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GENERAL INTEGRAL TYPE CONTRACTION MAPPING IN METRIC SPACE ENDOWED WITH A GRAPH

SHANTANU BHAUMIK

Abstract. The subject under discussion of this paper is to find out the condition for which a function satisfying general integral type contraction defined on a metric space endowed with a graph will be Picard operator. With appropriate examples we demonstrate that our result is more general than that of Banach G-contraction, Branciari and G-Ciric-Reich-Rus operator.

1. Introduction

In 2002, Branciari [2] gave a different version of the Banach contraction principle which is mainly integral type inequality for a single valued mapping and showed the following famous fixed point theorem.

Theorem 1 Let (X, d) be a complete metric space, $a \in (0, 1)$, and let $f : X \to X$ be a mapping such that for each $x, y \in X$,

$$\int_0^{d(fx,fy)} \varphi(t)dt \leq a \int_0^{d(x,y)} \varphi(t)dt, \tag{1}$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0, \infty)$, nonnegative, and such that

$$\forall\ \epsilon > 0,\ \int_0^{\epsilon} \varphi(t)dt > 0.$$

Then, f admits a unique fixed point $p \in X$ such that for each $x \in X$, $\lim\limits_{n\to\infty} f^n x = p$.

Theorem 1 is a generalization of the Banach-Caccioppoli principle [3].

In 2007, Jachymski [9] gave another version of the Banach contraction principle which is mainly known as Banach G-contraction. The main aim of this paper is to find the condition for which a function will be Picard operator. The first result in this direction was given by Ran and Reurings [17]. In this paper they shows that if a function f is continuous, monotone and defined on a complete

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metric space (X,d) endowed with partial ordering, it must be Picard operator if it satisfies Banach contraction principle and the additional condition: there exists $x_0 \in X$ with $x_0 \preceq f(x_0)$ or $f(x_0) \preceq x_0$. This particular direction was more generalized by Nieto and Rodriguez-Lopez [15] then by Petrusel and Rus [16] and so on.

Jachymski in his paper, instead of using the concept of partial order on complete metric space, used the concept of graph theory and obtained more general fixed point results. In this paper we further generalized these results with the help of general integral type contraction mapping in metric space endowed with a graph and obtained some progressively broad fixed point results.

2. Basic concept and mathematical preliminary

Definition 1 Let (X,d) be a metric space and f be a selfmapping on X. Then f is said to be a Picard operator (abbr., PO) if f has a unique fixed point x^* and $\lim_{n\to\infty} f^n(x) = x^*$ for all $x \in X$ and f is said to be weakly Picard operator (abbr., WPO) if for any $x \in X$, $\lim_{n\to\infty} f^n(x)$ exists (it may depend on x) and is a fixed point of f.

Definition 2 ([9]). Let (X,d) be a metric space and $\Delta = \{(x,x) : x \in X\}$. Consider a graph G such that its vertex set $V(G)$ coincide with X and the edge set $E(G)$ contains all loops i.e $\Delta \subseteq E(G)$. Assume that G has no parallel edges so, we can identify G with the pair $G(V(G), E(G))$.

By G^{-1} we denote the conversion of a graph G, that is the graph obtained from G by reversing the direction of the edges. Thus we have

$$V(G^{-1}) = V(G)$$
$$E(G^{-1}) = \{(x,y) \in X \times X : (y,x) \in E(G)\}$$

By $\tilde{G}$ we denote the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\tilde{G}$ as a directed graph for which the set of edges is symmetric. Under this convention,

$$E(\tilde{G}) = E(G) \cup E(G^{-1})$$

We call (V', E') a subgraph of G if $V' \subseteq V(G), E' \subseteq E(G)$ and for any edges $(x,y) \in E'$, $x, y \in V'$

If x and y are vertices of G, then a path in G from x to y of length $k \in \mathbb{N}$ is a finite sequence $\{x_n\}$ $(n \in \{0,1,2,\cdots,k\})$ of vertices such that $x_0 = x$, $x_k = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = \{1,2,\cdots,k\}$.

A graph G is connected if there is a path between any two vertices. G is weakly connected if $\tilde{G}$ is connected. If G is such that $E(G)$ is symmetric and x is a vertex in G, then the subgraph G_x consisting of all edges and vertices which are contained in some path beginning at x is called the component of G containing x. In this case $V(G_x) = [x]_G$, where $[x]_G$ is the equivalence class of the following relation R defined on $V(G)$ by the rule

yRz if there is a path in G from y to z.

