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ON STABILITY OF α - RADICAL RECIPROCAL FUNCTIONAL EQUATION

SHALINI TOMAR AND NAVNEET HOODA

Abstract. The goal of this paper is to study the stability of α - radical reciprocal functional equation

$$f(\sqrt[\alpha]{(k+1)x^\alpha + ky^\alpha}) + f(\sqrt[\alpha]{(k+1)x^\alpha - ky^\alpha}) = \frac{2(k+1)f(x)[f(y)]^2}{[(k+1)f(y)]^2 - [kf(x)]^2}$$

in quasi $-\beta-$ normed and (β,p)- Banach spaces, where k is any integer greater than zero.

1. Introduction

Foremost, let us recollect the annals in the stability theory for functional equations (FEs). The stability problem for the FEs pertaining the stability of group homomorphisms was initiated by Ulam[1]. The Ulam′s question was to an extent solved by Hyers[2]. Subsequently, Hyers result was extended by several mathematicians like Aoki[11], Th.M.Rassias[12] and Găvruta[9].

Radical FEs is one of the popular topics for investigation in the theory of stability. Now-a-days, lot of papers concerning the stability and the hyper-stability of the radical FEs in various spaces appeared (see in [13, 3, 4, 7, 8, 5] and references therein).
In this paper, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of all positive integers, real numbers and complex numbers respectively; we put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$ and $\mathbb{R}^+ = [0, \infty)$.

2. Preliminaries

Now, let us discuss various basic properties and concepts concerning b-metric spaces and quasi-β-normed spaces.

2.1. **Definitions** [10]**.**

(D1) Let X be a nonempty set, $\vartheta \geq 1$ and $d : X \times X \to \mathbb{R}^+$ be a function such that
  (QM1) $d(x, y) = 0$ iff $x = y$,
  (QM2) $d(x, y) = d(y, x)$,

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(QM3) $d(x,z) \leq \vartheta(d(x,y)+d(y,z))$
for all $x,y,z \in X$. Then d is called a $b-metric$ (or quasi-metric) on X and (X,d,ϑ) is called a *b-metric space* (or a quasi-metric space). The smallest possible ϑ is called the *module of concavity of d.*

(D2) If $\lim\limits_{n,m\to\infty} d(x_n,x_m)=0$ then, the sequence $\{x_n\}$ is called *Cauchy.*

(D3) The sequence $\{x_n\}$ is called *convergent* to x in (X,d,ϑ) if

$$\lim_{n\to\infty} d(x_n,x)=0.$$

(D4) The space (X,d,ϑ) is called complete if each Cauchy sequence is a convergent sequence.

(D5) Let X be a vector space over the field $K \in \{\mathbb{R},\mathbb{C}\}$ and β be a fixed real number with $0<\beta\leq 1$. A *quasi-β-norm* $||\cdot||_\beta$ is a real-valued function on X satisfying the following conditions:
(S1) $||x||_\beta \geq 0$ for all $x \in X$ and $||x||_\beta = 0$ if and only if $x=0$.
(S2) $||\rho x||_\beta = |\rho|^\beta |x||_\beta$ for all $\rho \in K$ and all $x \in X$.
(S3) There is a constant $\vartheta \geq 1$ such that

$$||x+y||_\beta \leq \vartheta(||x||_\beta + ||y||_\beta)$$

for all $x,y \in X$. The pair $(X,||x||_\beta,\vartheta)$ is called a *quasi-β-normed space*, if $||.||_\beta$ is a quasi-β-norm on X. The smallest possible ϑ is called the *modulus of concavity of* $||.||_\beta$. A *quasi-β-Banach space* is a complete quasi-β-normed space.

A quasi-β-norm $||.||_\beta$ is called a (β,p)- norm $(0<p\leq 1)$ if

$$||x+y||_\beta^p \leq ||x||_\beta^p + ||y||_\beta^p$$

for all $x,y \in X$. In this case, a quasi -β- Banach space is called a (β,p)-Banach space.

