Electronic Journal of Mathematical Analysis and Applications
Vol. 9(1) Jan. 2021, pp. 274-283.
ISSN: 2090-729X(online)
http://math-frac.org/Journals/EJMAA/

A NOTE ON (m, n)-PARANORMAL OPERATORS

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Abstract. In this paper, we prove properties of the class of (m, n)-paranormal operators (a generalization of paranormal operators) on Hilbert space. Equality of the approximate point spectrum and the joint approximate point spectrum, for (m, n)-paranormal operators has been proved under certain given conditions. Moreover, the point spectrum coincides with the joint point spectrum for the class of (m, n)-paranormal operators. We also discuss the SVEP, normaloid and subnormality for the same class of operators.

1. Introduction

Throughout this note, $B(\mathcal{H})$ be the C^*-algebra of all bounded linear operators acting on infinite dimensional separable complex Hilbert space $\mathcal{H}$. If $T \in B(\mathcal{H})$, then we shall write $N(T)$ and $R(T)$ for the null space and the range space of T, respectively. In this paper, $\mathbb{C}$ and $\mathbb{N}$ denote the set of all complex numbers and the set of all natural numbers, respectively. The orthogonal complement $S^\perp$ of a subset S of Hilbert space is defined by $S^\perp = \{x \in \mathcal{H} : \langle x, y\rangle = 0 \text{ for all } y \in S\}$.

For T, S in $B(\mathcal{H})$, $T \otimes S$ denotes the tensor product on the product space $\mathcal{H} \otimes \mathcal{H}$. If $T \in B(\mathcal{H})$, then we write $\sigma(T), \sigma_p(T), \sigma_{jp}(T), \sigma_a(T)$ and $\sigma_{ja}(T)$ for the spectrum, the point spectrum, the joint point spectrum, the approximate point spectrum and the joint approximate point spectrum of T, respectively. An operator T in $B(\mathcal{H})$ is said to be:

1) positive (denoted $T \geq 0$) if $\langle Tx, x\rangle \geq 0$, for all $x \in \mathcal{H}$.
2) if $T^*T - TT^* \geq 0$, or equivalently, $\|Tx\| \geq \|T^*x\|$ for all $x \in \mathcal{H}$[15].
3) paranormal if $\|Tx\|^2 \leq \|T^2x\|\|x\|$, for all $x \in \mathcal{H}$ [15, 13].
4) (m, n)-paranormal and $(m, n)^*$-paranormal if $\|Tx\|^{n+1} \leq m\|T^{n+1}x\|\|x\|^n$ and $\|T^*x\|^{n+1} \leq m\|T^{n+1}x\|\|x\|^n$, respectively for all x in $\mathcal{H}$, where m is a positive real number and n is a positive integer [10].
5) normaloid, if its spectral radius coincides with its norm, that is, $r(T) = \|T\|$, or equivalently, $\|T^n\| = \|T\|^n$ for every positive integer n.

2010 *Mathematics Subject Classification.* Primary 47B20, Secondary 47A10, 47A05.

Key words and phrases. (m, n)-paranormal operator, $(m, n)^*$-paranormal operator, hyponormal operator, single valued extension property.

Submitted Jan. 11, 2020. Revised April May 28, 2020.

2. (m,n)-paranormal operators

We begin this section with the following theorem for the class of (m,n)-paranormal operators.

Theorem 2.1. *Let $T \in B(\mathcal{H}\oplus\mathcal{H})$ be a (m,n)-paranormal operator defined by 2×2 matrix representation $T = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$. Then A is (m,n)-paranormal.*

Proof. By [10, Theorem 2.1], the following matrix

$$m^{\frac{2}{n+1}} T^{*n+1} T^{n+1} - (n+1)a^n T^* T + m^{\frac{2}{n+1}} n a^{n+1} I = \begin{bmatrix} Q & R \\ R^* & S \end{bmatrix}$$

is positive for each $a > 0$, where

$$Q = m^{\frac{2}{n+1}} A^{*n+1} A^{n+1} - (n+1)a^n A^* A + m^{\frac{2}{n+1}} n a^{n+1} I$$

$$R = m^{\frac{2}{n+1}} A^{*n+1} P - (n+1)a^n A^* C$$

and

$$S = m^{\frac{2}{n+1}} (P^* P + B^{*n+1} B^{n+1}) - (n+1)a^n (C^* C + B^* B) + m^{\frac{2}{n+1}} n a^{n+1} I$$

