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COMMON FIXED POINTS OF TWO PAIRS OF SELFMAPS SATISFYING A GERAGHTY-BERINDE TYPE CONTRACTION CONDITION IN b-METRIC SPACES

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Abstract. In this paper, we introduce Geraghty-Berinde type contraction for two pairs of selfmaps in b-metric spaces and we prove the existence of common fixed points under the assumptions that these two pairs of maps are weakly compatible and satisfying a Geraghty-Berinde type contraction condition in complete b-metric spaces. The same is extended to a sequence of selfmaps. Also, we prove the same with different hypotheses on two pairs of selfmaps in which one pair is compatible, reciprocally continuous and the other one is weakly compatible. Further, we also prove the same with different hypotheses on two pairs in which these selfmaps are satisfy b-(E.A)-property. We also discuss the importance of L in our contraction condition. Our theorems extend/generalize some of the results in literature to two pairs of self maps.

1. Introduction

The development of fixed point theory is based on the generalization of contraction conditions in one direction or/and generalization of ambient spaces of the operator under consideration on the other. Banach contraction principle plays an important role in solving nonlinear equations and it is one of the most useful result in fixed point theory. In the direction of generalization of contraction conditions, in 1973, Geraghty [22] proved a fixed point theorem, generalizing Banach contraction principle. Several authors proved later various results using Geraghty-type conditions. In continuation to the extensions of contraction maps, Berinde [13] introduced 'weak contractions' as a generalization of contraction maps. Berinde renamed 'weak contractions' as 'almost contractions' in his later work [14]. For more works on almost contractions and its generalizations, we refer Babu, Sandhya and Kameswari [10], Abbas, Babu and Alemayehu [2] and the related references cited in these papers.

The main idea of b-metric was initiated from the works of Bourbaki [17] and Bakhtin [12]. The concept of b-metric space or metric type space was introduced by Czerwik [18] as a generalization of metric space. Afterwards, many authors studied

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fixed point theorems for single-valued and multi-valued mappings in b-metric spaces, for more information we refer [4, 9, 11, 15, 16, 19, 26, 27, 28, 33, 34, 35, 36, 37].

In 2002, Aamari and Moutawakil [1] introduced the notion of property (E.A). Later, several authors apply this concept to prove the existence of common fixed points, we refer [3, 5, 6, 7, 29, 30, 31].

Definition 1.1 [18] Let X be a non-empty set.A function $d : X \times X \to [0, \infty)$ is said to be a b-metric if the following conditions are satisfied: for any $x, y, z \in X$

(i) $0 \leq d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) there exists $s \geq 1$ such that $d(x, z) \leq s[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) is called a b-metric space with coefficient s.

Every metric space is a b-metric space with $s = 1$. In general, every b-metric space is not a metric space.

Definition 1.2 [16] Let (X, d) be a b-metric space and let $\{x_n\}$ be a sequence in X.

(i) A sequence $\{x_n\}$ in X is called b-convergent if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. In this case, we write $\lim_{n\to\infty} x_n = x$.
(ii) A sequence $\{x_n\}$ in X is called b-Cauchy if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.
(iii) A b-metric space (X, d) is said to be a complete b-metric space if every b-Cauchy sequence in X is b-convergent in X.
(iv) A set $B \subset X$ is said to be b-closed if for any sequence $\{x_n\}$ in B such that $\{x_n\}$ is b-convergent to $z \in X$ then $z \in B$.

In general, a b-metric is not necessarily continuous.

Example 1.3 [23] Let $X = \mathbb{N} \cup \{\infty\}$. We define a mapping $d : X \times X \to [0, \infty)$ as follows:

$$d(m,n) = \begin{cases} 0 & \text{if } m = n, \\ |\frac{1}{m} - \frac{1}{n}| & \text{if one of } m, n \text{ is even and the other is even or } \infty, \\ 5 & \text{if one of } m, n \text{ is odd and the other is odd or } \infty, \\ 2 & \text{otherwise.} \end{cases}$$

Then (X, d) is a b-metric space with coefficient $s = \frac{5}{2}$.

Definition 1.4 [16] Let (X, d_X) and (Y, d_Y) be two b-metric spaces. A function $f : X \to Y$ is a b-continuous at a point $x \in X$, if it is b-sequentially continuous at x. i.e., whenever $\{x_n\}$ is b-convergent to x, fx_n is b-convergent to fx.

Definition 1.5 [24] Let A and B be selfmaps of a metric space (X, d). The pair (A, B) is said to be a compatible pair on X, if $\lim_{n\to\infty} d(ABx_n, BAx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$, for some $t \in X$.

