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# ON THE GLOBAL BEHAVIOR OF A HIGHER-ORDER NONAUTONOMOUS RATIONAL DIFFERENCE EQUATION

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Abstract. In this article, we study the global behavior of the following higher-order nonautonomous rational difference equation

$$y_{n+1} = \frac{\alpha_n + y_{n-l}}{\alpha_n + y_{n-k}}, \quad n = 0, 1, ...,$$

where $\{\alpha_n\}_{n\geq 0}$ is a convergent sequence of positive numbers, $k, l$ are nonnegative integers such that $l < k$, and the initial values $y_{-k}, ..., y_0$ are positive real numbers. We give sufficient conditions under which the unique equilibrium $\bar{y} = 1$ is globally asymptotically stable. Furthermore, we establish an oscillation result for positive solutions about the equilibrium point. As an application, we give some examples to illustrate our results.

## 1. Introduction

Difference equations have been studied intensively in the last few decades. Especially, there has been great interest in the study of the dynamics of rational difference equations, and many researchers have investigated the behavior of their solutions, (for example, see [1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18]).

In [11], Kocic and Ladas studied the following high order difference equation

$$y_{n+1} = \frac{a + by_n}{A + y_{n-k}}, \quad n \in \mathbb{N},, \tag{1}$$

$a, b, A$ are nonnegative real numbers and $k$ is a positive integer. They showed that the positive equilibrium point of the Eq. (1) is globally asymptotically stable. In addition, they showed that all positive solutions of Eq. (1) are oscillatory about the positive equilibrium point. These results were extended in [3] and [10] to the following nonautonomous analogues rational difference equation

$$y_{n+1} = \frac{\alpha_n + y_n}{\alpha_n + y_{n-k}}, \quad n = 0, 1, ... \tag{2}$$

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Precisely, Dekkar et al. [3] considered Eq. (2) in the case where $\{\alpha_n\}_{n\geq 0}$ is a periodic sequence of positive numbers with period $T$, while Kerker et al. [10] studied Eq. (2) when $\{\alpha_n\}_{n\geq 0}$ is a convergent sequence.

In this work, we study the global behavior of the more general rational difference equation

$$y_{n+1} = \frac{\alpha_n + y_{n-l}}{\alpha_n + y_{n-k}}, \quad n = 0, 1, ..., \tag{3}$$

where $\{\alpha_n\}_{n\geq 0}$ is a convergent sequence of positive numbers, $k, l$ are nonnegative integers such that $l < k$, and the initial values $y_{-k}, ..., y_0$ are nonnegative real numbers. We give sufficient conditions under which the unique equilibrium $\bar{y} = 1$ is globally asymptotically stable. Furthermore, we show that every positive solution of (3) is oscillatory about the equilibrium point $\bar{y} = 1$.

## 2. Preliminaries

In this preliminary section, we recall some notions and results about the theory of difference equations. For more details we refer readers to [4, 12].

Let $I$ be an interval of real numbers and let $f : \mathbb{N} \times I^{k+1} \longrightarrow I$ be a continuously differentiable function. Consider the difference equation

$$y_{n+1} = f(n, y_n, y_{n-1}, ..., y_{n-k}), \quad n \geq 0, \tag{4}$$

with $y_0, y_{-1}, ..., y_{-k} \in I$.

**Definition 1.** A point $\bar{y} \in I$ such that $\bar{y} = f(n, \bar{y}, \bar{y}, ..., \bar{y})$ for all $n \geq 0$, is called an *equilibrium point* of Eq. (4).

**Definition 2.** An equilibrium point $\bar{y}$ of (4) is said to be

1. *Stable* if, for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon)$ such that if $y_0, y_{-1}, ..., y_{-k} \in (\bar{y} - \delta, \bar{y} + \delta) \subset I$ then $|y_n - \bar{y}| < \varepsilon$, for all $n \geq -k$. Otherwise, the equilibrium $\bar{y}$ is called *unstable*.
2. *Attractive* if there exists $\mu > 0$ such that if $y_0, y_{-1}, ..., y_{-k} \in (\bar{y} - \mu, \bar{y} + \mu) \subset I$ then
$$\lim_{n\to\infty} y_n = \bar{y}.$$
If $\mu = \infty$, $\bar{y}$ is called *globally attractive*.
3. *Asymptotically stable* if it is stable and attractive.
4. *Globally asymptotically stable* if it is stable and globally attractive.