Here, we have

$$P = A^n C + A^{n-1} CB + A^{n-2} CB^2 + ... + ACB^{n-1} + CB^n$$

Since T is (m,n)-paranormal, so Q is positive for each $a > 0$. Hence, A is (m,n)-paranormal. □

Now, in the sequel of the above result, we have S is positive for each $a > 0$. Thus, we have

$$\begin{aligned} m^{\frac{2}{n+1}} B^{*n+1} B^{n+1} - (n+1)a^n B^* B + m^{\frac{2}{n+1}} n a^{n+1} I \\ \geq (n+1)a^n C^* C - m^{\frac{2}{n+1}} P^* P. \end{aligned}$$

Therefore, if we take $(n+1)a^n C^* C \geq m^{\frac{2}{n+1}} P^* P$ for each $a > 0$, then B is also (m,n)-paranormal. This is our next result.

Proposition 2.2. *Let $T \in B(\mathcal{H}\oplus\mathcal{H})$ be a (m,n)-paranormal operator defined by 2×2 matrix representation $T = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$. Then B is (m,n)-paranormal provided $(n+1)a^n C^* C \geq m^{\frac{2}{n+1}} P^* P$, for each $a > 0$.*

Remark 2.3. It is well known that $\begin{bmatrix} x & y \\ y^* & z \end{bmatrix}$ is positive if and only if $x \geq 0$, $z \geq 0$ and $y = x^{\frac{1}{2}} w z^{\frac{1}{2}}$ for some contraction w. Now, if we choose $Q = 0$ in Theorem 2.1, then we have $R = 0$, that is,

$$\begin{aligned} m^{\frac{2}{n+1}} A^{*n+1} (A^n C + A^{n-1} CB + A^{n-2} CB^2 + \cdots + ACB^{n-1} + CB^n) \\ = (n+1)a^n A^* C. \end{aligned}$$

Remark 2.4. In Theorem 2.1, if we set $C = 0$, then B is always (m,n)-paranormal.

Remark 2.5. If $T = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ on $\mathcal{H} = M \oplus M^{\perp}$ is (m,n)-paranormal and M be a closed invariant subspace of $\mathcal{H}$ under T, then T is (m,n)-paranormal on M.

In the following theorem, we show the relationship between (m,n)-paranormal and $(m,n+1)$-paranormal operators for $n \geq 2$.

Theorem 2.6. [16, Lemma 1] *Let T be a (m,n)-paranormal operator and for all unit vectors in $\mathcal{H}$, $\|T^n x\|\|Tx\| \leq \|T^{n+1}x\|$. Then T is $(m,n+1)$-paranormal.*

Conversely, if T is $(m,n+1)$-paranormal and $\|T^{n+1}x\|^n \leq m\|T^n x\|^{n+1}$ for all unit vectors in $\mathcal{H}$, then T is (m,n)-paranormal.

Proof. By using the (m,n)-paranormality of T and given condition, we have

$$\|Tx\|^{n+1} \leq m\|T^n x\|\|Tx\| \leq m\|T^{n+1}x\|,$$

that is,

$$\|Tx\|^{n+1} \leq m\|T^{n+1}x\|.$$

Conversely, with T $(m,n+1)$-paranormal and given condition, it follows that

$$\|Tx\|^{n(n+1)} \leq (m\|T^{n+1}x\|)^n \leq m^{n+1}\|T^n x\|^{n+1},$$

that is,

$$\|Tx\|^n \leq m\|T^n x\|.$$

Hence, the result holds. □

It is a natural question to ask whether an operator T is normaloid or not. The following example provides an operator which is (m,n)-paranormal but not normaloid for $m > 1$.

