(x, y) + d(x, Ty) + d(y, Tx)}, \right.$$
$$\left. \frac{d(x, Ty)d(x, y)}{1 + d(x, Tx) + d(y, Tx) + d(y, Ty)} \right\}$$

for some $k \in [0, 1)$
(iii) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2 x_0) \geq 1$.
(iv) Either T is continuous or X is α-regular.
Then T has a fixed point $x^* \in X$ and $\{T^n x_0\}$ converges to x^*. Further, if for all $x, y \in F(T)$, we have $\alpha(x, y) \geq 1$, then T has a unique fixed point in X.
Proof: Define $\psi(t) = kt$, clearly $\psi \in \Psi$. By theorem 2.2, T has a unique fixed point.

Corollary 3.6. Let $(X, d, \preceq)$ be a complete partially ordered rectangular b-metric space, and $T : X \to X$ be a nondecreasing mapping. Suppose that the following conditions are satisfied:
(i) $d(Tx, Ty) \leq kM(x, y)$ where

$$
M(x, y) \;=\; \max\left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Tx)d(y, Ty)}{1 + d(x, y) + d(x, Ty) + d(y, Tx)}, \right.
$$
$$
\left. \frac{d(x, Ty)d(x, y)}{1 + d(x, Tx) + d(y, Tx) + d(y, Ty)} \right\}
$$

for all $x, y \in X$ with $x \preceq y$ and some $k \in [0, 1)$.
(ii) There exists $x_0 \in X$ such that $x_0 \preceq Tx_0$ and $x_0 \preceq T^2 x_0$.
(iii) Either T is continuous or X is regular.
Then T has a fixed point $x^* \in X$ and $\{T^n x_0\}$ converges to x^*.

Proof: Define $\alpha : X \times X \to [0, \infty)$ **as**

$$
\alpha(x, y) \;=\; \left\{ \begin{array}{ll} 1, & \textbf{if } x \preceq y \textbf{ or } y \preceq x; \\ 0, & \textbf{otherwise} \end{array} \right.
$$

Corollary 3.5 **implies that** T **has a fixed point.**

References

[1] I. A. Bakhtin, The contraction principle in Quasi-metric spaces, Funct. Anal.30, 26-37, 1989.
[2] S. Czerwik, Contraction mappings in b-metric spaces, Acta Math. Inform. Univ. Ostraviensis, 1, 5-11, 1993.
[3] S. Czerwik, Nonlinear set valued contraction mappings in b-metric spaces, Atti Semin. Mat. Fis. Univ. Modena Reggio Emilia, 46, 263-276, 1998.
[4] A. Branciari, A fixed point theorem of Banach-Cassioppoli type on a class of generalized metric spaces, Publ. Math. Debrecen, 57, 31-37, 2000.
[5] R. George, S. *Radenović*, K.P. Reshma, S. Shukla, Rectangular b-metric spaces and contraction principles, Journal of Nonlinear Science and Applications, 8, 1005-1013, 2015.
[6] Hui-Shen Ding Vildan Ozturk, Stojan *Radenović* On some new fixed point results in b-rectangular metric spaces, Journal of Nonlinear Science and Applications, 8, 378-386, 2015.
[7] Ozgur Ege, Complex valued rectangular b-metric space and an application to linear equations, Journal of Nonlinear Science and Applications, 8, 1014-1021, 2015.
[8] Hui-Shen Ding, Mohammad Imdad, Stojan *Radenović*, Jelena *Vujaković*, On some fixed point results in b-metric, rectangular and b-rectangular metric spaces, Arab Journal of Mathematical Sciences, 22, 151-164, 2016.
[9] Zoran Kadelburg, Stojan *Radenović*, Pata-type common fixed point results in b-rectangular metric spaces, Journal of Nonlinear Science and Applications, 8, 944-954, 2015
[10] Hassen Aydi, Abdelbasset Felhi and Slah Sahmim, Common fixed points in rectangular b-metric spaces using (E.A) property, 8, 159-169, 2015.
[11] Hamed H Alsulami, Sumit Chandok, Mohamed-Aziz Taoudi and Inci M Erhan, Some fixed point theorems for (α, ψ) - rational type contractive mappings, Fixed Point Theory and Applications, 97, 2015-97, 2015.
[12] Bulbul Khomdram, Yumnam Rohen and Thokchom Chhatrajit Singh, Couple fixed point theorems in G_b- metric spaces satisfying some rational contractive conditions, Springer Plus, 5:1261, 2016.
[13] Bulbul Khomdram, Yumnam Rohen, Quadruple common fixed point theorems in G_b- metric spaces, International Journal of Pure and Applied Mathematics, 109(2), 279-293, 2016
[14] Nabil Mlaiki and Yumnam Rohen, Some couple fixed point theorems in partially ordered A_b metric space, Journal of Nonlinear Science and Applications, 10, 1731-1743, 2017.

[15] Yumnam Rohen, Tatjana *Došenović*, Stojan *Radenović*, A Note on the paper "A Fixed Point Theorems in S_b Metric spaces", Filomat 31:11, 3335-3346, 2017.

[16] M. Priyobarta, Yumanm Rohen and Nabil Mlaiki, Complex valued S_b Metric Spaces, Journal of Mathematical Analysis, 8(3), 13-24, 2017.

[17] Nabil Mlaiki, Aymen Mukheimer, Yumnam Rohen, Nizar Souayah, Thabet Abdeljawad, Fixed point theorems for α contractive mappings in S_b Metric Spaces, Journal of Mathematical Analysis, 8(5), 40-46, 2017.

[18] Y. Rohen , N. Priyobarta, N. Mlaiki and N. Souayah, Couple fixed point theorems in A_b Metric Spaces satisfying rational inequality, Journal of Inequalities and Special Functions, Vol.9(2), 41-56, 2018.

[19] D. Dhamodharan, Yumnam Rohen and A.Hojat Ansari, Fixed point theorems of C-class functions in S_b metric space, Results in Fixed Point Appl., Volume 2018, Article ID 2018018, 20 pages, 2018.

[20] Bulbul Khomdram and Yumnam Rohen, Some common Couple Fixed Point Theorems in S_b metric Spaces, Fasciculi Mathematici, 60(1), 79-92, 2018.

[21] M.S Khan, N. Priyobarta and Yumnam Rohen, Fixed points of generalised rational $\alpha^* - \psi$-Geraghty contraction for multivalued mappings, J. Adv. Math. Stud. 12(2), 156-169, 2019.

[22] Lishram Shanjit and Yumnam Rohen, Best Proximity Point theorems in b- metric space satisfying rational contractions, Journal of nonlinear analysis and applications (Accepted).

[23] Samet B., Vetro C., Vetro P., Fixed point theorms for (α, ψ)-contractive type mappings, Nonlinear Anal. 75, 2154-2165, 2012.

[24] Kirk W.A., Shahzad N., Generalized metrics and Caristi's theorem, Fixed Point Theory Appl.2013, Article ID 129, 2013.

Langpoklakpam Premila Devi
Department of Mathematics, Manipur University, Imphal-795003, Manipur, India
E-mail address: abelang.premi@gmail.com

Laishram Shambhu Singh
D.M.College of Science, Imphal-795001, Manipur, India
E-mail address: laishramshambhu11@gmail.com