Clearly G_x is connected.

Definition 3 ([9]). A mapping $f : X \to X$ is called orbitally continuous if for all $x \in X$ and any sequence $(k_n)_{n\in\mathbb{N}}$ of positive integers, $f^{k_n}x \to y \in X$ implies $f(f^{k_n}x) \to fy$ as $n \to \infty$.

Definition 4 ([9]). A mapping $f : X \to X$ is called G-continuous if for given $x \in X$ and a sequence $(x_n)_{n\in\mathbb{N}}$, $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$ implies $f(x_n) \to f(x)$.

Definition 5 ([9]). A mapping $f : X \to X$ is called orbitally G-continuous if for given $x, y \in X$ and a sequence $(k_n)_{n\in\mathbb{N}}$ of positive integers, $f^{k_n}x \to y$ and $(f^{k_n}x, f^{k_{n+1}}x) \in E(G)$ for $n \in \mathbb{N}$ implies $f(f^{k_n}x) \to fy$.

Definition 6 ([5]). A mapping $f : X \to X$ is called a Banach G-contraction if:

(a) $\forall\ x, y \in X\ \Big((x, y) \in E(G) \Rightarrow (fx, fy) \in E(G)\Big)$;

(b) there exits $\alpha \in (0, 1)$ such that for each $(x, y) \in E(G)$, we have

$$d(fx, fy) \le \alpha d(x, y).$$

Definition 7 ([5]). Let (X, d) be a metric space. The operator $f : X \to X$ is called a G-Ciric-Reich-Rus operator if:

(a) f is edge preserving, i.e. $\forall\ x, y \in X\ \Big((x, y) \in E(G) \Rightarrow (fx, fy) \in E(G)\Big)$;

(b) there exist $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + \beta + \gamma < 1$ such that for each $(x, y) \in E(G)$, the following inequality holds:

$$d(fx, fy) \le \alpha d(x, y) + \beta d(x, fx) + \gamma d(y, fy).$$

3. Main Results

Throughout this section, we assume that (X, d) is a metric space, and $\mathscr{G}$ denotes the set of all directed graph G such that $V(G) = X$, $\Delta \subseteq E(G)$ and the graph G has no parallel edges. The set of all fixed points of a mapping f is denoted by Fix$f$. Instead of writing general integral type G-contraction we will write G_T contraction.

Definition 8 A mapping $f : X \to X$ is called a general integral type G-contraction (abbr. G_T-contraction) if:

(a) f is edge preserving, i.e. $\forall\ x, y \in X\ \Big((x, y) \in E(G) \Rightarrow (fx, fy) \in E(G)\Big)$;

(b) there exist $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + 2\beta + 2\gamma < 1$ such that for each $(x, y) \in E(G)$, the following inequality holds:

$$\int_0^{d(fx,fy)} \phi(t)dt \le \alpha \int_0^{d(x,y)} \phi(t)dt + \beta \int_0^{d(x,fy)} \phi(t)dt + \gamma \int_0^{d(y,fx)} \phi(t)dt, \tag{2}$$

where $\phi : [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0, \infty)$, nonnegative, and such that $\forall\ \epsilon > 0$, $\int_0^{\epsilon} \phi(t)dt > 0$.

Remark 1 It follows from (a) of Definition 8 that $(f(V(G)), (f \times f)(E(G)))$ is a subgraph of G where $(f \times f)(x, y) = (fx, fy)$ for all $x, y \in X$.

Remark 2 Taking all other conditions of the Definition 8 as same if we put $\beta = \gamma = 0$ and $\phi(t) = 1$ then from inequality (2) we only get

$$d(fx, fy) \le \alpha d(x, y). \tag{3}$$

Which is Banach G-contraction.

Remark 3 Taking all other conditions of the Definition 8 as same if we only put $\beta = \gamma = 0$ then from (2) we get

$$\int_0^{d(fx,fy)} \phi(t)dt \leq \alpha \int_0^{d(x,y)} \phi(t)dt. \tag{4}$$

Which we call Branciari G-contraction. (2) is more general than that of (3) and (4) because (3) and (4) are derived from (2).

