MATHEMATICAL MODEL WITH VARIABLE POPULATION SIZE

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ABSTRACT. In this paper, we have discussed the stability analysis of the model when the vaccination is considered in the system. Here we calculate the basic reproduction number and discuss the stability analysis. First, the disease - free equilibrium and second, the endemic equilibrium. The disease - free equilibrium exists and stable when the basic reproduction number is less than or equal to unity and hence the disease disappear from the system and the endemic equilibrium exists and stable when basic reproduction number is greater than unity and hence the disease exists permanently in the system.

1. INTRODUCTION

In Mathematical models, the total population size remains constant if the disease spreads very quickly [1]. If the disease has modeled for many years, the births are approximately balanced by the natural deaths. But if the deaths due to the disease and the births are not balanced by the deaths, then the total population size must be non – constant or the total population size must vary. Here, we consider the model, the SIR type model with vaccination. In variable population size models, the epidemiological and demographic process interacts to be the different behaviors which do not occur in that type of models with constant population size mathematical model. In the world, the new infectious diseases are emerging and some diseases are eliminated. Diseases are reemerging because the infectious agents evolved and adapted the environment. Also, these diseases affect the lives and also lead to mortality. Various models in which disease – related deaths and disease reduced reproduction can affect the population size were formulated in 1981 by Anderson and May [2, 3, 4] and also Greenhalgh [5, 6, 7, 8]. The mathematical modeling is a tool to test the different strategies that might be helpful to control and eliminate the disease. The mathematical models help us to know the transmission dynamics of the disease. Here mathematical model is formulated and the corresponding basic reproduction number is calculated, which determine whether the disease die out from the system or not. The stability of equilibrium points are investigated in the model. We use the stability analysis by Routh – Hurwitz

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stability criterion [9].

Presently, the world is suffering from corona virus (COVID - 19), caused by Severe Acute Respiratory Syndrome Corona virus – 2 (SARS – CoV - 2), emerged out of Wuhan city of China at the end of 2019. The pandemic, which has rapidly spread over 210 countries, continues to inflict severe public health and social – economic burden in many parts of the world. It has as of June 26, 2020, accounted for over 9,473,214 confirmed cases and about 484,249 deaths globally, reported by WHO. There is currently no safe and effective vaccine or antiviral for use against the pandemic in humans. Consequently, the control and mitigation efforts against the pandemic are focused on implementing non – pharmaceutical interventions, such as social distancing, community lock down, contact tracing, quarantine of suspected cases, isolation of confirmed cases and the use of face mask in public [10, 11]. Social distancing maintaining a physical distance of 2 meters (or, 6 feet) from other humans in public gathering, community lock down entails implementing the stay – at – home or so that people stay at, and work from home, the closure of schools and non – essential business and services, avoiding large public or private gatherings etc. COVID – 19 pandemic is a global concern, due to its high spreading and alarming fatality rate. Mathematical models can play a decisive role in mitigating the spread and predicting the growth of the epidemic. Mathematical modeling performs a vital part in estimating and controlling the recent outbreak of corona virus disease 2019 (COVID - 19). In this epidemic, most countries impose severe intervention measures to contain the spread of COVID – 19. The policymakers are forced to make difficult decisions to leverage between health and economic development. How and when to make clinical and public health decisions in an epidemic situation in a challenging question? The most appropriate solution is based on scientific evidence, which is mainly dependent on data and models. So one of the most critical problems during this crisis is whether we can develop reliable epidemiological models to forecast the evolution of the virus and estimate the effectiveness of various intervention measures and their impacts on the economy. There are numerous types of mathematical model for epidemiological diseases.

There are some mathematical models in the literature that type to describe the dynamics of the evolution of COVID – 19. The phenomenological models are presented in [12], which were validated with outbreaks of other diseases different from COVID – 19, trying to generate and assess short – term forecasts of the cumulative number of reported cases. Other work [13] proposes SEIR type models with little variations and some of them incorporate stochastic components. COVID – 19 is a disease caused by a new virus, which is generating worldwide emergency situation and needs a model taking into account its known specific characteristics.

2. Formulation of the model

A population is divided into four compartments –

S (t) = Number of susceptible at any time t,

T (t) = number of individuals who get vaccine at any time t,

I (t) = number of individuals which is infected after any time t, and

R (t) = number of individual who recovered from the infected compartment and get the immunity from the disease.