In this paper, we will achieve general solutions of ensuing radical reciprocal functional equation

$$f(\sqrt[\alpha]{(k+1)x^\alpha+ky^\alpha}) + f(\sqrt[\alpha]{(k+1)x^\alpha-ky^\alpha}) = \frac{2(k+1)f(x)[f(y)]^2}{[(k+1)f(y)]^2-[kf(x)]^2} \quad (1)$$

and ratify generalized Hyers-Ulam-Rassias stability problem in Quasi - β - Banach Space and (β, p)- Banach spaces.

3. General Solution of (1)

In this section, taking inspiration from [6] we provide the general solution of FE (1).

3.1. **Theorem.** Let Y be a linear space. A function $f:\mathbb{R}\to Y$ satisfies the FE (1) if and only if $f(x)=F(x^\alpha)$, $x\in\mathbb{R}$ for some function $F:\mathbb{R}\to Y$ such that

$$F(\sqrt[\alpha]{(k+1)x^\alpha+ky^\alpha}) + F(\sqrt[\alpha]{(k+1)x^\alpha-ky^\alpha}) = \frac{2(k+1)F(x)[F(y)]^2}{[(k+1)F(y)]^2-[kF(x)]^2} \quad (2)$$

Proof: Indeed, it is easy to check that if $f:\mathbb{R}\to Y$ is of type $f(x)=F(x^\alpha)$ and satisfies (2) then, it is a solution to (1). Also, if $f:\mathbb{R}\to Y$ is a solution of (1), then assume $F_0(x)=f(\sqrt[\alpha]{x})$ for $x\in\mathbb{R}$. From (1) we obtain that

$$F_0((k+1)x^\alpha+ky^\alpha)+F_0((k+1)x^\alpha-ky^\alpha)$$

$$
\begin{aligned}
&= f(\sqrt[\alpha]{(k+1)x^\alpha + ky^\alpha}) + f(\sqrt[\alpha]{(k+1)x^\alpha - ky^\alpha}) \\
&= \frac{2(k+1)f(x)[f(y)]^2}{[(k+1)f(y)]^2 - [kf(x)]^2} \\
&= \frac{2(k+1)F_0(x^\alpha)[F_0(y^\alpha)]^2}{[(k+1)F_0(y^\alpha)]^2 - [kF_0\ (x^\alpha)]^2}
\end{aligned}
$$

for all $x, y \in \mathbb{R}$. Hence there is a function $F : \mathbb{R} \to Y$ with $F(x) = F_0(x)$ for all $x \in \mathbb{R}$. This completes the proof.

4. Stability of (1) in Quasi$-\beta-$ Normed Space

In this section, we ratify the generalized Hyers-Ulam-Rassias stability of radical functional equations (1) in quasi $-\beta-$ normed spaces using direct method. Let X be a normed space and $\varsigma : \mathbb{R}^2 \to \mathbb{R}^+ \cup \{0\}$ be a function such that

$$||D_r(x,y)|| \le \varsigma(x,y) \tag{3}$$

where,

$$
\begin{aligned}
D_r(x,y) = {}& f(\sqrt[\alpha]{(k+1)x^\alpha + ky^\alpha}) + f(\sqrt[\alpha]{(k+1)x^\alpha - ky^\alpha}) \\
&- \frac{2(k+1)f(x)[f(y)]^2}{[(k+1)f(y)]^2 - [kf(x)]^2}
\end{aligned} \tag{4}
$$

for all $x, y \in \mathbb{R}$ and function $f : \mathbb{R} \to X$.

4.1. **Theorem.** Let X be a quasi-β-Banach space and $f : \mathbb{R} \to X$ be a α - radical reciprocal function with f(0) = 0. If $\varsigma : \mathbb{R}^2 \to \mathbb{R}^+ \cup \{0\}$ satisfies the following

$$\Psi(x) = \sum_{i=0}^{\infty} (2k+1)^{i\beta} \varsigma(\sqrt[\alpha]{(2k+1)}^i x, \sqrt[\alpha]{(2k+1)}^i x) < \infty \tag{5}$$

and

$$\lim_{n\to\infty} (2k+1)^{n\beta} \varsigma((\sqrt[\alpha]{(2k+1)})^n x, (\sqrt[\alpha]{(2k+1)})^n y) = 0 \tag{6}$$

for all $x, y \in \mathbb{R}$, then there subsists an unique α - radical reciprocal mapping $R_{re} : \mathbb{R} \to X$ satisfying the functional equation (1) and the following inequality:

$$||f(x) - R_{re}(x)|| \le \vartheta(2k+1)^\beta \Psi(x) \tag{7}$$

for all $x \in \mathbb{R}$.