Example 1 As $\Delta \subseteq E(G)$ so any constant function $f : X \to X$ is a G_T-contraction for every $G \in \mathscr{G}$.

Example 2 Let $X = \{0, 1, 2, 3, 4, 5, 6\}$ and $d(x, y) = |x - y|$ and the function $f : X \to X$ is defined by

$$fx = \begin{cases} x-4 & \text{if } x \in \{5,6\}, \\ x-2 & \text{if } x \in \{3,4\}, \\ x-1 & \text{if } x = 2, \\ x & \text{if } x \in \{0,1\}. \end{cases}$$

Define the graph G by $V(G) = X$ and $E(G) = \Delta \cup \{(1, 2), (3, 5), (5, 6)\}$. It is easy to see that f preserves edges. Now $d(f3, f5) = 0, \quad d(f1, f2) = 0, \quad d(f5, f6) = 1, \quad d(5, f6) = 3, \quad \text{and} \quad d(6, f5) = 5$.

Since $d(f5, f6) = 1 = d(5, 6)$ so f is not a Banach G-contraction. Now the function $\phi : [0, \infty) \to [0, \infty)$ is defined by $\phi(t) = t$, then it is easily verified that f is a G_T-contraction with constant $\alpha = \frac{1}{2}$, $\beta = \frac{1}{8}$ and $\gamma = \frac{1}{16}$ but f is not a Branciari.

Lemma 1 Let (X, d) be a metric space endowed with a graph G and $f : X \to X$ be a G_T-contraction. If $x \in X$ satisfies the condition $(x, fx) \in E(G)$ then we have

$$\int_0^{d(f^n x, f^{n+1} x)} \phi(t)dt \ \leq \ r^n \int_0^{d(x,fx)} \phi(t)dt, \tag{5}$$

where $r := \frac{(\alpha+\beta)}{(1-\beta)} < 1$.

Proof. Let $x \in X$ with $(x, fx) \in E(G)$. An easy induction shows that $(f^n x, f^{n+1} x) \in E(G)$ for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$

$$\begin{aligned} \int_0^{d(f^n x, f^{n+1} x)} \phi(t)dt \ &\leq \ \alpha \int_0^{d(f^{n-1} x, f^n x)} \phi(t)dt + \beta \int_0^{d(f^{n-1} x, f^{n+1} x)} \phi(t)dt \\ &\quad + \gamma \int_0^{d(f^n x, f^n x)} \phi(t)dt \\ &\leq \ (\alpha + \beta) \int_0^{d(f^{n-1} x, f^n x)} \phi(t)dt + \beta \int_0^{d(f^n x, f^{n+1} x)} \phi(t)dt. \end{aligned}$$

Hence

$$\int_0^{d(f^n x, f^{n+1} x)} \phi(t)dt \ \leq \ \frac{(\alpha+\beta)}{(1-\beta)} \int_0^{d(f^{n-1} x, f^n x)} \phi(t)dt \ \leq \ r \int_0^{d(f^{n-1} x, f^n x)} \phi(t)dt,$$

where $r := \frac{(\alpha+\beta)}{(1-\beta)} < 1$. So we get

$$\int_0^{d(f^n x, f^{n+1}x)} \phi(t)dt \ \le \ r^n \int_0^{d(x,fx)} \phi(t)dt.$$

Definition 9 Let (X,d) be a metric space endowed with a graph G and $f : X \to X$ be a mapping. We say that the graph G is f-connected if for all vertices x, y of G with $(x,y) \notin E(G)$ there exists a path in G, $(x_i)_{i=0}^N$ from x to y such that $x_0 = x,\ x_N = y$ and $(x_i, fx_i) \in E(G)$ for all $i = 1, \cdots, N-1$. A graph G is weakly connected if $\tilde{G}$ is f-connected.