Also, the notation of parameters is -

β represents the rate of contact at which the individual leaves the compartment,

γ represents the rate of recovery from the disease,
α represents the rate at which the susceptible individual gets vaccine,
μ represents the birth rate at which the individuals enters to the susceptible compartment,
β_1 represents the transmission rate for the individuals getting vaccine.
Also the individual gets vaccine may be infected before gaining immunity.
γ_1 represents the rate for the individuals getting vaccinated to obtain immunity and recovered from the disease.
Also, μ_S represents the natural death rate of the susceptible individuals,
μ_T represents the natural death rate of the vaccinated individuals
μ_R represents the natural death rate of the recovered individuals, and
μ_I represents the total death rate of the infected individuals.
Here we assume that β_1 may be assuming to be less than β because the vaccinating individuals may have some partial immunity during the process or they may recognize the transmission characters of the disease.

3. The Mathematical Model diagram

The diagram given below shows the population transfer from one compartment to another compartment with variable population size -
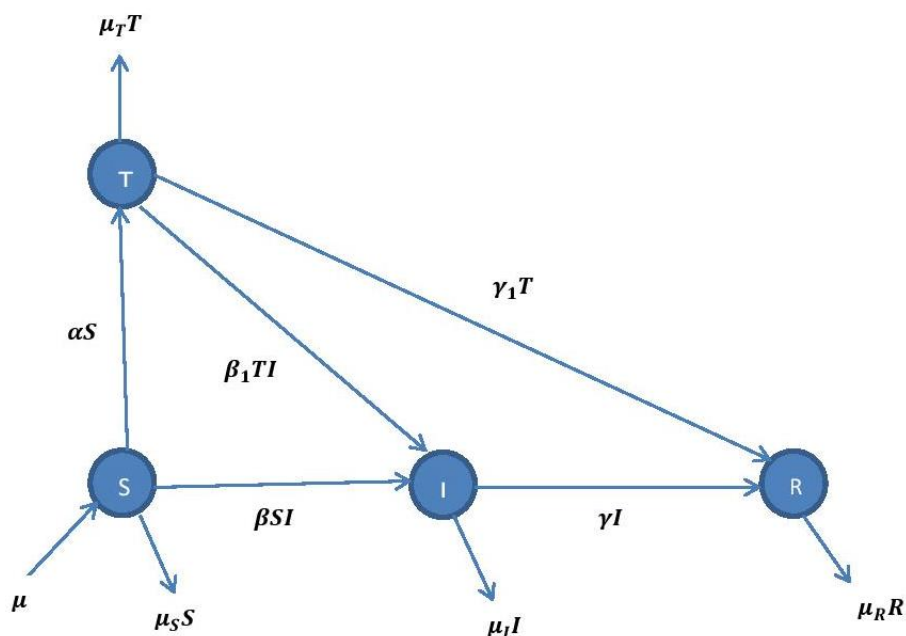

Figure 1. Mathematical Model diagram

4. Model Equations

By taking the above conditions, we get, the system of differential equations of the model is given by –

$\frac{dS}{dt} = \mu - \mu_S S - \beta SI - \alpha S$

$\frac{dT}{dt} = \alpha S - \beta_1 TI - \gamma_1 T - \mu_T T ....(1)$

$\frac{dI}{dt} = \beta SI + \beta_1 TI - \gamma I - \mu_I I$

$\frac{dR}{dT} = \gamma_1 T + \gamma I - \mu_R R$

Where sum of S, T, I and R are equal to unity and also S(0), T(0), I(0) all are positive and R(0) = 0 and all the other given parameters are positive.

We use the following theorems to established the existence and uniqueness of solution for the model -

5. Existence and Uniqueness of solution for the above model

The general first - order ODE is in the form:

$\frac{dx}{dt} = f(t,x), x(t_0) = x_0 ....(1')$

Then the question is –
(i) under what conditions the solution of equation $(1')$ *exists*?
(ii) *Under what condition there is a unique solution to the equation* $(1')$?

Solution -

Theorem - 1 (Uniqueness of solution)

Let D denotes the domain:

$$|t - t_0| \le a, ||x - x_0|| \le b ....(2')$$

where $x = (x_1, x_2, ..., x_n), x_0 = (x_{10}, x_{20}, ..., x_{n0})$
and suppose that $f(t, x)$ satisfies the Lipschitz conditions:
$||f(t, x_1) - f(t, x_2)|| \le k ||x_1 - x_2|| ....(3')$

and whatever the pair (t, x_1) *and* (t, x_2) *belongs to the domain D,*
where k is used to represent a positive constant.
Then, there exist a constant $\delta > 0$ *such that there exists a unique continuous vector*
solution $x(t)$ *of the system* $(1')$ *in the interval* $|t - t_0| \le \delta$. It is important to note
that condition $(3')$ *is satisfied that* $\frac{\partial f_i}{\partial x_j}, i, j = 1, 2, 3, ..., n$ *be continuous and bounded in the domain D.*

Lemma - If f(t, x) has continuous partial derivative $\frac{\partial f_i}{\partial x_j}$, i, j = 1, 2, 3,...,n
on a bounded closed convex domain R (i.e. convex set of real number), where
R denotes the real numbers, then it satisfies a Lipschitz condition in R. Let 1
$\le \varepsilon \le R ....(4')$,
so we do for a bounded solution of the form $0 < R < \infty$.