Definition 1.6 [25] Let X be a nonempty set. Let $A, B : X \to X$ be two selfmaps. If $Ax = Bx$ implies that $ABx = BAx$ for x in X, then we say that the pair (A, B) is weakly compatible.

Definition 1.7 [32] Two selfmappings A and B of a metric space (X,d) are called reciprocally continuous if $\lim\limits_{n\to\infty} ABx_n = Az$ and $\lim\limits_{n\to\infty} BAx_n = Bz$ whenever $\{x_n\}$ is a sequence in X such that $\lim\limits_{n\to\infty} Ax_n = \lim\limits_{n\to\infty} Bx_n = z$ for some $z \in X$.

Definition 1.8 [29] Two selfmappings A and B of a b-metric space (X,d) are said to satisfy b-(E.A)-property if there exists a sequence $\{x_n\}$ in X such that $\lim\limits_{n\to\infty} Ax_n = \lim\limits_{n\to\infty} Bx_n = z$ for some $z \in X$.

In 1973, Geraghty [22] introduced a class of functions

$$\mathfrak{S} = \{\beta : [0,\infty) \to [0,1)/ \lim_{n\to\infty} \beta(t_n) = 1 \implies \lim_{n\to\infty} t_n = 0\}.$$

Theorem 1.9 [22] Let (X,d) be a complete metric space. Let $T : X \to X$ be a selfmap satisfying the following: there exists $\beta \in \mathfrak{S}$ such that

$$d(Tx,Ty) \leq \beta(d(x,y))d(x,y) \ for\ all\ x,y \in X.$$

Then T has a unique fixed point.

We denote

$$\mathfrak{B} = \{\alpha : [0,\infty) \to [0,\frac{1}{s})/ \lim_{n\to\infty} \alpha(t_n) = \frac{1}{s} \implies \lim_{n\to\infty} t_n = 0\}.$$

In 2011, Dukic, Kadelburg and Radenović [20] extended Theorem 1.9 to the case of b-metric spaces as follows.

Theorem 1.10 [20] Let (X,d) ba a complete b-metric space with coefficient $s \geq 1$ and let $T : X \to X$ be a selfmap of X. Suppose that there exists $\alpha \in \mathfrak{B}$ such that

$$d(Tx,Ty) \leq \alpha(d(x,y))d(x,y) \ for\ all\ x,y \in X.$$

Then T has a unique fixed point in X.

Throughout this paper, we denote

$$\mathfrak{F} = \{\beta : [0,\infty) \to [0,\frac{1}{s})/ \limsup_{n\to\infty} \beta(t_n) = \frac{1}{s} \implies \lim_{n\to\infty} t_n = 0\}$$

and $\mathbb{N}$, the set of all natural numbers.

The following lemma is useful in proving our main results.

Lemma 1.11 [3] Let (X,d) be a b-metric space with coefficient $s \geq 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are b-convergent to x and y respectively, then we have

$$\frac{1}{s^2}d(x,y) \leq \liminf_{n\to\infty} d(x_n,y_n) \leq \limsup_{n\to\infty} d(x_n,y_n) \leq s^2 d(x,y)$$

In particular, if $x = y$, then we have $\lim\limits_{n\to\infty} d(x_n,y_n) = 0$. Moreover for each $z \in X$ we have

$$\frac{1}{s}d(x,z) \leq \liminf_{n\to\infty} d(x_n,z) \leq \limsup_{n\to\infty} d(x_n,z) \leq sd(x,z).$$

In 2019, Faraji, Savić and Radenović [21] proved the following two theorems.

Theorem 1.12 [21] Let (X, d) be a complete b-metric space with parameter $s \geq 1$. Let $T : X \to X$ be a selfmap satisfying: there exists $\beta \in \mathfrak{F}$ such that

$$d(Tx, Ty) \leq \beta(M(x, y))M(x, y) \ for \ all \ x, y \in X,$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2s}(d(x, Ty) + d(y, Tx))\}$$

Then T has a unique fixed point.

Theorem 1.13 [21] Let (X, d) be a complete b-metric space with parameter $s \geq 1$. Let $T, S : X \to X$ be selfmaps on X which satisfy: there exists $\beta \in \mathfrak{F}$ such that

$$sd(Tx, Sy) \leq \beta(M(x, y))M(x, y) \ for \ all \ x, y \in X,$$

where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Sy)\}$. If T or S are continuous, then T and S have a unique common fixed point.

In 2019, Babu and Babu [8] proved a common fixed theorem for a pair of almost Geraghty contraction type maps.