Lemma 2 Let (X,d) be a metric space endowed with a graph G and $f : X \to X$ be a G_T-contraction such that the graph G is f-connected. If $x \in X$ with $(x, fx) \notin E(G)$ satisfies the condition $(x_i, fx_i) \in E(G)$ for all $i = 1, \cdots, N-1$ then we have

$$\int_0^{d(f^n x, f^{n+1}x)} \phi(t)dt \ \le \ p^n q(x) + np^{n-1}s(x), \tag{6}$$

where $p := \frac{(\alpha+\beta+2\gamma)}{(1-\beta)}$ and $q(x) := \sum\limits_{i=1}^{N} \int_0^{d(x_{i-1},x_i)} \phi(t)dt$ and

$s(x) := \frac{(\beta+\gamma)}{(1-\beta)} \sum\limits_{i=2}^{N} \int_0^{d(x_{i-1},fx_{i-1})} \phi(t)dt$.

Proof. Since $(x, fx) \notin E(G)$, there exists a path in G, $(x_i)_{i=0}^N$ from x to fx such that $x_0 = x, x_N = fx$ with $(x_{i-1}, x_i) \in E(G)$ for all $i = 1, \cdots, N$ and $(x_i, fx_i) \in E(G)$ for all $i = 1, \cdots, N-1$. Then by the triangle inequality and (2) we get

$$\begin{aligned}
\int_0^{d(f^n x, f^{n+1}x)} \phi(t)dt \ &\le \ \sum_{i=1}^{N} \int_0^{d(f^n x_{i-1}, f^n x_i)} \phi(t)dt \\
&\le \alpha \sum_{i=1}^{N} \int_0^{d(f^{n-1}x_{i-1}, f^{n-1}x_i)} \phi(t)dt \\
&+ \beta \sum_{i=1}^{N} \int_0^{d(f^{n-1}x_{i-1}, f^n x_i)} \phi(t)dt \\
&+ \gamma \sum_{i=1}^{N} \int_0^{d(f^{n-1}x_i, f^n x_{i-1})} \phi(t)dt \\
&\le (\alpha+\beta+\gamma) \sum_{i=1}^{N} \int_0^{d(f^{n-1}x_{i-1}, f^{n-1}x_i)} \phi(t)dt \\
&+ \beta \sum_{i=1}^{N} \int_0^{d(f^{n-1}x_i, f^n x_i)} \phi(t)dt + \gamma \sum_{i=1}^{N} \int_0^{d(f^{n-1}x_{i-1}, f^n x_{i-1})} \phi(t)dt \\
&\le (\alpha+\beta+\gamma) \sum_{i=1}^{N} \int_0^{d(f^{n-1}x_{i-1}, f^{n-1}x_i)} \phi(t)dt \\
&+ \beta \int_0^{d(f^n x, f^{n+1}x)} \phi(t)dt + \beta \sum_{i=1}^{N-1} \int_0^{d(f^{n-1}x_i, f^n x_i)} \phi(t)dt
\end{aligned}$$

$$
\begin{aligned}
&+\gamma\int_0^{d(f^{n-1}x,f^nx)}\phi(t)dt+\gamma\sum_{i=2}^{N}\int_0^{d(f^{n-1}x_{i-1},f^nx_{i-1})}\phi(t)dt\\
&\leq(\alpha+\beta+2\gamma)\sum_{i=1}^{N}\int_0^{d(f^{n-1}x_{i-1},f^{n-1}x_i)}\phi(t)dt\\
&+\beta\int_0^{d(f^nx,f^{n+1}x)}\phi(t)dt+(\beta+\gamma)r^{n-1}\sum_{i=2}^{N}\int_0^{d(x_{i-1},fx_{i-1})}\phi(t)dt.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\int_0^{d(f^nx,f^{n+1}x)}\phi(t)dt &\leq \sum_{i=1}^{N}\int_0^{d(f^nx_{i-1},f^nx_i)}\phi(t)dt\\
&\leq\frac{(\alpha+\beta+2\gamma)}{(1-\beta)}\sum_{i=1}^{N}\int_0^{d(f^{n-1}x_{i-1},f^{n-1}x_i)}\phi(t)dt\\
&+\frac{(\beta+\gamma)}{(1-\beta)}r^{n-1}\sum_{i=2}^{N}\int_0^{d(x_{i-1},fx_{i-1})}\phi(t)dt.
\end{aligned}
$$