**Definition 3.** A solution $\{y_n\}_{n\geq -k}$ of Eq. (4) is called *nonoscillatory* if there exists $p \geq -k$ such that either

$$y_n > \bar{y}, \quad \forall n \geq p \qquad \text{or} \qquad y_n < \bar{y}, \quad \forall n \geq p,$$

and it is called *oscillatory* if it is not nonoscillatory.

Finally, we state the comparison principle for nonautonomous difference equations.

**Lemma 1.** *Let $z \geq 0$ be a real number, $g(n, z)$ be a nondecreasing function with respect to $z$ for any fixed natural number $n \geq n_0$, $n_0 \in \mathbb{N}$. Suppose that for $n \geq n_0$, we have*

$$x_{n+1} \leq g(n, x_n),$$
$$y_{n+1} \geq g(n, y_n).$$

*Then,*

$$x_{n_0} \le y_{n_0}$$

*implies that*

$$x_n \le y_n, \qquad \forall n \ge n_0.$$

## 3. Global asymptotic stability

In this section, we investigate the global asymptotic stability of the equilibrium point. Throughout this paper, we use the following notations

$$\alpha = \lim \alpha_n, \qquad a = \inf\{\alpha_n\} \qquad \text{and} \qquad A = \sup\{\alpha_n\}.$$

We have the following lemma.

**Lemma 2.** *Assume that*

$$a > 1. \tag{5}$$

*Then, every positive solution is bounded.*

*Proof.* We have

$$y_{n+1} = \frac{\alpha_n + y_{n-l}}{\alpha_n + y_{n-k}} \le \frac{A}{a} + \frac{1}{a} y_{n-l},$$

which gives (see [4, p. 77])

$$y_n \le \frac{A}{a-1} + a^{-\frac{n}{l+1}} \sum_{i=0}^{l} c_i n^i \underset{n\to\infty}{\to} \frac{A}{a-1}.$$

Then, by Lemma 1, there exists $M > 0$, such that

$$y_n \le M, \qquad \forall n \ge k.$$

□

We begin with the following local stability result.

**Theorem 1.** *Assume that (5) holds. Then, $\bar{y} = 1$ is stable.*

*Proof.* Let $\varepsilon > \frac{A}{a-1}$ such that

$$y_{-k}, ..., y_0 \in [0, \varepsilon[.$$

Hence,

$$\begin{aligned} y_{n+1} = \frac{\alpha_n + y_{n-l}}{\alpha_n + y_{n-k}} &\le \frac{A + y_{n-l}}{a + y_{n-k}} \le \frac{A + y_{n-l}}{a} \\ &< \frac{A + \varepsilon}{a} < \varepsilon, \qquad \forall n \ge 0. \end{aligned}$$

This implies that $y_n \in [0, \varepsilon[$, $\forall n \ge -k$. □

In the next theorem, we show the global attractivity of the equilibrium point.

**Theorem 2.** *Assume that (5) holds. Then, $\bar{y} = 1$ is globally attractive.*

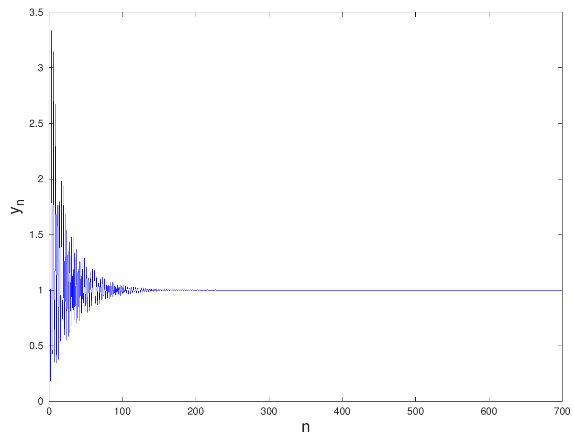


FIGURE 1. Plot of the solution $\{y_n\}_{n\geq 0}$ of Eq. (9) for the initial values $y_{-4} = 10$, $y_{-3} = 0.9$, $y_{-2} = 6.5$, $y_{-1} = 1.9$ and $y_0 = 0.1$.