Example 2.7. Let $\mathcal{H} = l^2(\mathbb{N},\mathbb{C})$. Define weighted shift operator T by $T(e_k) = w_k e_{k+1}$ for all positive integers k, with non zero weights w_k and orthonormal basis e_k, where

$$w_k = 1 \text{if } k = 1 2 \text{if } k = 2 3 \text{if } k \geq 3$$

Equivalently, for $x \in l^2(\mathbb{N},\mathbb{C})$, we have

$$T(x_1, x_2, ...) = (0, x_1, 2x_2, 3x_3, 3x_4, ...).$$

By [10, Theorem 2.9], T is (m,n)-paranormal if and only if

$$|w_k|^{n-1} \leq m|w_{k+1}||w_{k+2}|\cdots|w_{k+n-1}|, \tag{2.1}$$

for $n \geq 2$, all positive integers k and all unit vectors. Note that the inequality (2.1) is satisfied for all $m \geq 1$ by weighted sequences. Hence, T is (m,n)-paranormal. Now, $\|T\| = \sup|w_k|$ and so it is easy to see that $\|T\| = 3$. It is well known that $0 \leq r(T) \leq \|T\|$. Thus, $r(T) \leq 3$.

Now, we claim that $r(T) < 3$. Suppose if possible, $r(T) = 3$. Then there exists $\lambda \in \mathbb{C}$ such that $|\lambda| = 3$ and $T - \lambda I$ is not invertible. Note that

$$(T - \lambda I)(x_1, x_2, ...) = (-\lambda x_1, x_1 - \lambda x_2, 2x_2 - \lambda x_3, 3x_3 - \lambda x_4, 3x_4 - \lambda x_5, ...)$$

It is easy to see that $T - \lambda I$ is one one and onto. Hence, $T - \lambda I$ is invertible, which is a contradiction. Therefore, $\lambda \notin \sigma(T)$ and $r(T) \neq 3$. Thus, $r(T) < 3$. Hence, T is not normaloid.

To the sequel, we sketch the following theorem which shows that a (m,n)-paranormal operator is normaloid for $m \leq 1$.

Theorem 2.8. [16, Proposition 1] *If an operator T is (m,n)-paranormal for $m \leq 1$, then T is normaloid.*

The proof of the next theorem is similar to that of [11, Theorem 2.3].

Theorem 2.9. *Let $\mathcal{H}$ be the direct sum of countably many isomorphic copies of Hilbert spaces $\mathcal{H}_i$. If T_i is (m,n)-paranormal operator on $\mathcal{H}_i$ for each i, then the direct sum of T_i is also (m,n)-paranormal.*

Lemma 2.10. [10, Theorem 2.9] *Let an operator $T\colon l^2(\mathbb{Z},\mathbb{C}) \longrightarrow l^2(\mathbb{Z},\mathbb{C})$ be defined by $T(e_k) = w_{k-1}e_{k-1}$ with non zero weights (w_k), and the orthonormal basis (e_k). Then T is (m,n)-paranormal if and only if*

$$|w_{k-1}|^{n-1} \leq m|w_{k-2}||w_{k-3}|\cdots|w_{k-n}|$$

holds for all integers k, unit vectors and $n \geq 2$.

In the following example, we show that the inverse of (m,n)-paranormal operator need not be (m,n)-paranormal.

Example 2.11. Let $\mathcal{H} = l^2(\mathbb{Z},\mathbb{C})$ and T be a weighted shift operator on $\mathcal{H}$ defined by $Te_k = w_k e_{k+1}$ with non zero weights w_k, and the orthonormal basis e_k for all integers k, where

$$w_k = \frac{1}{2}\text{if } k \leq 02\text{if } k = 14\text{if } k \geq 2.$$

Equivalently, T is defined by

$$T(..., x_{-1}, x_0, x_1, ...) = (..., \frac{1}{2}x_{-1}, \frac{1}{2}x_0, 2x_1, 4x_2, 4x_3, ...)$$

By [10, Theorem 2.9], T is (m,n)-paranormal if and only if

$$|w_k|^{n-1} \leq m|w_{k+1}||w_{k+2}|...|w_{k+n-1}|, \tag{2.2}$$

for unit vectors and $n \geq 2$. Thus, (2.2) holds for $m \geq 1$. It is straightforward to see that T is invertible. Also,

$$T^{-1}(..., y_{-1}, y_0, y_1, ...) = (..., 2y_0, 2y_1, \frac{y_2}{2}, \frac{y_3}{4}, \frac{y_4}{4}, ...),$$

that is,

$$T^{-1}e_k = \alpha_{k-1}e_{k-1}$$

with weighted sequence

$$\alpha_k = 2\text{if } k \leq 0\frac{1}{2}\text{if } k = 1\frac{1}{4}\text{if } k \geq 2$$