Since $0<r=\frac{(\alpha+\beta)}{(1-\beta)}<\frac{(\alpha+\beta+2\gamma)}{(1-\beta)}=p<1$, so from above we get

$$
\begin{aligned}
\int_0^{d(f^nx,f^{n+1}x)}\phi(t)dt &\leq \sum_{i=1}^{N}\int_0^{d(f^nx_{i-1},f^nx_i)}\phi(t)dt\\
&\leq p\sum_{i=1}^{N}\int_0^{d(f^{n-1}x_{i-1},f^{n-1}x_i)}\phi(t)dt\\
&+\frac{(\beta+\gamma)}{(1-\beta)}p^{n-1}\sum_{i=2}^{N}\int_0^{d(x_{i-1},fx_{i-1})}\phi(t)dt\\
&\leq p^2\sum_{i=1}^{N}\int_0^{d(f^{n-2}x_{i-1},f^{n-2}x_i)}\phi(t)dt\\
&+2\frac{(\beta+\gamma)}{(1-\beta)}p^{n-1}\sum_{i=2}^{N}\int_0^{d(x_{i-1},fx_{i-1})}\phi(t)dt\\
&\cdots\cdots\cdots\\
&\leq p^n\sum_{i=1}^{N}\int_0^{d(x_{i-1},x_i)}\phi(t)dt\\
&+n\frac{(\beta+\gamma)}{(1-\beta)}p^{n-1}\sum_{i=2}^{N}\int_0^{d(x_{i-1},fx_{i-1})}\phi(t)dt.
\end{aligned}
$$

Hence

$$
\int_0^{d(f^nx,f^{n+1}x)}\phi(t)dt \;\leq\; p^nq(x)+np^{n-1}s(x), \tag{7}
$$

where $p:=\frac{(\alpha+\beta+2\gamma)}{(1-\beta)}$, $q(x):=\sum\limits_{i=1}^{N}\int_0^{d(x_{i-1},x_i)}\phi(t)dt$ and

$s(x) := \frac{(\beta+\gamma)}{(1-\beta)} \sum\limits_{i=2}^{N} \int_0^{d(x_{i-1}, fx_{i-1})} \phi(t)dt.$

Lemma 3 Let (X, d) be a metric space endowed with a graph G and $f : X \to X$ be a G_T-contraction. For all $x \in X$ the sequence $(f^n x)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Proof. Let $x \in X$ is fixed. We discuss two cases.

Case 1. If $(x, fx) \in E(G)$ then by Lemma 1 we get

$$\int_0^{d(f^n x, f^{n+1}x)} \phi(t)dt \ \leq \ r^n \int_0^{d(x,fx)} \phi(t)dt, \tag{8}$$

for all $n \in \mathbb{N}$, where $r := \frac{(\alpha+\beta)}{(1-\beta)} < 1$.

Let $m, n \in \mathbb{N}$, $n > m$. Then using triangular inequality we get

$$d(f^m x, f^n x) \ \leq \ \sum_{i=m}^{n-1} d(f^i x, f^{i+1}x). \tag{9}$$

Therefore

$$\begin{aligned}\int_0^{d(f^m x, f^n x)} \phi(t)dt \ &\leq \ \sum_{i=m}^{n-1} \int_0^{d(f^i x, f^{i+1}x)} \phi(t)dt \\ &\leq (r^m + r^{m+1} + \cdots + r^{n-1}) \int_0^{d(x,fx)} \phi(t)dt \\ &= r^m(1 + r + \cdots + r^{n-m-1}) \int_0^{d(x,fx)} \phi(t)dt \\ &\leq \frac{r^m}{1-r} \int_0^{d(x,fx)} \phi(t)dt.\end{aligned}$$

Letting $m \to \infty$ on both sides of the above inequality and using the property that, $\int_0^{\epsilon} \phi(t)dt > 0$, $\forall\ \epsilon > 0$, it follows that the sequence $(f^n x)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Case 2. $(x, fx) \notin E(G)$ then by Lemma 2 we get

$$\int_0^{d(f^n x, f^{n+1}x)} \phi(t)dt \ \leq \ p^n q(x) + np^{n-1} s(x), \tag{10}$$

where $p := \frac{(\alpha+\beta+2\gamma)}{(1-\beta)}$, $q(x) := \sum\limits_{i=1}^{N} \int_0^{d(x_{i-1}, x_i)} \phi(t)dt$ and

$s(x) := \frac{(\beta+\gamma)}{(1-\beta)} \sum\limits_{i=2}^{N} \int_0^{d(x_{i-1}, fx_{i-1})} \phi(t)dt.$

Since $0 < p < 1$, so from (10) we get

$$\sum_{n=0}^{\infty} \int_0^{d(f^n x, f^{n+1}x)} \phi(t)dt \leq q(x) \sum_{n=0}^{\infty} p^n + s(x) \sum_{n=0}^{\infty} np^{n-1} \leq \frac{q(x)}{(1-p)} + \frac{s(x)}{(1-p)^2} < \infty,$$

and a standard argument shows that the sequence $(f^n x)_{n\in\mathbb{N}}$ is a Cauchy sequence.