We now prove the following existence theorem -
Theorem - (2) (Existence of solution) -

Let D denote the domain defined in $(2^{'})$ *such that* $(3^{'})$ *and* $(4^{'})$ *holds. Then there exist a solution of model system of equations* (1) *which is bounded in the domain D.*
Proof -

Let, $f_1 = \mu - \mu_S S - \beta S I - \alpha S.....(e)$
$f_2 = \alpha S - \beta_1 T I - \gamma_1 T - \mu_T T..........(f)$
$f_3 = \beta S I + \beta_1 T I - \gamma I - \mu_I I...........(g)$
$f_4 = \gamma_1 T + \gamma I - \mu_R R......................(h)$

We shows that $\frac{\partial f_i}{\partial x_j}$, i, j = 1, 2, 3,4 are continuous and bounded, i.e. the partial derivatives are continuous and bounded.
From equation (e), we get,

$\frac{\partial f_1}{\partial S} = (-\mu_S - \beta I - \alpha)$,and $\left|\frac{\partial f_1}{\partial S}\right| = |(-\mu_S - \beta I - \alpha)| < \infty$

$\frac{\partial f_1}{\partial T} = 0$, and $\left|\frac{\partial f_1}{\partial T}\right| = |0| < \infty$

$\frac{\partial f_1}{\partial I} = -\beta S$, and $\left|\frac{\partial f_1}{\partial I}\right| = |-\beta S| < \infty$

$\frac{\partial f_1}{\partial R} = 0$, and $\left|\frac{\partial f_1}{\partial R}\right| = |0| < \infty$

Similarly, From equation (f), we get,
$\frac{\partial f_2}{\partial S} = \alpha$, and $\left|\frac{\partial f_2}{\partial S}\right| = |\alpha| < \infty$

$\frac{\partial f_2}{\partial T} = (-\beta_1 I - \gamma_1 - \mu_T)$, and $\left|\frac{\partial f_2}{\partial T}\right| = |(-\beta_1 I - \gamma_1 - \mu_T)| < \infty$

$\frac{\partial f_2}{\partial I} = -\beta_1 T$, and $\left|\frac{\partial f_2}{\partial I}\right| = |-\beta_1 T| < \infty$

$\frac{\partial f_2}{\partial R} = 0$, and $\left|\frac{\partial f_2}{\partial R}\right| = |0| < \infty$

Similarly, From equation (g), we get,
$\frac{\partial f_3}{\partial S} = \beta I$, and $\left|\frac{\partial f_3}{\partial S}\right| = |\beta I| < \infty$

$\frac{\partial f_3}{\partial T} = \beta_1 I$, and $\left|\frac{\partial f_3}{\partial T}\right| = |\beta_1 I| < \infty$

$\frac{\partial f_3}{\partial I} = \beta S + \beta_1 T - \gamma - \mu_I$, and $\left|\frac{\partial f_3}{\partial I}\right| = |\beta S + \beta_1 T - \gamma - \mu_I| < \infty$

$\frac{\partial f_3}{\partial R} = \alpha$, and $\left|\frac{\partial f_3}{\partial R}\right| = |\alpha| < \infty$

And finally, From equation (h), we get,
$\frac{\partial f_4}{\partial S} = 0$, and $\left|\frac{\partial f_4}{\partial S}\right| = |0| < \infty$

$\frac{\partial f_4}{\partial T} = \gamma_1$, and $\left|\frac{\partial f_4}{\partial T}\right| = |\gamma_1| < \infty$

$\frac{\partial f_4}{\partial I} = \gamma$, and $\left|\frac{\partial f_4}{\partial I}\right| = |\gamma| < \infty$

$\frac{\partial f_4}{\partial R} = -\mu_R$, and $\left|\frac{\partial f_4}{\partial R}\right| = |-\mu_R| < \infty$

Here, we have established that all these partial derivatives are continuous and bounded, hence, by Theorem - 1, we can say that there exist a unique solution of system of equation (1) in the region D.