Now, we claim that T^{-1} is not (m,n)-paranormal. By using Lemma 2.10, T^{-1} is (m,n)-paranormal if and only if

$$|\alpha_{k-1}|^{n-1} \leq m|\alpha_{k-2}||\alpha_{k-3}|\cdots|\alpha_{k-n}|, \tag{2.3}$$

for unit vectors, all k and $n \geq 2$. If we choose $n = 3, k = 4$ and $m = \frac{1}{3}$, then (2.3) fails to hold. Hence, the claim holds.

The proof of the following theorem is similar to [11, Theorem 2.6].

Theorem 2.12. *Let T be an (m,n)-paranormal operator. Then $T \otimes I$ and $I \otimes T$ are also (m,n)-paranormal.*

The following example shows that the tensor product of two (m,n)-paranormal operators need not be (m,n)-paranormal.

Example 2.13. For each positive integer k, assume that $H_k = \mathbb{R}\times\mathbb{R}$. Let $\mathcal{H}$ be a Hilbert space such that $\mathcal{H} = \oplus_{k=1}^{\infty} H_k$. Now, we choose A and B to be positive operators on H_k such that $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and $B^4 = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}$. We simply assign the operator T on $\mathcal{H}$ as:

$$T(x_1, x_2, ...) = (0, Ax_1, Ax_2, ..., Ax_n, Bx_{n+1}, Bx_{n+2}, ...).$$

Thus, the adjoint of T is given by

$$T^*(x_1, x_2, ...) = (Ax_2, Ax_3, ..., Ax_{n+1}, Bx_{n+2}, Bx_{n+3}, ...).$$

For $x = (..., 0, 0, x_n, 0, 0, ...)$ in $\mathcal{H}$, by [10, Theorem 2.1], an operator T is $(2^{\frac{3}{2}}, 2)$-paranormal if and only if

$$2AB^4A - 3a^2A^2 + 4a^3I \geq 0,$$

for each $a > 0$. Now,

$$2AB^4A - 3a^2A^2 + 4a^3I = \begin{bmatrix} 4a^3 - 12a^2 + 32 & 8 \\ 8 & 4a^3 - 12a^2 + 24 \end{bmatrix}$$

is positive for each $a > 0$. Thus, T is $(2^{\frac{3}{2}}, 2)$-paranormal. Similarly, by [10, Theorem 2.1], the operator $T \otimes T$ is $(2^{\frac{3}{2}}, 2)$-paranormal if and only if

$$2(AB^4A \otimes AB^4A) - 3a^2(A^2 \otimes A^2) + 4a^3(I \otimes I) \geq 0,$$

for each $a > 0$. For $a = 10$, the operator

$$2(AB^4A \otimes AB^4A) - 3a^2(A^2 \otimes A^2) + 4a^3(I \otimes I) = \begin{bmatrix} -288 & 128 & 128 & 32 \\ 128 & -416 & 32 & 96 \\ 128 & 32 & -416 & 96 \\ 32 & 96 & 96 & -512 \end{bmatrix}$$

is not positive. Hence, our claim holds.

Embry [12] has proved that an operator T is subnormal if and only if $\sum_{i,j=0}^{k} \left\langle T^{i+j}x_i, T^{i+j}x_j \right\rangle$ is non-negative for all finite collection of vectors $x_0,\ x_1, \cdots, x_k$. In the following theorem, by using this characterization, we prove that a (m, n) paranormal is subnormal under certain conditions.