The main result of this paper is given by the following theorem.

Theorem 2 Let (X, d) be a complete metric space endowed with a graph G and $f : X \to X$ be a G_T-contraction. We suppose that

(i) G is f-connected;

(ii) for any sequence $(x_n)_{n\in\mathbb{N}}$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ then there is a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ with $(x_{k_n}, x) \in E(G)$ for all $n \in \mathbb{N}$.

Then f is a PO.

Proof. Since the sequence $(f^n x)_{n\geq 0}$ is a Cauchy sequence for all $x \in X$ according to Lemma 3, so the sequence $(f^n x)_{n\geq 0}$ is convergent.
Let $x, y \in X$ then $(f^n x)_{n\geq 0} \to x^*$ and $(f^n y)_{n\geq 0} \to y^*$, as $n \to \infty$.
Now we consider the following two cases

Case 1. If $(x, y) \in E(G)$, we have $(f^n x, f^n y) \in E(G)$ for all $n \in \mathbb{N}$, then

$$\begin{aligned}\int_0^{d(f^n x, f^n y)} \phi(t)dt \ &\leq \ \alpha \int_0^{d(f^{n-1}x, f^{n-1}y)} \phi(t)dt + \beta \int_0^{d(f^{n-1}x, f^n y)} \phi(t)dt \\ &+ \gamma \int_0^{d(f^{n-1}y, f^n x)} \phi(t)dt,\end{aligned}$$

for all $n \in \mathbb{N}$. Taking $n \to \infty$ on both sides of the above inequality we get

$$\int_0^{d(x^*, y^*)} \phi(t)dt \ \leq \ \alpha \int_0^{d(x^*, y^*)} \phi(t)dt + \beta \int_0^{d(x^*, y^*)} \phi(t)dt + \gamma \int_0^{d(y^*, x^*)} \phi(t)dt,$$

or

$$(1 - \alpha - \beta - \gamma) \int_0^{d(x^*, y^*)} \phi(t)dt \ \leq \ 0.$$

Since $0 < (1 - \alpha - \beta - \gamma) < 1$ and $\forall\ \epsilon > 0$, $\int_0^{\epsilon} \phi(t)dt > 0$ so we obtain $x^* = y^*$.

Case 2. If $(x, y) \notin E(G)$, then there is a path in G, $(x_i)_{i=0}^N$ from x to y such that $x_0 = x, x_N = y$ with $(x_{i-1}, x_i) \in E(G)$ for all $i = 1, \cdots, N$ and $(x_i, fx_i) \in E(G)$ for all $i = 1, \cdots, N-1$. Then $(f^n x_{i-1}, f^n x_i) \in E(G)$ for all $n \in \mathbb{N}$ and $i = 1, \cdots, N$ and by triangle inequality we get

$$\begin{aligned}\int_0^{d(f^n x, f^n y)} \phi(t)dt &\leq \sum_{i=1}^N \int_0^{d(f^n x_{i-1}, f^n x_i)} \phi(t)dt \\ &\leq \alpha \sum_{i=1}^N \int_0^{d(f^{n-1}x_{i-1}, f^{n-1}x_i)} \phi(t)dt + \beta \sum_{i=1}^N \int_0^{d(f^{n-1}x_{i-1}, f^n x_i)} \phi(t)dt \\ &+ \gamma \sum_{i=1}^N \int_0^{d(f^{n-1}x_i, f^n x_{i-1})} \phi(t)dt \\ &\leq (\alpha + \beta + \gamma) \sum_{i=1}^N \int_0^{d(f^{n-1}x_{i-1}, f^{n-1}x_i)} \phi(t)dt \\ &+ \beta \sum_{i=1}^N \int_0^{d(f^{n-1}x_i, f^n x_i)} \phi(t)dt + \gamma \sum_{i=1}^N \int_0^{d(f^{n-1}x_{i-1}, f^n x_{i-1})} \phi(t)dt.\end{aligned}$$