Proof: Replacing (x,y) with (x,x) in (3), we get

$$||f(\sqrt[\alpha]{(2k+1)}x) - \frac{1}{(2k+1)} f(x)|| \le \varsigma(x,x) \tag{8}$$

for all $x \in \mathbb{R}$. Next, by using property (S2) from definition (D5), we can say that

$$||(2k+1)f(\sqrt[\alpha]{(2k+1)}x) - f(x)|| \le (2k+1)^\beta \varsigma(x,x) \tag{9}$$

for all $x \in \mathbb{R}$. Then, by the iterative method, we get

$$
\begin{aligned}
&||(2k+1)^n f((\sqrt[\alpha]{(2k+1)})^n x) - f(x)|| \\
&\le \vartheta(2k+1)^\beta \sum_{i=0}^{n-1} (2k+1)^{i\beta} \varsigma(\sqrt[\alpha]{(2k+1)}^i x, \sqrt[\alpha]{(2k+1)}^i x)
\end{aligned} \tag{10}
$$

for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}^+$. Next we have, for all $n, m \in \mathbb{Z}^+$, with $n > m \geq 0$, we have

$$||(2k+1)^m f((\sqrt[\alpha]{(2k+1)})^m x) - (2k+1)^n f((\sqrt[\alpha]{(2k+1)})^n x)||$$
$$\leq \vartheta(2k+1)^\beta \sum_{i=m}^{n-1} (2k+1)^{i\beta} \varsigma(\sqrt[\alpha]{(2k+1)}^i x, \sqrt[\alpha]{(2k+1)}^i x) \tag{11}$$

for all $x \in \mathbb{R}$. Thus, by (11) we can assert that sequence $\{(2k+1)^n f((\sqrt[\alpha]{(2k+1)})^n x)\}$ is a Cauchy in X and so it converges. Thus, we can expound a mapping as $R_{re}(x) : \mathbb{R} \to ?X$ as

$$R_{re}(x) = \lim_{n\to\infty} (2k+1)^n f((\sqrt[\alpha]{(2k+1)})^n x)$$

for all $x \in \mathbb{R}$. Next, using (3) and (6) take in account that

$$||D_{R_{re}}(x,y)|| = \lim_{n\to\infty} (2k+1)^{\beta n} ||D_r((\sqrt[\alpha]{(2k+1)})^n x, (\sqrt[\alpha]{(2k+1)})^n y)||$$

$$\leq \lim_{n\to\infty} (2k+1)^{\beta n} \varsigma((\sqrt[\alpha]{(2k+1)})^n x, (\sqrt[\alpha]{(2k+1)})^n y) = 0$$

for all $x, y \in \mathbb{R}$ and for that reason mapping R_{re} is α-radical reciprocal mapping. Next, assuming limit as n approaches to infinity in (11) with $m = 0$, we assert that the mapping R_{re} satisifies (7) near the approximate function f of (1).

Uniqueness: To prove uniqueness, assume now that there is R'_{re} as another radical reciprocal mapping satisfying (7) and (1). Also, we can easily say that

$$R_{re}((\sqrt[\alpha]{(2k+1)})^n x) = \frac{1}{(2k+1)^n} R_{re}(x)$$

$$\textit{and} \qquad R'_{re}((\sqrt[\alpha]{(2k+1)})^n x) = \frac{1}{(2k+1)^n} R'_{re}(x)$$

for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Thus,

$$||R_{re}(x) - R'_{re}(x)||$$

$$= ||(2k+1)^n R_{re}((\sqrt[\alpha]{(2k+1)})^n x) - (2k+1)^n R'_{re}((\sqrt[\alpha]{(2k+1)})^n x)||$$

$$\leq \vartheta(2k+1)^{n\beta}(||R_{re}((\sqrt[\alpha]{(2k+1)})^n x) - f((\sqrt[\alpha]{(2k+1)})^n x)||$$

$$+||f((\sqrt[\alpha]{(2k+1)})^n x) - R'_{re}((\sqrt[\alpha]{(2k+1)})^n x)||)$$

$$\leq 2\vartheta^2(2k+1)^{\beta(n+1)} \sum_{i=0}^{\infty} (2k+1)^{i\beta} \varsigma(\sqrt[\alpha]{(2k+1)}^{i+n} x, \sqrt[\alpha]{(2k+1)}^{i+n} x)$$

for all $x \in X$. Therefore, as $n \to \infty$, in the above inequality, we have $R'_{re} = R_{re}$ for all $x \in X$, thus by proving claimed uniqueness of R_{re} proof of the theorem completes.