*Proof.* Let $\{y_n\}_{n\geq -k}$ be an arbitrary positive solution of Eq. (3). Set

$$I = \liminf_{n\to\infty} y_n \qquad \text{and} \qquad S = \limsup_{n\to\infty} y_n,$$

which by Theorem 2 exist. Then, Eq. (3) yields

$$S \leq \frac{\alpha + S}{\alpha + I} \qquad \text{and} \qquad I \geq \frac{\alpha + I}{\alpha + S}.$$

Hence,

$$\alpha + I(1-\alpha) \leq IS \leq \alpha + S(1-\alpha)$$

and so

$$I \geq S.$$

Thus, the sequence $\{y_n\}_{n\geq -k}$ is convergent to a limit $l$. By taking limits on both sides of Eq. (3) we find that $l = 1$. □

From Theorems 1 and 2 we obtain the following result.

**Theorem 3.** *Assume that $a > 1$. Then, the unique equilibrium point of Eq. (3) is globally asymptotically stable.*

Here are some illustrative examples:

**Example 1.** We consider the following third order difference equation

$$y_{n+1} = \frac{(n+2)/(n+1) + y_{n-3}}{(n+2)/(n+1) + y_{n-4}} \tag{6}$$

with the initial values $y_{-4} = 10$, $y_{-3} = 0.9$, $y_{-2} = 6.5$, $y_{-1} = 1.9$ and $y_0 = 0.1$. From Theorem 2, the equilibrium point of equation (6) is globally asymptotically stable, see Fig. 1.

**Example 2.** We consider the following rational difference equation

$$y_{n+1} = \frac{(5n+7)/(3n+2) + y_{n-4}}{(5n+7)/(3n+2) + y_{n-11}}, \tag{7}$$

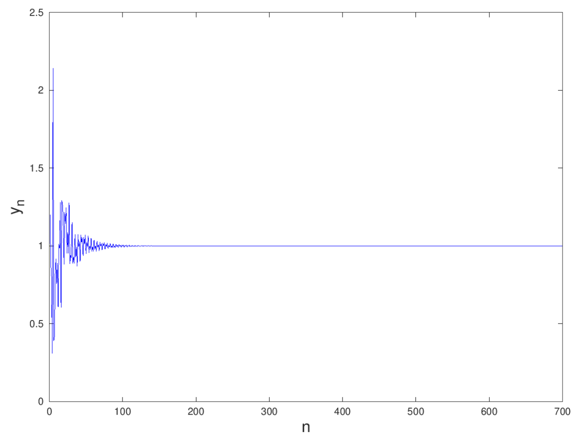


FIGURE 2. Plot of the solution $\{y_n\}_{n\geq 0}$ of Eq. (10) for the initial values $y_{-11} = 3.3$, $y_{-10} = 2$, $y_{-9} = 7$, $y_{-8} = 0.1$, $y_{-7} = 5.3$, $y_{-6} = 4.5$, $y_{-5} = 0.8$, $y_{-4} = 2.5$, $y_{-3} = 1.1$, $y_{-2} = 0.7$, $y_{-1} = 2.5$ and $y_0 = 1.2$.

with the initial values $y_{-11} = 3.3$, $y_{-10} = 2$, $y_{-9} = 7$, $y_{-8} = 0.1$, $y_{-7} = 5.3$, $y_{-6} = 4.5$, $y_{-5} = 0.8$, $y_{-4} = 2.5$, $y_{-3} = 1.1$, $y_{-2} = 0.7$, $y_{-1} = 2.5$ and $y_0 = 1.2$. By Theorem 2, $\bar{y} = 1$ is globally asymptotically stable, see Fig. 2.

## 4. Oscillation

In this section, we study the oscillatory behavior of positive solutions of Eq. (3).