By Lemma 3 we have, the sequence $(f^n x_i)_{n\geq 0}$ is convergent and using the continuity of distance we have, the sequence $(d(f^n x_{i-1}, f^n x_i))_{n\in\mathbb{N}}$ is convergent and let $\lim_{n\to\infty}\int_0^{d(f^n x_{i-1},f^n x_i)}\phi(t)dt = t_i$ for all $i = 1,\cdots,N$. Taking $n \to \infty$ on both sides of the above inequality we get $t_i = 0$ for all $i = 1,\cdots,N$ that is $d(x^*,y^*) \leq 0$, hence $x^* = y^*$.

Therefore, for all $x \in X$ there exists a unique $x^* \in X$ such that

$$\lim_{n\to\infty} f^n x = x^*.$$

Now we will prove that $x^* \in \text{Fix}f$. Since the graph G is f-connected, so there is at least one $x_0 \in X$ such that $(x_0, fx_0) \in E(G)$ so $(f^n x_0, f^{n+1}x_0) \in E(G)$ for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} f^n x_0 = x^*$, therefore by (ii) of Theorem 2 there is a subsequence $(f^{k_n}x_0)_{n\in N}$ with $(f^{k_n}x_0, x^*) \in E(G)$ for all $n \in \mathbb{N}$. Then for all $n \in \mathbb{N}$ we get

$$\begin{aligned}\int_0^{d(x^*,fx^*)}\phi(t)dt &\leq \int_0^{d(x^*,f^{k_n+1}x_0)}\phi(t)dt + \int_0^{d(f^{k_n+1}x_0,fx^*)}\phi(t)dt\\ &\leq \int_0^{d(x^*,f^{k_n+1}x_0)}\phi(t)dt + \alpha\int_0^{d(f^{k_n}x_0,x^*)}\phi(t)dt\\ &+\ \beta\int_0^{d(f^{k_n}x_0,fx^*)}\phi(t)dt + \gamma\int_0^{d(x^*,f^{k_n+1}x^*)}\phi(t)dt.\end{aligned}$$

Now, letting $n \to \infty$ on both sides of the above inequality we get

$$\int_0^{d(x^*,fx^*)}\phi(t)dt \ \leq\ \beta\int_0^{d(x^*,fx^*)}\phi(t)dt,$$

or

$$(1-\beta)\int_0^{d(x^*,fx^*)}\phi(t)dt \ \leq\ 0.$$

Since $0 < (1-\beta) < 1$ and $\forall\ \epsilon > 0$, $\int_0^{\epsilon}\phi(t)dt > 0$, so $fx^* = x^*$, that is, $x^* \in \text{Fix}f$. For uniqueness, if we have $fy = y$ for some $y \in X$, then from above, we must have $f^n y = x^*$ as $n \to \infty$, so $y = x^*$ and therefore, f is a PO.

Corollary 1 Let (X,d) be a complete metric space endowed with a graph G and $f : X \to X$ be a mapping satisfies the following conditions

(i) G is f-connected;

(ii) for any sequence $(x_n)_{n\in\mathbb{N}}$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ then there is a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ with $(x_{k_n}, x) \in E(G)$ for all $n \in \mathbb{N}$;

(iii) there exist $\alpha, \beta \in (0,1)$ with $\alpha + 2\beta < 1$ such that for each $(x,y) \in E(G)$, the following inequality holds:

$$\int_0^{d(fx,fy)}\phi(t)dt \ \leq\ \alpha\int_0^{d(x,y)}\phi(t)dt + \beta\int_0^{d(x,fy)}\phi(t)dt,$$

where $\phi : [0,\infty) \to [0,\infty)$ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0,\infty)$, nonnegative, and such that $\forall\ \epsilon > 0$, $\int_0^{\epsilon}\phi(t)dt > 0$. Then f is a PO.

Proof. It is clear that f is a G_T-contraction with constant $\gamma = 0$, so by Theorem 2 it is proved that f is a PO.