6. Computation of the Equilibrium points and Basic Reproduction Number

There are two types of equilibrium points of epidemic mathematical models, that is, first is the disease – free equilibrium point, i.e. the point at which the disease does not spread in an area because the infected population is equal to zero i.e. I = 0 for t $\rightarrow \infty$, *while the endemic equilibrium point is the point at which the disease must spread when I >* 0 *for* t $\rightarrow \infty$.
The equilibrium point of the system in system of equation of (1), exists when,

$\frac{dS}{dt} = 0$, $\frac{dT}{dt} = 0$. $\frac{dI}{dt} = 0$, $\frac{dR}{dt} = 0$. Then

$\mu - \mu_S S - \beta SI - \alpha S = 0$
$\alpha S - \beta_1 TI - \gamma_1 T - \mu_T T = 0$............(2)
$\beta SI + \beta_1 TI - \gamma I - \mu_I I = 0$
$\gamma_1 T + \gamma I - \mu_R R = 0$

Now, from third equation of system of equations (2), we get,
$(\beta S + \beta_1 T - \gamma - \mu_I)I = 0$
$\Rightarrow I = 0$ *or,* $(\beta S + \beta_1 T - \gamma - \mu_I) = 0$
Then there are two cases arises -

7. Case - I (For the Disease – free Equilibrium point)

If I = 0, then it is the necessary condition for the disease free – equilibrium point.
Then from the first equation of the system of equations (2), we get,

$\mu - \mu_S S - \alpha S = 0$
Which gives $S = \frac{\mu}{\alpha + \mu_S}$

Again from the second equation of the system of equations (2), we get,

$\alpha S - \gamma_1 T - \mu_T T = 0$

Which gives $T = \frac{\alpha S}{\gamma_1 + \mu_T}$
Putting the value of S, we get,
$T = \frac{\alpha \mu}{(\gamma_1 + \mu_T)(\alpha + \mu_S)}$

Again from fourth equation of the system of equation (2), we get,
$\gamma_1 T - \mu_R R = 0$

Which gives $R = \frac{\gamma_1 T}{\mu_R}$

Putting the value of T, we get,

$R = \frac{\gamma_1 \alpha \mu}{\mu_R(\gamma_1+\mu_T)(\alpha+\mu_S)}$

Thus, we get the disease – free equilibrium point as –

$E_1 = (S_1, T_1, I_1, R_1) = (\frac{\mu}{\alpha+\mu_S}, \frac{\alpha\mu}{(\gamma_1+\mu_T)(\alpha+\mu_S)}, 0, \frac{\gamma_1\alpha\mu}{\mu_R(\gamma_1+\mu_T)(\alpha+\mu_S)})$

8. CASE - II (FOR THE ENDEMIC EQUILIBRIUM POINT)

If $I > 0$, then it is the necessary condition for the endemic equilibrium point, $E_2 = (S_2, T_2, I_2, R_2)$. Then, from the first equation of the system of equations (2), we get,

$$\mu - \mu_S S - \beta SI - \alpha S = 0$$

Which gives $S_2 = \frac{\mu}{(\alpha+\mu_S+\beta I_2)}$

Now form second equation of the system of equation (2), we get,

$$\alpha S - \beta_1 TI - \gamma_1 T - \mu_T T = 0$$

Which gives $T_2 = \frac{\alpha S}{(\gamma_1+\mu_T+\beta_1 I_2)}$

Putting the value of S_2 here, we get,

$T_2 = \frac{\alpha\mu}{(\gamma_1+\mu_T+\beta_1 I_2)(\alpha+\mu_S+\beta I_2)}$

Also from the fourth equation of the system of equations (2), we get,

$\gamma_1 T + \gamma I - \mu_R R = 0$

Which gives $R_2 = \frac{\gamma_1 T_2+\gamma I_2}{\mu_R}$

Now, from the third equation of the system of equations (2), we get,

If $I \neq 0$, then for $(\beta S + \beta_1 T - \gamma - \mu_I)I = 0$

We have, $(\beta S + \beta_1 T - \gamma - \mu_I) = 0$

That is, $(\beta S_2 + \beta_1 T_2 - \gamma - \mu_I) = 0$

Putting the values of S_2 and T_2, we get,

$$\frac{\mu\beta}{(\alpha+\mu_S+\beta I_2)} + \frac{\beta_1\alpha\mu}{(\gamma_1+\mu_T+\beta_1 I_2)(\alpha+\mu_S+\beta I_2)} - (\gamma+\mu_I) = 0$$

$$\Rightarrow (\gamma+\mu_I) - \frac{\mu\beta}{(\alpha+\mu_S+\beta I_2)} - \frac{\beta_1\alpha\mu}{(\gamma_1+\mu_T+\beta_1 I_2)(\alpha+\mu_S+\beta I_2)}$$

Let $k = (\gamma_1+\mu_T+\beta_1 I_2)(\alpha+\mu_S+\beta I_2)$

Then the above equation becomes

$(\gamma+\mu_I)k - \frac{\mu\beta k}{(\alpha+\mu_S+\beta I_2)} - \beta_1\alpha\mu = 0$