Definition 1.14 [8] Let (X, d) be a b-metric space with coefficient $s \geq 1$, and let f and g be selfmaps of X. If there exist $\beta \in \mathfrak{F}$ and $L \geq 0$ such that

$$sd(fx, gy) \leq \beta(M(x, y))M(x, y) + LN(x, y) \ for \ all \ x, y \in X,$$

where $M(x, y) = \max\{d(x, y), d(x, fx), d(y, gy)\}$
and $N(x, y) = \min\{d(x, fx), d(x, gy), d(y, fx)\}$
then the (f, g) is called an almost Geraghty contraction type maps.

Theorem 1.15 [8] Let (X, d) be a complete b-metric space with coefficient $s \geq 1$ and let (f, g) be an almost Geraghty contraction type pair of maps. If either f or g is b-continuous then f and g have a unique common fixed point in X.

Motivated by works of Babu and Babu [8], in Section 2, we introduce Geraghty-Berinde type contraction for two pairs of selfmaps in b-metric spaces and we prove the existence of common fixed points under the assumptions that these two pairs of maps are weakly compatible and satisfying a Geraghty-Berinde type contraction condition in complete b-metric spaces. The same is extended to a sequence of selfmaps. Also, we prove the same with different hypotheses on two pairs of selfmaps in which one pair is compatible, reciprocal continuous and the other one is weakly compatible. Further, we also prove the same with different hypotheses on two pairs in which these selfmaps are satisfy b-(E.A)-property. We also discuss the importance of L in our contraction condition. Our theorems extend some of the results in literature to two pairs of self maps. We draw some corrollaries from our results and provide examples in support of our results.

2. Main Results

The following we introduce Geraghty-Berinde type contraction maps in b-metric spaces.

Definition 2.1 Let (X,d) be a b-metric space with coefficient $s \geq 1$ and $A,B,S,T : X \to X$ be selfmaps. If there exist $\beta \in \mathfrak{F}$ and $L \geq 0$ such that

$$sd(Ax,By) \leq \beta(M(x,y))M(x,y) + LN(x,y) \tag{1}$$

for all $x,y \in X$, where $M(x,y) = \max\{d(Sx,Ty), d(Ax,Sx), d(By,Ty)\}$ and $N(x,y) = \min\{d(Ax,Sx), d(Ax,Ty), d(By,Sx)\}$.
Then we call A,B,S and T are Geraghty-Berinde type contraction maps.

Example 2.2 Let $X = [0,1]$ and let $d : X \times X \to [0,\infty)$ defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ (x+y)^2 & \text{if } x \neq y. \end{cases}$$

Then clearly (X,d) is a complete b-metric space with $s = 2$.
We define $A,B,S,T : X \to X$ by $A(x) = \frac{1+x^2}{2}, B(x) = \frac{x}{8}$,
$S(x) = \frac{1+x^2}{4}, T(x) = \frac{x}{4}$ for all $x \in X$.
We define $\beta : [0,\infty) \to [0,\frac{1}{s})$ by $\beta(t) = \frac{1}{2+t}$. Then clearly $\beta \in \mathfrak{F}$
Then we have
$sd(Ax,By) = 2(\frac{1+x^2}{2} + \frac{y}{4})^2 \leq \frac{16}{32+9(1+x^2)^2}\frac{9}{16}(1+x^2)^2 + 5(\frac{1+x^2}{4} + \frac{y}{8})^2$.
Therefore A,B,S and T are Geraghty-Berinde type contraction maps with $L = 5$.

Here we observe that if we take $L = 0$ then the inequality (1) fails to hold.
For, we choose $x = 0, y = 1$.
Then $sd(Ax,By) = \frac{9}{8}$ and $M(x,y) = \frac{9}{16}$.
Therefore $sd(Ax,By) = \frac{9}{8} \not\leq \beta(\frac{9}{16})\frac{9}{16} = \beta(M(x,y))M(x,y)$ for all $\beta \in \mathfrak{F}$.

Remark 2.3 If we take $L = 0$ in the inequality (1) then we say that A,B,S and T are Geraghty contraction type maps.