4.2. **Theorem.** Let X be a quasi-β-Banach space and $f : \mathbb{R} \to X$ be a α - radical reciprocal function with f(0) = 0. If $\varsigma : \mathbb{R}^2 \to \mathbb{R}^+ \cup \{0\}$ satisfies the following

$$\Psi(x) = \sum_{i=0}^{\infty} \Big(\frac{1}{(2k+1)^{\beta}}\Big)^i \varsigma\Big(\frac{x}{\sqrt[\alpha]{(2k+1)}^i}, \frac{x}{\sqrt[\alpha]{(2k+1)}^i}\Big) < \infty \tag{12}$$

and

$$\lim_{n\to\infty} \frac{1}{(2k+1)^n}\varsigma\Big(\frac{x}{(\sqrt[\alpha]{(2k+1)})^n}, \frac{y}{(\sqrt[\alpha]{(2k+1)})^n}\Big) = 0 \tag{13}$$

for all $x, y \in \mathbb{R}$, then there exists a unique α-radical reciprocal mapping $R_{re} : \mathbb{R} \to X$ satisfying the functional equation (1) and the following inequality:

$$||f(x) - R_{re}(x)|| \leq \vartheta(2k+1)^{\beta}\Psi(x) \tag{14}$$

for all $x \in \mathbb{R}$.

Proof: The proof follows from previous theorem by replacing x by $\dfrac{x}{\sqrt[\alpha]{(2k+1)}}$ in (9).

4.3. **Corollary.** Let r and s be any real numbers with $r + s \neq -\alpha\beta$ and $\lambda \geq 0$. Assume that $f : \mathbb{R} \to X$ be a mapping such that

$$||D_r(x, y)|| \leq \lambda|x|^r|y|^s \tag{15}$$

for all $x, y \in \mathbb{R}$, then there exists a unique α-radical reciprocal mapping $R_{re} : \mathbb{R} \to X$ satisfying the functional equation (1) and the following inequality:

$$||f(x) - R_{re}(x)|| \leq \frac{\vartheta\lambda}{(2k+1)^{-\beta} - (2k+1)^{\frac{r+s}{\alpha}}}|x|^{s+r} \tag{16}$$

for all $x \in \mathbb{R}$.

Proof: Proof follows from theorem 4.1 by taking

$$\varsigma(x, y) = \lambda|x|^r|y|^s.$$

4.4. **Corollary.** Let r and s be any real numbers with $r, s \neq -\alpha\beta$ and $\lambda \geq 0$. Assume that $f : \mathbb{R} \to X$ be a mapping such that

$$||D_r(x, y)|| \leq \lambda(|x|^r + |y|^s) \tag{17}$$

for all $x, y \in \mathbb{R}$, then there exists a unique radical reciprocal mapping $R_{re} : \mathbb{R} \to X$ satisfying the functional equation (1) and the following inequality:

$$||f(x) - R_{re}(x)|| \leq \vartheta\lambda\Big(\frac{|x|^r}{(2k+1)^{-\beta} - (2k+1)^{\frac{r}{\alpha}}} + \frac{|x|^s}{(2k+1)^{-\beta} - (2k+1)^{\frac{s}{\alpha}}}\Big) \tag{18}$$

for all $x \in \mathbb{R}$.

Proof: Proof follows from theorem 4.1 by taking

$$\varsigma(x, y) = \lambda(|x|^r + |y|^s).$$

5. Stability of (1) in (β,p)-Banach Space

In this section, we ratify the generalized Hyers-Ulam-Rassias stability of radical functional equations (1) in (β,p)- Banach spaces using direct method.