Putting the value of k in second term, we get,
$(\gamma + \mu_I)k - \mu\beta(\gamma_1 + \mu_T + \beta_1 I_2) - \beta_1 \alpha\mu = 0$

Since, $k = (\gamma_1 + \mu_T + \beta_1 I_2)(\alpha + \mu_S + \beta I_2)$
$= \alpha\gamma_1 + \alpha\mu_T + \alpha\beta_1 I_2 + \mu_S\gamma_1 + \mu_S\mu_T + \mu_S\beta_1 I_2 + \beta I_2\gamma_1 + \beta I_2\mu_T + \beta\beta_1 I_2^2$
$= \beta\beta_1 I_2^2 + [(\alpha + \mu_S)\beta_1 + (\gamma_1 + \mu_T)\beta]I_2 + (\alpha\gamma_1 + \alpha\mu_T + \mu_S\gamma_1 + \mu_S\mu_T)$
$= \beta\beta_1 I_2^2 + [(\alpha + \mu_S)\beta_1 + (\gamma_1 + \mu_T)\beta]I_2 + (\alpha + \mu_S)(\gamma_1 + \mu_T)$

Putting the value of k in the above equation
$(\gamma + \mu_I)k - \mu\beta(\gamma_1 + \mu_T + \beta_1 I_2) - \beta_1\alpha\mu = 0$

we get,
$(\gamma + \mu_I)[\beta\beta_1 I_2^2 + [(\alpha + \mu_S)\beta_1 + (\gamma_1 + \mu_T)\beta]I_2 + (\alpha + \mu_S)(\gamma_1 + \mu_T)] - \mu\beta(\gamma_1 + \mu_T + \beta_1 I_2) - \beta_1\alpha\mu = 0$
i.e. $(\gamma + \mu_I)\beta\beta_1 I_2^2 + [(\gamma + \mu_I)\{(\alpha + \mu_S)\beta_1 + (\gamma_1 + \mu_T)\beta\}I_2 - \mu\beta\beta_1 I_2] + (\gamma + \mu_I)(\alpha + \mu_S)(\gamma_1 + \mu_T) - \mu\beta(\gamma_1 + \mu_T) - \beta_1\alpha\mu = 0$

So, we can write the above equation as -
$a_1 I_2^2 + a_2 I_2 + a_3[1 - (\frac{\mu\beta}{(\alpha+\mu_S)(\gamma+\mu_I)} + \frac{\beta_1\alpha\mu}{(\alpha+\mu_S)(\gamma+\mu_I)(\gamma_1+\mu_T)})] = 0$

Where,
$a_1 = (\gamma + \mu_I)\beta\beta_1$, which is positive.
$a_2 = (\gamma + \mu_I)\{(\alpha + \mu_S)\beta_1 + (\gamma_1 + \mu_T)\beta\} - \mu\beta\beta_1$
$a_3 = (\gamma + \mu_I)(\alpha + \mu_S)(\gamma_1 + \mu_T)$, which is positive.

If we suppose, $C = (\frac{\mu\beta}{(\alpha+\mu_S)(\gamma+\mu_I)} + \frac{\beta_1\alpha\mu}{(\alpha+\mu_S)(\gamma+\mu_I)(\gamma_1+\mu_T)})$

So, the above equation becomes,
$a_1 I_2^2 + a_2 I_2 + a_3(1 - C) = 0 ....(a)$

Then if we solve this equation (a), we get,
$I_2 = \frac{-a_2 \pm \sqrt{a_2^2 - 4a_1 a_3(1-C)}}{2a_1}$

So, the proportion of the infected population must be positive. Hence, the existence of the endemic equilibrium point of infected population depends on the value of C. So, from equation (a), we have, positive root for $I_2 > 0 \; if \; C > 0$.

The value of a positive endemic equilibrium point lies in the value of C, so that it can be defined the value of basic reproduction number. So,

The basic reproduction number is $R_0 = (\frac{\mu\beta}{(\alpha+\mu_S)(\gamma+\mu_I)} + \frac{\beta_1\alpha\mu}{(\alpha+\mu_S)(\gamma+\mu_I)(\gamma_1+\mu_T)})$

For epidemiology, R_0 *shows about the disease spread and control. If $R_0 \leq 1$, then the disease−free equilibrium is stable in this case, and the disease dies out or disappear from the system. And if $R_0 > 1$, then the endemic equilibrium exists, and so the disease permanently exist in the system. So, it is observed that I_2 exist if $R_0 > 1$. Thus the endemic equilibrium point if $R_0 > 1$ is−*

$$E_2 = (S_2, T_2, I_2, R_2) = \left(\frac{\mu}{(\alpha+\mu_S+\beta I_2)}, \frac{\alpha\mu}{(\gamma_1+\mu_T+\beta_1 I_2)(\alpha+\mu_S+\beta I_2)}, I_2, \frac{\gamma_1 T + \gamma I_2}{\mu_R}\right)$$

So, the existence of disease – free equilibrium point is not depends on R_0. *Thus if $R_0 \leq$
1, there exist unique equilibrium of system* (1), *that is, the disease –free equilibrium
point E_1. Also, if $R_0 > 1$, then there are two equilibrium points of system* (1)
that is, the disease –free equilibrium point E_1 and the endemic equilibrium point E_2.