Theorem 2.14. *If an operator T is (m, n) paranormal and partial isometry with $m \leq 1$ and $\|T^n\|^2 \leq \frac{1}{m^{\frac{2}{n+1}}}$, then T is subnormal.*

Proof. As T is (m, n)-paranormal, so by [10, Theorem 2.1], we have

$$m^{\frac{2}{n+1}} T^{*n+1}T^{n+1} - (n + 1)a^nT^*T + m^{\frac{2}{n+1}} na^{n+1}I \geq 0,$$

for each $a > 0$. Also, it follows that

$$T^*T(m^{\frac{2}{n+1}} T^{*n+1}T^{n+1} - (n + 1)a^nT^*T + m^{\frac{2}{n+1}} na^{n+1}I)T^*T \geq 0 \tag{2.4}$$

for each $a > 0$. Since T is a partial isometry, $TT^*T = T$ by [14, Corollary 3, Problem 98]. Now, take $a = 1$ in (2.4). Then we have

$$m^{\frac{2}{n+1}} T^{*n+1}T^{n+1} - ((n + 1) - m^{\frac{2}{n+1}} n)T^*T \geq 0,$$

that is,

$$T^*T \leq \frac{m^{\frac{2}{n+1}}}{(n+1) - nm^{\frac{2}{n+1}}} T^{*n+1}T^{n+1},$$

equivalently,

$$\begin{aligned}\|Tx\|^2 &\leq \frac{m^{\frac{2}{n+1}}}{(n+1) - nm^{\frac{2}{n+1}}} \|T^{n+1}x\|^2 \\ &\leq m^{\frac{2}{n+1}} \|T^{n+1}x\|^2 \\ &\leq m^{\frac{2}{n+1}} \|T^n\|^2 \| \|Tx\|^2 \\ &\leq ||Tx||^2.\end{aligned}$$

Therefore, we have

$$T^*T = m^{\frac{2}{n+1}} T^{*n+1}T^{n+1} \text{ for all } n. \tag{2.5}$$

Further, let $x_0, \; x_1, ..., x_k$ be a finite collection of vectors, then by using (2.5) we get

$$\begin{aligned}m^4 \sum_{i,j=0}^{k} \left\langle T^{i+j}x_i, T^{i+j}x_j \right\rangle &= m^4 \big(\langle x_0, x_0 \rangle + \langle T^*Tx_0, x_1 \rangle + \langle T^*Tx_1, x_0 \rangle \big) \\ &\quad + \sum_{\substack{i,j=0 \\ i+j \neq 0,1}}^{k} m^4 m^{\frac{-2}{i+j}} \left\langle m^{\frac{2}{i+j}} T^{*i+j} T^{i+j} x_i, x_j \right\rangle \\ &= m^4 \big(\langle x_0, x_0 \rangle + \langle T^*Tx_0, x_1 \rangle + \langle T^*Tx_1, x_0 \rangle \big) \\ &\quad + \sum_{\substack{i,j=0 \\ i+j \neq 0,1}}^{k} m^4 m^{\frac{-2}{i+j}} \left\langle T^*Tx_i, x_j \right\rangle.\end{aligned}$$

Since T^*T is a projection by [14, Problem 98], we have

$$\begin{aligned}m^4 \sum_{i,j=0}^{k} \left\langle T^{i+j}x_i, T^{i+j}x_j \right\rangle &= m^4 \big(\langle x_0, x_0 \rangle + \langle T^*Tx_0, T^*Tx_1 \rangle + \langle T^*Tx_1, T^*Tx_0 \rangle \big) \\ &\quad + \sum_{\substack{i,j=0, \\ i+j \neq 0,1}}^{k} m^4 m^{\frac{-2}{i+j}} \left\langle (T^*T)^{i+j} x_i, (T^*T)^{i+j} x_j \right\rangle \\ &= m^4 \big(\langle x_0, x_0 \rangle + \langle T^*Tx_0, T^*Tx_1 \rangle + \langle T^*Tx_1, T^*Tx_0 \rangle \big) \\ &\quad + \sum_{i+j=2} m^3 \left\langle (T^*T)^2 x_i, (T^*T)^2 x_j \right\rangle + \cdots \\ &\quad + \sum_{i+j=2k-1} m^{2(\frac{4k-3}{2k-1})} \left\langle (T^*T)^{2k-1} x_i, (T^*T)^{2k-1} x_j \right\rangle \\ &\quad + \sum_{i+j=2k} m^{(\frac{4k-1}{k})} \left\langle (T^*T)^{2k} x_i, (T^*T)^{2k} x_j \right\rangle.\end{aligned}$$