**Theorem 4.** *Every nonoscillatory positive solution of (3) tends to $\bar{y} = 1$ as $n \to \infty$.*

*Proof.* Assume that

$$y_n > 1, \quad \text{for all } n \geq n_0 \quad \text{for} \quad n_0 \geq -k.$$

The case where $y_n < 1$ is similar and will be omitted. So, for $n \geq n_0 + k$, we have

$$y_{n+1} = y_{n-l}\frac{(\alpha_n / y_{n-l} + 1)}{\alpha_n + y_{n-k}} < y_{n-l}\frac{\alpha_n + 1}{\alpha_n + y_{n-k}} < y_{n-l}. \tag{8}$$

Next, setting

$$z_n^i = y_{n_0+i+n(l+1)}, \qquad \text{for all,} \quad n \geq 0 \text{ and } i = 0, ..., l,$$

it follows from (8) that

$$z_{n+1}^i < z_n^i, \quad \text{for all } n \geq 0 \text{ and } i = 0, ..., l,$$

and so the subsequences $\left(z_n^i\right)$ are convergent to a limit, say $l_i$. Therefore, for $i = 0, ..., l$, we have

$$z_{n+1}^i = \frac{\alpha_{n_0+i+n(l+1)} + z_{n-l}^i}{\alpha_{n_0+i+n(l+1)} + z_{n-k}^i}.$$

By taking limits on both sides, as $n \to \infty$, we get

$$l_i = \frac{\alpha + l_i}{\alpha + l_i} = 1.$$

Thus, we have
$$l = 1.$$
□

Now, we state and prove the main result of this section.

**Theorem 5.** *Every positive solution of (3) oscillates about $\bar{y} = 1$.*

*Proof.* Assume that Eq. (3) has a nonoscillatory solution. Then, there exists $n_0 \geq -k$ such that
$$y_n > 1, \quad \text{for all } n \geq n_0$$
or
$$y_n < 1, \quad \text{for all } n \geq n_0.$$
Suppose that $y_n > 1, \quad \forall\, n \geq n_0$. Let $p$ such that
$$y_{n_0+k+p} = \max \{y_{n_0+k+i},\ i = 0, ..., l\}.$$
Therefore, there exists $m \in \mathbb{N}$ and $j \in \{0, ..., l\}$, such that
$$k - l + p = j \bmod (l + 1).$$
Hence, we get
$$y_{n_0+2k+p+1} = \frac{\alpha_{n_0+2k+p} + y_{n_0+k+k-l+p}}{\alpha_{n_0+2k+p} + y_{n_0+k+p}}$$
$$= \frac{\alpha_{n_0+2k+p} + y_{n_0+k+m(l+p)+j}}{\alpha_{n_0+2k+p} + y_{n_0+k+p}}.$$
By (8), we have two situations to contemplate:

*Case 1:* If $m = 0$, we have
$$y_{n_0+2k+p+1} = \frac{\alpha_{n_0+2k+p} + y_{n_0+k+j}}{\alpha_{n_0+2k+p} + y_{n_0+k+p}}$$
$$\leq \frac{\alpha_{n_0+2k+p} + y_{n_0+k+p}}{\alpha_{n_0+2k+p} + y_{n_0+k+p}} = 1.$$
*Case 2:* If $m \geq 1$, we have
$$y_{n_0+2k+p+1} < \frac{\alpha_{n_0+2k+p} + y_{n_0+k+j}}{\alpha_{n_0+2k+p} + y_{n_0+k+p}}$$
$$\leq \frac{\alpha_{n_0+2k+p} + y_{n_0+k+p}}{\alpha_{n_0+2k+p} + y_{n_0+k+p}} = 1.$$
Thus, in both cases we have a contradiction, and the proof is complete. □

**Remark 1.** By virtue of Theorem 2 and Theorem 5, when $a > 1$, all positive solutions of Eq. (3) tend to the equilibrium point while oscillating.

To confirm our result on the oscillatory behavior of the positive solutions of Eq. (3), we consider the two following numerical examples.