Corollary 2 Let (X, d) be a complete metric space endowed with a graph G and $f : X \to X$ be a mapping satisfies the following conditions

(i) G is f-connected;

(ii) for any sequence $(x_n)_{n\in\mathbb{N}}$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ then there is a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ with $(x_{k_n}, x) \in E(G)$ for all $n \in \mathbb{N}$;

(iii) there exist $\alpha, \gamma \in (0, 1)$ with $\alpha + 2\gamma < 1$ such that for each $(x, y) \in E(G)$, the following inequality holds:

$$\int_0^{d(fx,fy)} \phi(t)dt \ \leq \ \alpha \int_0^{d(x,y)} \phi(t)dt + \gamma \int_0^{d(y,fx)} \phi(t)dt,$$

where $\phi : [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0, \infty)$, nonnegative, and such that $\forall\ \epsilon > 0$, $\int_0^\epsilon \phi(t)dt > 0$. Then f is a PO.

Proof. It is clear that f is a G_T-contraction with constant $\beta = 0$, so the conclusion is obtained by Theorem 2 that f is a PO.

4. Example

Example 3 Let $X = [0, 1]$ be endowed with the Euclidean metric $d(x, y) = |x - y|$. Define the graph G by $V(G) = X$ and $E(G) = \{(x, y) \in [0, 1) \times [0, 1)|x \geq y\} \cup \{(1, 1)\}$ and the function $f : X \to X$ is defined by $fx = \frac{x}{4}$ for all $x \in (0, 1]$ and $f0 = \frac{1}{3}$ then (X, d) is a complete metric space and G is weakly f-connected and the function $\phi : [0, \infty) \to [0, \infty)$ is defined by $\phi(t) = t$.

Now

$$\int_0^{d(fx,fy)} \phi(t)dt \ = \ \frac{(x-y)^2}{32},$$

and

$$\int_0^{d(x,y)} \phi(t)dt \ = \ \frac{(x-y)^2}{2}, \qquad \int_0^{d(x,fy)} \phi(t)dt \ = \ \frac{(4x-y)^2}{32},$$

$$\int_0^{d(y,fx)} \phi(t)dt \ = \ \frac{(4y-x)^2}{32}.$$

So f is a G_T-contraction with constant $\alpha = \frac{1}{8}, \beta = \frac{1}{12}, \gamma = \frac{1}{6}$, but the condition (ii) of the Theorem 2 is not satisfied. Clearly, $\lim\limits_{n\to\infty} f^n x = 0$ for all $x \in X$, but f has no fixed point.

Example 4 Let $X = \{0, 2, 4, 8, 16\}$ be endowed with the Euclidean metric $d(x, y) = |x - y|$ then (X, d) is a complete metric space. Define the graph G by $V(G) = X$ and $E(G) = \Delta \cup \{(0, 2), (2, 4), (4, 8), (8, 16)\}$ and the function $f : X \to X$ is defined by

$$fx \ = \ \begin{cases} \frac{x}{2} & \text{if} \ \ x \in \{4, 8, 16\}, \\ 0 & \text{if} \ \ x = \{0, 2\}. \end{cases}$$

It is easy to see that f preserves edges and G is weakly f-connected.
Now the function $\phi : [0, \infty) \to [0, \infty)$ is defined by $\phi(t) = t$.

Then

$$\int_0^{d(f2,f4)} \phi(t)dt \ = \ 2,$$

and

$$\int_0^{d(2,4)} \phi(t)dt \ = \ 2, \ \int_0^{d(2,f4)} \phi(t)dt \ = \ 0, \ \int_0^{d(4,f2)} \phi(t)dt \ = \ 8.$$

So f is a G_T-contraction with constant $\alpha = \frac{1}{2}, \beta = \frac{1}{16}, \gamma = \frac{1}{6}$. As $d(f2, f4) = d(2, 4) = d(2, f2) = d(4, f4) = 2$ so f is neither a Banach G-contraction nor a G-Ciric-Reich-Rus operator. Now it can be easily verified that all the conditions of the Theorem 2 are satisfied. Clearly, 0 is the only fixed point of f and $\lim_{n\to\infty} f^n x = 0$ for all $x \in X$. So f is a PO.

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Shantanu Bhaumik
Department of Mathematics, Dr. C. V. Raman University, Bilaspur (C.G.), INDIA
E-mail address: `santbhk@gmail.com`