As $m \leq 1$, we obtain the following relation

$$m^3 \geq \cdots \geq m^{2(\frac{4k-3}{2k-1})} \geq m^{(\frac{4k-1}{k})} \geq m^4.$$

By using the above relation we obtain

$$m^4 \sum_{i,j=0}^{k} \left\langle T^{i+j}x_i, T^{i+j}x_j \right\rangle \quad \geq \quad m^4 \sum_{i,j=0}^{k} \left\langle (T^*T)^{i+j}x_i, (T^*T)^{i+j}x_j \right\rangle .$$

Since T^*T is self adjoint, T^*T is subnormal and so we have

$$\sum_{i,j=0}^{k} \left\langle T^{i+j}x_i, T^{i+j}x_j \right\rangle \geq 0.$$

Hence, the result holds. □

In the sequel, we give the following example to show that there also exists a (m, n)-paranormal operator, which is not subnormal.

Example 2.15. Let T be an operator defined by $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ as a 2×2 matrix $T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. By [10, Theorem 2.1], T is $(2^{\frac{3}{2}}, 2)$ paranormal if and only if $2T^{*3}T^3 - 3a^2T^*T + 4a^3I \geq 0$, for each $a \geq 0$.

It is easy to see that the matrix

$$2T^{*3}T^3 - 3a^2T^*T + 4a^3I = \begin{bmatrix} 6a^2 + 4a^3 & 12 - 6a^2 \\ 12 - 6a^2 & 75 - 15a^2 + 4a^3 \end{bmatrix}$$

is positive for each $a > 0$. Hence, T is $(2^{\frac{3}{2}}, 2)$ is paranormal.

We next move to show that T is not subnormal. Consider

$$\sum_{i,j=0}^{1} \left\langle T^{i+j}x_i, T^{i+j}x_j \right\rangle = \langle x_0, x_0 \rangle + \langle Tx_0, Tx_1 \rangle + \langle Tx_1, Tx_0 \rangle + \langle T^2x_1, T^2x_1 \rangle.$$

Choose $x_0 = \begin{bmatrix} x_0' \\ x_0'' \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $x_1 = \begin{bmatrix} x_1' \\ x_1'' \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{-1}{4} \end{bmatrix}$. Then

$$\sum_{i,j=0}^{1} \left\langle T^{i+j}x_i, T^{i+j}x_j \right\rangle = \frac{-55}{16}$$

is not positive. Hence, T is not subnormal.

3. Spectral properties

In this section we prove some spectral properties on (m, n)-paranormal operators. The spectrum set of an operator $T \in B(\mathcal{H})$, denoted by $\sigma(T)$, is the set of complex number λ such that $T - \lambda I$ is not invertible. A complex number λ is said to be in the point spectrum $\sigma_p(T)$ of T if there is a nonzero $x \in \mathcal{H}$ such that $(T - \lambda)x = 0$. If in addition, $(T^* - \bar{\lambda})x = 0$, then λ is said to be in the joint point spectrum $\sigma_{jp}(T)$ of T. Analogously, a complex number λ is said to be in the approximate point spectrum $\sigma_a(T)$ of T if there is a sequence (x_n) of unit vectors in $\mathcal{H}$ such that $(T - \lambda)x_n \to 0$. If in addition, $(T^* - \bar{\lambda})x_n \to 0$, then λ is said to be in the joint approximate point spectrum $\sigma_{ja}(T)$ of T. In general, $\sigma_p(T) \neq \sigma_{jp}(T)$, $\sigma_a(T) \neq \sigma_{ja}(T)$.

Some researchers showed that, for some classes of nonnormal operators T , the nonzero points of its point spectrum and joint point spectrum are identical, the nonzero points of its approximate point spectrum and joint approximate point spectrum are identical [8, 9, 19, 20, 21]. The reader can refer to the recent papers

[1, 2, 3, 4, 5, 6, 7, 17, 18, 22, 23] for the spectrum and fine spectrum of certain linear operators represented by a triangle matrix over some sequence spaces.

The proof of the following theorem is similar to [11, Theorem 3.1].