9. Stability Analysis of equilibrium points

In Case – I and Case – II, We have determined the disease – free and the endemic equilibrium points. Since, the last equation of the system of equations (1) is independent of the other equations, thus, it is sufficient to study only the remaining equations as follows –

$$\frac{dS}{dt} = \mu - \mu_S S - \beta SI - \alpha S$$

$$\frac{dT}{dt} = \alpha S - \beta_1 TI - \gamma_1 T - \mu_T T ....(3)$$

$$\frac{dI}{dt} = \beta SI + \beta_1 TI - \gamma I - \mu_I I$$

For the system of equation (3), we study the following results -

Theorem 3

If $R_0 = \left(\frac{\mu\beta}{(\alpha+\mu_S)(\gamma+\mu_I)} + \frac{\beta_1\alpha\mu}{(\alpha+\mu_S)(\gamma+\mu_I)(\gamma_1+\mu_T)}\right)$

Then, the Disease – free equilibrium point is asymptotically stable if $R_0 \leq 1$, *and if $R_0 >$
1, then the equilibrium point is unstable.*

Proof -
The Jacobian of the system (3) is –

$$J(S, T, I) = \begin{bmatrix} -(\mu_S + \beta I + \alpha) & 0 & -\beta S \\ \alpha & -(\beta_1 I + \gamma_1 + \mu_T) & -\beta_1 T \\ \beta I & \beta_1 I & \beta S + \beta_1 T - (\gamma + \mu_I) \end{bmatrix}$$

Now the jacobian matrix at the point is

$$(S_1, T_1, I_1) = \left(\frac{\mu}{\alpha+\mu_S}, \frac{\alpha\mu}{(\gamma_1+\mu_T)(\alpha+\mu_S)}, 0\right)$$

$$\text{So, } J(S_1, T_1, I_1) = \begin{bmatrix} -(\mu_S + \alpha) & 0 & -\beta S_1 \\ \alpha & -(\gamma_1 + \mu_T) & -\beta_1 T_1 \\ 0 & 0 & \beta S_1 + \beta_1 T_1 - (\gamma + \mu_I) \end{bmatrix}$$

Then the characteristic equation is

$$|\lambda - J(S_1, T_1, I_1)| = 0$$

$$\Rightarrow \begin{vmatrix} \lambda + (\mu_S + \alpha) & 0 & \beta S_1 \\ -\alpha & \lambda + (\gamma_1 + \mu_T) & \beta_1 T_1 \\ 0 & 0 & \lambda - \{\beta S_1 + \beta_1 T_1 - (\gamma + \mu_I)\} \end{vmatrix} = 0$$

$$\Rightarrow [\lambda + (\mu_S + \alpha)][\{\lambda + (\gamma_1 + \mu_T)\}\{\lambda - \{\beta S_1 + \beta_1 T_1 - (\gamma + \mu_I)\}\}] = 0$$

So, the roots of the cubic characteristic equation is -
$\lambda_1 = -(\mu_S + \alpha) < 0$
$\lambda_2 = -(\gamma_1 + \mu_T) < 0$
$\lambda_3 = \beta S_1 + \beta_1 T_1 - (\gamma + \mu_I)$

$= \frac{\beta\mu}{\alpha+\mu_S} + \frac{\beta_1\alpha\mu}{(\gamma_1+\mu_T)(\alpha+\mu_S)} - (\gamma + \mu_I)$
$= (\gamma + \mu_I)[(\frac{\mu\beta}{(\alpha+\mu_S)(\gamma+\mu_I)} + \frac{\beta_1\alpha\mu}{(\alpha+\mu_S)(\gamma+\mu_I)(\gamma_1+\mu_T)}) - 1]$
$= (\gamma + \mu_I)(R_0 - 1)$

So, if $R_0 < 1$, *then all the eigenvalues of $J(E_1)$ are negative, then by the Routh $-Hurwitz$ Stability Criterion, the Equilibrium point E_1 is asymptotically stable.*
Also, if $R_0 > 1$, then there exist a positive eigenvalue, then, the point E_1 is unstable.