Let A,B,S and T be mappings from a metric space (X,d) into itself and satisfying

$$A(X) \subseteq T(X) \text{ and } B(X) \subseteq S(X) \tag{2}$$

Now, by (1), for any $x_0 \in X$, there exists $x_1 \in X$ such that $y_0 = Ax_0 = Tx_1$. In the same way for this x_1, we can choose a point $x_2 \in X$ such that $y_1 = Bx_1 = Sx_2$ and so on. In general, we can define a sequence $\{y_n\} \in X$ such that

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \text{ and } y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \text{ for } n = 0,1,2,\ldots. \tag{3}$$

Lemma 2.4 Let (X,d) be a b-metric space with coefficient $s \geq 1$. Suppose that A,B,S and T are Geraghty-Berinde type contraction maps. Then we have the following:
(i) If $A(X) \subseteq T(X)$ and the pair (B,T) is weakly compatible, and if x is a common fixed point of A and S then x is a common fixed point of A,B,S and T and it is unique.
(ii) If $B(X) \subseteq S(X)$ and the pair (A,S) is weakly compatible, and if x is a common fixed point of B and T then x is a common fixed point of A,B,S and T and it is

unique.
Proof. First, we assume that *(i)* holds. Let x be a common fixed point of A and S. Then $Ax = Sx = x$.
Since $A(X) \subseteq T(X)$, there exists $y \in X$ such that $Ty = x$.
Therefore $Ax = Sx = Ty = x$.
We now prove that $Ax = By$.
Suppose that $Ax \neq By$.
We consider,

$$sd(Ax, By) \leq \beta(M(x,y))M(x,y) + LN(x,y) \tag{4}$$

where,
$M(x,y) = \max\{d(Sx,Ty), d(Ax,Sx), d(By,Ty)\} = d(Ax,By)$ and
$N(x,y) = \min\{d(Ax,Sx), d(Ax,Ty), d(By,Sx)\} = 0$.
From the inequality (4), we have
$sd(Ax, By) \leq \beta(d(Ax,By))d(Ax,By) < \frac{d(Ax,By)}{s}$,
which is a contradiction.
Therefore $Ax = By = Sx = Ty = x$.
Since the pair (B,T) is weakly compatible and $Ty = By$, we have
$BTy = TBy$. i.e., $Bx = Tx$.
Now, we prove that $Bx = x$. If $Bx \neq x$, then

$$sd(Bx, x) = sd(x, Bx) = sd(Ax, Bx) \leq \beta(M(x,x))M(x,x) + LN(x,x) \tag{5}$$

where
$M(x,x) = \max\{d(Sx,Tx), d(Ax,Sx), d(Bx,Tx)\} = d(x,Bx)$ and
$N(x,x) = \min\{d(Ax,Sx), d(Ax,Tx), d(Bx,Sx)\} = 0$
From the inequality (5), we have
$sd(Bx,x) \leq \beta(d(Bx,x))d(Bx,x) < \frac{d(Bx,x)}{s}$,
it is a contradiction.
Hence, $Bx = x$.
Therefore $Ax = Bx = Sx = Tx = x$.
Therefore, x is a common fixed point of A, B, S and T.
If x' is also a common fixed point of A, B, S and T with $x \neq x'$, then

$$sd(x, x') = sd(Ax, Bx') \leq \beta(M(x,x'))M(x,x') + LN(x,x') \tag{6}$$

where
$M(x,x') = \max\{d(Sx,Tx'), d(Ax,Sx), d(Bx',Tx')\} = d(x,x')$ and
$N(x,x') = \min\{d(Ax,Sx), d(Ax,Tx'), d(Bx',Sx)\} = 0$
From the inequality (6),we have
$sd(x,x') \leq \beta(d(x,x'))d(x,x') < \frac{d(x,x')}{s}$,
which is a contradiction. Therefore, $x = x'$.
Hence, x is the unique common fixed point of A, B, S and T.
The proof of (ii) is similar to (i) and hence is omitted.

Lemma 2.5 *Let A, B, S and T be selfmaps of a b-metric space (X,d) and satisfy* (2) *and are Geraghty-Berinde type contraction maps. Then for any $x_0 \in X$, the sequence $\{y_n\}$ defined by* (3) *is b-Cauchy in X.*
Proof. Let $x_0 \in X$ and let $\{y_n\}$ be a sequence defined by (3).
Assume that $y_n = y_{n+1}$ for some n.
Case (i): n even.