Before continuing let us recall about sub-additive and sub-quadratic functions. Let $\theta : A \to B$ be a function with a domain A and a codomain $(B, \leq)$ both of which are closed under addition and gratifies the ensuing property:

$$\theta(x+y) \leq \theta(x) + \theta(y)$$

for all $x, y \in A$ is labeled as sub-additive function. Let $\delta \in \{-1, 1\}$ be fixed and there exists a constant K with $0 < K < 1$ such that the function $\theta : A \to B$ satisfies

$$\delta\theta(x+y) \leq \theta K^{\theta}(\theta(x) + \theta(y))$$

for all $x, y \in A$, then we say that θ is contractively sub-additive or expansively super-additive if $\delta = 1$ or $\delta = -1$, respectively. Also it follows that θ satisfies the following properties:

$$\theta(2^{\delta}x) \leq 2^{\delta}K\theta(x),$$
$$\theta(2^{\delta j}x) \leq (2^{\delta}K)^{j}\theta(x)$$

for all $x \in A$ and $j \geq 1$.

Also, a function $\phi : A \to B$ with $\phi(0) = 0$ which gratifies the ensuing property:

$$\phi(x+y) + \phi(x-y) \leq 2\phi(x) + 2\phi(y)$$

for all $x, y \in A$ is labeled as a sub-quadratic function. Again, if there exists a constant K with $0 < K < 1$ such that the function $\phi : A \to B$ with $\phi(0) = 0$ satisfies

$$\delta\phi(x+y) + \delta\phi(x-y) \leq 2\delta K^{\delta}(\phi(x) + \phi(y))$$

for all $x, y \in A$, then we say that ϕ is contractively sub-quadratic or expansively super-quadratic if $\delta = 1$ or $\delta = -1$, respectively. Also, ϕ satisfies the following properties:

$$\phi(2^{\delta}x) \leq 4^{\delta}K\phi(x),$$
$$\phi(2^{\delta j}x) \leq (4^{\delta}K)^{j}\phi(x)$$

for all $x \in A$ and $j \geq 1$.

From now on, we establish the modified Hyers- Ulam-Rassias stability of equation (1) in a (β, p)-Banach space X by using subadditive and subquadratic functions.

5.1. **Theorem.** Let $f : \mathbb{R} \to X$ be a α - radical reciprocal function with the property f(0) = 0 and ψ is contractively sub-additive with a constant L satisfying $(2k+1)^{(1+\alpha\beta)}L < 1$. Then there exists a unique function $F : \mathbb{R} \to X$ which satisfies (1) and the inequality

$$||f(x) - R_{re}(x)|| \leq \frac{(2k+1)^{-\alpha\beta}}{\sqrt[p]{(2k+1)^{-\alpha\beta p} - ((2k+1)L)^{p}}}\Phi(x) \tag{19}$$

for all $x \in \mathbb{R}$.

Proof: Using (11) from theorem 4.1 we get that

$$||f(x) - (2k+1)^{\alpha}f((2k+1)x)||$$
$$\leq (2k+1)^{\beta p}\sum_{i=0}^{\alpha-1}[(2k+1)^{i\beta}\varsigma(\sqrt[\alpha]{(2k+1)}^{i}x, \sqrt[\alpha]{(2k+1)}^{i}x)]^{p} \tag{20}$$

for all $x \in \mathbb{R}$. Let us assign

$$\Phi(x) = (2k+1)^{\beta p} \sum_{i=0}^{\alpha-1} [(2k+1)^{i\beta} \varsigma(\sqrt[\alpha]{(2k+1)}^i x, \sqrt[\alpha]{(2k+1)}^i x)]^p.$$