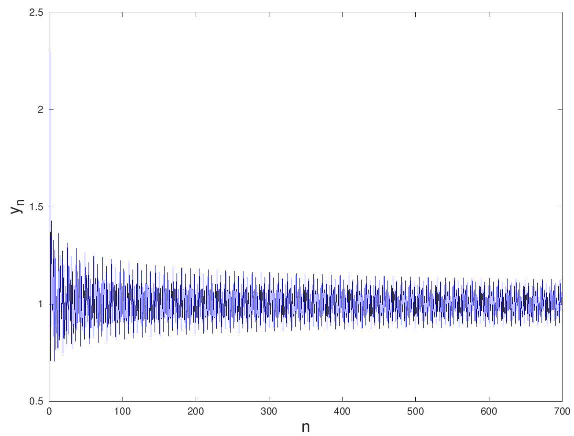


FIGURE 3. Plot of the solution $\{y_n\}_{n\geq 0}$ of Eq. (9) for the initial values $y_{-4} = 1.4$, $y_{-3} = 0.9$, $y_{-2} = 0.7$, $y_{-1} = 1.5$ and $y_0 = 2.3$.

with the initial values $y_{-3} = 7$, $y_{-2} = 1.2$, $y_{-1} = 0.3$ and $y_0 = 2.1$. The solution of Eq. (9) is oscillatory about $\bar{y} = 1$, see Fig. 3.

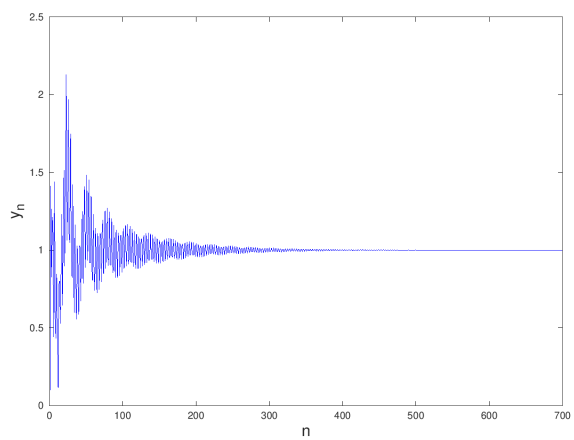


FIGURE 4. Plot of the solution $\{y_n\}_{n\geq 0}$ of Eq. (10) for the initial values $y_{-11} = 1.02$, $y_{-10} = 3.5$, $y_{-9} = 10.1$, $y_{-8} = 1.2$, $y_{-7} = 3.3$, $y_{-6} = 0.5$, $y_{-5} = 2.8$, $y_{-4} = 2.5$, $y_{-3} = 1.9$, $y_{-2} = 2.7$, $y_{-1} = 12.5$ and $y_0 = 0.1$.

**Example 3.** We consider the following fifth order difference equation

$$y_{n+1} = \frac{1/(n+1) + y_{n-2}}{1/(n+1) + y_{n-4}}, \tag{9}$$

with the initial values $y_{-4} = 1.4$, $y_{-3} = 0.9$, $y_{-2} = 0.7$, $y_{-1} = 1.5$ and $y_0 = 2.3$. The solution of Eq. (9) is oscillatory about the equilibrium point $\bar{y} = 1$, see Fig. 3

**Example 4.** We consider the following rational difference equation

$$y_{n+1} = \frac{(n+10)/(8n+9) + y_{n-3}}{(9n+10)/(8n+9) + y_{n-11}}, \tag{10}$$

with the initial values $y_{-11} = 1.02$, $y_{-10} = 3.5$, $y_{-9} = 10.1$, $y_{-8} = 1.2$, $y_{-7} = 3.3$, $y_{-6} = 0.5$, $y_{-5} = 2.8$, $y_{-4} = 2.5$, $y_{-3} = 1.9$, $y_{-2} = 2.7$, $y_{-1} = 12.5$ and $y_0 = 0.1$. Since $a = 10/9$, by Theorem 2 and Theorem 5 the solution of Eq. (10) tends to the equilibrium point by oscillating about it, see Fig. 4.

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