Theorem 4

The Endemic equilibrium point is asymptotically stable if $R_0 > 1$, *and if $R_0 \leq$* 1, *then the equilibrium point is unstable.*

Proof –
Here, the endemic point E_2 *exists when $R_0 > 1$, so the Jacobian at the point (S_2, T_2, I_2) is–*

So, J$(S_2, T_2, I_2) = \begin{bmatrix} -(\mu_S + \beta I_2 + \alpha) & 0 & -\beta S_2 \\ \alpha & -(\beta_1 I_2 + \gamma_1 + \mu_T) & -\beta_1 T_2 \\ \beta I_2 & \beta_1 I_2 & \beta S_2 + \beta_1 T_2 - (\gamma + \mu_I) \end{bmatrix}$

Since, $(S_2, T_2, I_2) = (\frac{\mu}{(\alpha+\mu_S+\beta I_2)}, \frac{\alpha\mu}{(\gamma_1+\mu_T+\beta_1 I_2)(\alpha+\mu_S+\beta I_2)}, I_2)$

Then, $S_2 = \frac{\mu}{\alpha+\mu_S+\beta I_2}$
$\Rightarrow (\alpha + \mu_S + \beta I_2) = \frac{\mu}{S_2}$

Also, $T_2 = \frac{\alpha\mu}{(\gamma_1+\mu_T+\beta_1 I_2)(\alpha+\mu_S+\beta I_2)}$
$= \frac{\alpha S_2}{(\gamma_1+\mu_T+\beta_1 I_2)}$

Which gives $(\gamma_1 + \mu_T + \beta_1 I_2) = \frac{\alpha S_2}{T_2}$

Also for equilibrium endemic point,
$\beta S_2 + \beta_1 T_2 - (\gamma + \mu_I) = 0$ as $I > 0$ from (case - II).

So, Putting the value of $(\alpha + \mu_S + \beta I_2)$ and $(\gamma_1 + \mu_T + \beta_1 I_2)$ in the Jacobian J(S_2, T_2, I_2), *we get,*

J$(S_2, T_2, I_2) = \begin{bmatrix} -\frac{\mu}{S_2} & 0 & -\beta S_2 \\ \alpha & -\frac{\alpha S_2}{T_2} & -\beta_1 T_2 \\ \beta I_2 & \beta_1 I_2 & 0 \end{bmatrix}$

Then the Characteristic equation is -

$|\lambda - J(S_2, T_2, I_2)| = 0$

$\Rightarrow \begin{vmatrix} \lambda + \frac{\mu}{S_2} & 0 & \beta S_2 \\ -\alpha & \lambda + \frac{\alpha S_2}{T_2} & \beta_1 T_2 \\ -\beta I_2 & \beta_1 I_2 & \lambda \end{vmatrix} = 0$

$\Rightarrow (\lambda + \frac{\mu}{S_2})\{(\lambda + \frac{\alpha S_2}{T_2})\lambda + \beta_1^2 I_2 T_2\} + \beta S_2\{\alpha\beta_1 I_2 + \beta I_2(\lambda + \frac{\alpha S_2}{T_2})\} = 0$

$\Rightarrow \lambda^3 + \frac{\mu\lambda^2}{S_2} + \frac{\lambda^2\alpha S_2}{T_2} + \frac{\lambda\alpha\mu}{T_2} + \lambda\beta_1^2 T_2 I_2 + \frac{\mu\beta_1^2 T_2 I_2}{S_2} + \alpha\beta_1\beta S_2 I_2 + \lambda\beta^2 S_2 I_2 + \frac{\alpha\beta^2 S_2^2 I_2}{T_2} = 0$

$\Rightarrow \lambda^3 + \lambda^2(\frac{\mu}{S_2} + \frac{\alpha S_2}{T_2}) + \lambda(\frac{\alpha\mu}{T_2} + \beta_1^2 T_2 I_2 + \beta^2 S_2 I_2) + (\frac{\mu\beta_1^2 T_2 I_2}{S_2} + \alpha\beta_1\beta S_2 I_2 + \frac{\alpha\beta^2 S_2^2 I_2}{T_2}) = 0$

It can be written as -
$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0$

Where, $b_1 = (\frac{\mu}{S_2} + \frac{\alpha S_2}{T_2}) > 0$

$b_2 = (\frac{\alpha\mu}{T_2} + \beta_1^2 T_2 I_2 + \beta^2 S_2 I_2) > 0$

$b_3 = (\frac{\mu\beta_1^2 T_2 I_2}{S_2} + \alpha\beta_1\beta S_2 I_2 + \frac{\alpha\beta^2 S_2^2 I_2}{T_2}) > 0$