Then it follows from (20) that

$$\begin{aligned} &||(2k+1)^{\alpha m} f((2k+1)^m x) - (2k+1)^{\alpha n} f((2k+1)^n x)|| \\ &\leq \sum_{i=m}^{n-1} ((2k+1)^{\alpha\beta})^{pi} \Phi((2k+1)^i x)^p \\ &\leq \Phi(x)^p \sum_{i=m}^{n-1} ((2k+1)^{(1+\alpha\beta)} L)^{pi} \end{aligned} \tag{21}$$

for all $x \in \mathbb{R}$ and integers $n > m \geq 0$. Thus, the sequence $\{(2k+1)^{\alpha n} f((2k+1)^n x)\}$ is a Cauchy in (β, p) - Banach Space X^p and so it converges. Thus, we can expound a mapping as $R_{re}(x) : \mathbb{R} \to ?X$ as

$$R_{re}(x) = \lim_{n\to\infty} (2k+1)^{\alpha n} f((2k+1)^n x)$$

for all $x \in \mathbb{R}$. Next, we get that

$$\begin{aligned} ||D_{R_{re}}(x,y)||^p &= \lim_{n\to\infty} (2k+1)^{\alpha\beta np} ||D_r((\sqrt[\alpha]{(2k+1)})^n x, (\sqrt[\alpha]{(2k+1)})^n y)^p|| \\ &\leq \varsigma(x,y)^p \lim_{n\to\infty} ((2k+1)^{(1+\alpha\beta)} L)^{pn} = 0 \end{aligned}$$

for all $x, y \in \mathbb{R}$ and for that reason mapping R_{re} is α-radical reciprocal mapping. Assuming limit as n approaches to infinity in (21) with $m = 0$, we assert that

$$||f(x) - R_{re}(x)|| \leq \frac{(2k+1)^{-\alpha\beta}}{\sqrt[p]{(2k+1)^{-\alpha\beta p} - ((2k+1)L)^p}} \Phi(x)$$

Uniqueness: To prove uniqueness, assume now that there is R'_{re} as another radical reciprocal mapping satisfying (19) and (1). Thus,

$$\begin{aligned} &||R'_{re}(x) - (2k+1)^{\alpha n} f((2k+1)^n x)||^p \\ &= (2k+1)^{\alpha\beta pn} ||R'_{re}((2k+1)^n x) - f((2k+1)^n x)||^p \\ &\leq \frac{\Phi(x)^p}{(2k+1)^{-\alpha\beta p} - ((2k+1)L)^p} ((2k+1)^{(1+\alpha\beta)} L)^{pn} \end{aligned}$$

for all $x \in \mathbb{R}$. Therefore, as $n \to \infty$, in the above inequality, we have $R'_{re} = R_{re}$ for all $x \in \mathbb{R}$, thus by proving claimed uniqueness of R_{re} proof of the theorem completes.

5.2. **Theorem.** Let $f : \mathbb{R} \to X$ be a α - radical reciprocal function with the property f(0) = 0 and ψ is expansively super-additive with a constant L satisfying $(2k+1)^{-(1+\alpha\beta)} L < 1$. Then there exists a unique function $F : \mathbb{R} \to X$ which satisfies (1) and the inequality

$$||f(x) - R_{re}(x)|| \leq \frac{(2k+1)^{-\alpha\beta}}{\sqrt[p]{((2k+1)L^{-1})^p - (2k+1)^{-\alpha\beta p}}} \Phi(x) \tag{22}$$

for all $x \in \mathbb{R}$.

Proof: Using (20) from theorem 5.1 we get that

$$||(2k+1)^{-\alpha(m+1)}f((2k+1)^{-m-1}x)-(2k+1)^{-\alpha m}f((2k+1)^{-m}x)|| \\ \leq (2k+1)^{-\alpha\beta p(m+1)}\phi((2k+1)^{-(m+1)}x) \quad (23)$$

for all $x\in\mathbb{R}$. For integers m,n with $n>m\geq 0$, we can assert that

$$||(2k+1)^{-\alpha m}f((2k+1)^{-m}x)-(2k+1)^{-\alpha n}f((2k+1)^{-n}x)|| \\ \leq \phi(x)^p\sum_{i=m+1}^{n}((2k+1)^{-\alpha\beta-1}L)^{pi} \quad (24)$$

for all $x\in\mathbb{R}$. The remaining part of the proof follows from Theorem 5.1. Hence the proof.