Theorem 3.1. *If $T \in B(\mathcal{H})$ is a (m,n)-paranormal and hyponormal operator, then $\sigma_a(T) = \sigma_{ja}(T)$ for unit vectors.*

If an operator T is (m,n)-paranormal but not hyponormal, then the above result does not holds. We prove it in the following example.

Example 3.2. Let $T : l^2(\mathbb{N},\mathbb{C}) \longrightarrow l^2(\mathbb{N},\mathbb{C})$ be weighted shift operator defined by $T(e_k) = w_k e_{k-1}$, that is,

$$T(x_1, x_2, ...) = (x_2, x_3, ...)$$

with weighted sequence (w_k) such that $w_k = 1$ for all positive integers k and adjoint of T is given by

$$T^*(x_1, x_2, ...) = (0, x_1, x_2, ...).$$

By [10, Theorem 2.9], T is (m,n)-paranormal for unit vectors and $m \geq 1$. Since it is easy to check that $\|Tx\| \ngeq \|T^*x\|$ for some x in $\mathcal{H}$, T is not hyponormal. Next, we move to prove that $\sigma_a(T) \neq \sigma_{ja}(T)$. Now, we choose $0 = \lambda \in \sigma_a(T)$, a unit vector $x = (1, 0, 0, ...)$ and a sequence $\{x_n\}_{n=1}^{\infty} = \{x, x, ...\}$. Then

$$\|(T - \lambda I)x_n\| = \|Tx_n\| = 0 \text{ as } n \longrightarrow \infty$$

$$\text{but } \|(T - \lambda I)^* x_n\| = \|T^* x_n\| = \|(0, 1, 0, 0, \cdots)\| \nrightarrow 0 \text{ as } n \longrightarrow \infty.$$

This shows that $0 \notin \sigma_{ja}(T)$.

Definition 3.3. An operator T is said to have single valued extension property (abbreviated as SVEP) at $\gamma_0 \in \mathbb{C}$, if for every open neighborhood G of γ_0, the only analytic function $f : G \to \mathcal{H}$ which satisfies the equation $(T - \gamma I)f(\gamma) = 0$ for all $\gamma \in G$ is the function $f = 0$.

An operator T has SVEP if T has SVEP at every $\gamma \in \mathbb{C}$.

We can prove the following theorem in a similar fashion as that of [11, Theorem 3.4].

Theorem 3.4. *Let $T \in B(\mathcal{H})$ be a (m,n)-paranormal and hyponormal operator. Then T has SVEP.*

The proof of following proposition is similar to [11, Proposition 3.5].

Proposition 3.5. *Let $T \in B(\mathcal{H})$ be a (m,n)-paranormal and hyponormal operator. Then $N(T - \lambda I) \subseteq N(T^* - \bar{\lambda} I)$ for unit vectors and for all $\lambda \in \mathbb{C}$.*

The proof of the following theorem is similar to Theorem 3.1.

Theorem 3.6. *If T is a (m,n)-paranormal and hyponormal operator, then*

$$\sigma_p(T) = \sigma_{jp}(T)$$

for all unit vectors.

Proposition 3.7. *If T is a (m,n)-paranormal operator for $m \leq 1$ and $\{x_k\}$ is a sequence of unit vectors in $\mathcal{H}$, which satisfies $\lim_{k\to\infty} \|Tx_k\| = \|T\|$, then $\lim_{k\to\infty} \|T^{n+1}x_k\| = \|T\|^{n+1}$.*

Proof. As T is (m,n)-paranormal, so we have the inequality

$$\|Tx\|^{n+1} \leq m\|T^{n+1}x\|\|x\|^n.$$

For $m \leq 1$, we have

$$\begin{aligned} \|Tx_k\|^{n+1} &\leq m\|T^{n+1}x_k\| \\ &\leq \|T\|\ \|T\| \cdots \|T\| \\ &= \|T\|^{n+1}. \end{aligned}$$

Equivalently,

$$\|Tx_k\|^{n+1} \leq \|T^{n+1}x_k\| \leq \|T\|^{n+1}$$

As $\lim_{k\to\infty} \|Tx_k\| = \|T\|$, therefore by using squeeze principal, we have

$$\lim_{k\to\infty} \|T^{n+1}x_k\| = \|T\|^{n+1}.$$

□

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