Hence, $b_1 b_2 - b_3 = (\frac{\mu}{S_2} + \frac{\alpha S_2}{T_2})(\frac{\alpha\mu}{T_2} + \beta_1^2 T_2 I_2 + \beta^2 S_2 I_2) - (\frac{\mu\beta_1^2 T_2 I_2}{S_2} + \alpha\beta_1\beta S_2 I_2 + \frac{\alpha\beta^2 S_2^2 I_2}{T_2})$

$= \frac{\alpha\mu^2}{T_2 S_2} + \frac{\mu\beta_1^2 T_2 I_2}{S_2} + \mu\beta^2 I_2 + \frac{\alpha^2\mu S_2}{T_2^2} + \alpha S_2\beta_1^2 I_2 + \frac{\alpha S_2^2\beta^2 I_2}{T_2} - \frac{\mu\beta_1^2 T_2 I_2}{S_2} - \alpha\beta_1\beta S_2 I_2 - \frac{\alpha\beta^2 S_2^2 I_2}{T_2}$

$= \frac{\alpha\mu^2}{T_2 S_2} + \mu\beta^2 I_2 + \frac{\alpha^2\mu S_2}{T_2^2} + \alpha S_2\beta_1^2 I_2 - \alpha\beta_1\beta S_2 I_2$

Putting the value of $\mu = \mu_S S_2 + \beta S_2 I_2 + \alpha S_2$ from the first equation of the system of equations (2), we get,

$= \frac{\alpha\mu^2}{T_2 S_2} + (\mu_S S_2 + \beta S_2 I_2 + \alpha S_2)\beta^2 I_2 + \frac{\alpha^2\mu S_2}{T_2^2} + \alpha S_2\beta_1^2 I_2 - \alpha\beta_1\beta S_2 I_2$

$= \frac{\alpha\mu^2}{T_2 S_2} + \mu_S\beta^2 I_2 S_2 + \beta^3 S_2 I_2^2 + \alpha S_2\beta^2 I_2 + \frac{\alpha^2\mu S_2}{T_2^2} + \alpha S_2\beta_1^2 I_2 - 2\alpha\beta_1\beta S_2 I_2 + \alpha\beta_1\beta S_2 I_2$

$= \frac{\alpha\mu^2}{T_2 S_2} + (\mu_S\beta^2 I_2 S_2 + \beta^3 S_2 I_2^2) + \frac{\alpha^2\mu S_2}{T_2^2} + (\alpha S_2\beta^2 I_2 + \alpha S_2\beta_1^2 I_2 - 2\alpha\beta_1\beta S_2 I_2) + \alpha\beta_1\beta S_2 I_2$

$= \frac{\alpha\mu^2}{T_2 S_2} + (\mu_S + \beta I_2)\beta^2 I_2 S_2 + \frac{\alpha^2\mu S_2}{T_2^2} + \alpha S_2 I_2(\beta - \beta_1)^2 + \alpha\beta_1\beta S_2 I_2 > 0$

Hence, we have, $b_1 > 0, b_2 > 0, b_3 > 0$. *Also by the Routh $-$Hurwitz stability criterion, the necessary and sufficient condition for stability of the endemic equilibrium is that the coefficients are positive and the Hurwitz determinants H_i are also positive. For the third degree polynomial, the Hurwitz determinant are $H_1 = b_1 > 0$, $H_2 = b_3 > 0$, $H_3 = (b_1 b_2 - b_3) > 0$. So, by Routh$-$Hurwitz stability criterion, roots have negative real parts if $(b_1 b_2 - b_3) > 0$. Thus, it satisfies all the conditions of the Routh $-$Hurwitz stability criterion. Therefore, the endemic equilibrium point*

$E_2 = (S_2, T_2, I_2)$ *is asymptotically stable. Thus the Endemic equilibrium point is asymptotically stable if* $R_0 > 1$, *and if* $R_0 \leq 1$, *then the equilibrium point is unstable.*

10. CONCLUSION

Here, in this paper, we have determined the basic reproduction number for the model from the result of the endemic equilibrium point analysis.
When the vaccination is introduced in the system of population then we have studied the effect of it in the system. Along with the development of knowledge in the field of health, control of an epidemic disease can be done by vaccination action. Here there exists two equilibrium points, first the disease - free equilibrium point E_1 *and second the endemic equilibrium point* E_2. *So, we have found that if* $R_0 \leq$ 1, *then the disease* − *free equilibrium point* E_1 *is asymptotically stable i.e.*
if we consider the long time period, then the disease disappears from the system.
Also if $R_0 > 1$, *then the endemic equilibrium point is asymptotically stable,*
i.e. if we consider the long time period then the disease must present in the system.

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