5.3. **Theorem.** Let $f:\mathbb{R}\to X$ be a α - radical reciprocal function with the property f(0) = 0 and ψ is contractively sub-quadratic with a constant L satisfying $(2k+1)^{(2+\alpha\beta)}L<1$. Then there exists a unique function $F:\mathbb{R}\to X$ which satisfies (1) and the inequality

$$||f(x)-R_{re}(x)||\leq\frac{(2k+1)^{-\alpha\beta}}{\sqrt[p]{(2k+1)^{-\alpha\beta p}-((2k+1)^2L)^p}}\Phi(x) \quad (25)$$

for all $x\in\mathbb{R}$.

Proof: Using (21) from theorem 5.1 we get that

$$||(2k+1)^{\alpha m}f((2k+1)^{m}x)-(2k+1)^{\alpha n}f((2k+1)^{n}x)|| \\ \leq\sum_{i=m}^{n-1}((2k+1)^{\alpha\beta})^{pi}\Phi((2k+1)^{i}x)^p \\ \leq\Phi(x)^p\sum_{i=m}^{n-1}((2k+1)^{(2+\alpha\beta)}L)^{pi} \quad (26)$$

for all $x\in\mathbb{R}$ and integers $n>m\geq 0$. Thus, the sequence $\{(2k+1)^{\alpha n}f((2k+1)^n x)\}$ is a Cauchy in (β,p)- Banach Space X^p and so it converges. Thus, we can expound a mapping as $R_{re}(x):\mathbb{R}\to ?X$ as

$$R_{re}(x)=\lim_{n\to\infty}(2k+1)^{\alpha n}f((2k+1)^n x)$$

for all $x\in\mathbb{R}$. Next, we get that

$$||D_{R_{re}}(x,y)||^p=\lim_{n\to\infty}(2k+1)^{\alpha\beta np}||D_r((\sqrt[\alpha]{(2k+1)})^n x,(\sqrt[\alpha]{(2k+1)})^n y)^p|| \\ \leq\varsigma(x,y)^p\lim_{n\to\infty}((2k+1)^{(2+\alpha\beta)}L)^{pn}=0$$

for all $x,y\in\mathbb{R}$ and for that reason mapping R_{re} is α-radical reciprocal mapping. Assuming limit as n approaches to infinity in (26) with *m=0*, we assert that

$$||f(x)-R_{re}(x)||\leq\frac{(2k+1)^{-\alpha\beta}}{\sqrt[p]{(2k+1)^{-\alpha\beta p}-((2k+1)^2L)^p}}\Phi(x)$$

Uniqueness: To prove uniqueness, assume now that there is R'_{re} as another radical reciprocal mapping satisfying (25) and (1). Thus,

$$||R'_{re}(x)-(2k+1)^{\alpha n}f((2k+1)^n x)||^p \\ =(2k+1)^{\alpha\beta pn}||R'_{re}((2k+1)^n x)-f((2k+1)^n x)||^p$$

$$\leq \frac{\Phi(x)^p}{(2k+1)^{-\alpha\beta p} - ((2k+1)L)^p}((2k+1)^{(2+\alpha\beta)}L)^{pn}$$

for all $x \in \mathbb{R}$. Therefore, as $n \to \infty$, in the above inequality, we have $R'_{re} = R_{re}$ for all $x \in \mathbb{R}$, thus by proving claimed uniqueness of R_{re} proof of the theorem completes.

5.4. **Theorem.** Let $f : \mathbb{R} \to X$ be a α - radical reciprocal function with the property f(0) = 0 and ψ is expansively super-quadratic with a constant L satisfying $(2k+1)^{-(2+\alpha\beta)}L < 1$. Then there exists a unique function $F : \mathbb{R} \to X$ which satisfies (1) and the inequality

$$||f(x) - R_{re}(x)|| \leq \frac{(2k+1)^{-\alpha\beta}}{\sqrt[p]{((2k+1)^2L^{-1})^p - (2k+1)^{-\alpha\beta p}}}\Phi(x) \tag{27}$$

for all $x \in \mathbb{R}$.

Proof: The proof can be easily generated using theorems 5.2 and 5.3.

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Shalini Tomar
Assistant Professor, Department of Mathematics, Kanya Mahavidyalaya, Kharkhoda (Sonepat), Haryana, India
E-mail address: s_saroha30@yahoo.com

Navneet Hooda
Professor, Department of Mathematics, DCRUST, Murthal (Sonepat), Haryana, India
E-mail address: nawneethooda@gmail.com