We write $n = 2m, m \in \mathbb{N}$.
Now we consider

$$\begin{aligned} sd(y_{n+1}, y_{n+2}) &= sd(y_{2m+1}, y_{2m+2}) \\ &= sd(y_{2m+2}, y_{2m+1}) \\ &= sd(Ax_{2m+2}, Bx_{2m+1}) \\ &\leq \beta(M(x_{2m+2}, x_{2m+1}))M(x_{2m+2}, x_{2m+1}) + LN(x_{2m+2}, x_{2m+1}) \end{aligned} \tag{7}$$

where

$$\begin{aligned} M(x_{2m+2}, x_{2m+1}) &= \max\{d(Sx_{2m+2}, Tx_{2m+1}), d(Ax_{2m+2}, Sx_{2m+2}), d(Bx_{2m+1}, Tx_{2m+1})\} \\ &= \max\{d(y_{n+1}, y_n), d(y_{n+2}, y_{n+1}), d(y_{n+1}, y_n)\} \\ &= d(y_{n+1}, y_{n+2}) \text{ and } N(x_{2m+2}, x_{2m+1}) = 0 \end{aligned}$$

From the inequality (7), we have
$sd(y_{n+1}, y_{n+2}) \leq \beta(d(y_{n+1}, y_{n+2}))d(y_{n+1}, y_{n+2}) < \frac{d(y_{n+1}, y_{n+2})}{s} < d(y_{n+1}, y_{n+2})$,
Therefore, $(s-1)d(y_{n+1}, y_{n+2}) \leq 0$ which implies that $y_{n+2} = y_{n+1} = y_n$.
In general, we have $y_{n+k} = y_n$ for $k = 0, 1, 2, \dots$.
Case (ii): n odd.
We write $n = 2m + 1$ for some $m \in \mathbb{N}$.
We now consider

$$\begin{aligned} sd(y_{n+1}, y_{n+2}) &= d(y_{2m+2}, y_{2m+3}) \\ &= d(Ax_{2m+2}, Bx_{2m+3}) \\ &\leq \beta(M(x_{2m+2}, x_{2m+3}))M(x_{2m+2}, x_{2m+3}) + LN(x_{2m+2}, x_{2m+3}) \end{aligned} \tag{8}$$

where

$$\begin{aligned} M(x_{2m+2}, x_{2m+3}) &= \max\{d(Sx_{2m+2}, Tx_{2m+3}), d(Ax_{2m+2}, Sx_{2m+2}), d(Bx_{2m+3}, Tx_{2m+3})\} \\ &= \max\{d(y_{2m+1}, y_{2m+2}), d(y_{2m+2}, y_{2m+1}), d(y_{2m+3}, y_{2m+2})\} \\ &= \max\{d(y_n, y_{n+1}), d(y_{n+1}, y_n), d(y_{n+2}, y_{n+1})\} \\ &= d(y_{n+1}, y_{n+2}) \text{ and} \end{aligned}$$

$$\begin{aligned} N(x_{2m+2}, x_{2m+3}) &= \min\{d(Ax_{2m+2}, Sx_{2m+2}), d(Ax_{2m+2}, Tx_{2m+3}), d(Bx_{2m+3}, Sx_{2m+2})\} \\ &= \min\{d(y_{2m+2}, y_{2m+1}), d(y_{2m+2}, y_{2m+2}), d(y_{2m+3}, y_{2m+1})\} \\ &= 0 \end{aligned}$$

From the inequality (8), We have
$sd(y_{n+1}, y_{n+2}) \leq \beta(d(y_{n+1}, y_{n+2}))d(y_{n+1}, y_{n+2}) < \frac{d(y_{n+1}, y_{n+2})}{s}$,
which is a contradiction if $y_{n+1} \neq y_{n+2}$
Therefore $y_{n+2} = y_{n+1} = y_n$.
In general, we have $y_{n+k} = y_n$ for $k = 1, 2, 3, \dots$.
From Case (i) and Case (ii), we have $y_{n+k} = y_n$ for $k = 0, 1, 2, \dots$.
Therefore, $\{y_{n+k}\}$ is a constant sequence and hence $\{y_n\}$ is b- Cauchy.
Now we assume that $y_n \neq y_{n+1}$, for all $n \in \mathbb{N}$.
If n is odd, then $n = 2m + 1$ for some $m \in \mathbb{N}$.

We now consider

$$\begin{aligned} sd(y_{n+1}, y_{n+2}) &= sd(y_{2m+2}, y_{2m+3}) \\ &= sd(Ax_{2m+2}, Bx_{2m+3}) \\ &\leq \beta(M(x_{2m+2}, x_{2m+3}))M(x_{2m+2}, x_{2m+3}) + LN(x_{2m+2}, x_{2m+3}) \end{aligned} \tag{9}$$

where

$$\begin{aligned} M(x_{2m+2}, x_{2m+3}) &= \max\{d(Sx_{2m+2}, Tx_{2m+3}), d(Ax_{2m+2}, Sx_{2m+2}), d(Bx_{2m+3}, Tx_{2m+3})\} \\ &= \max\{d(y_{2m+1}, y_{2m+2}), d(y_{2m+2}, y_{2m+1}), d(y_{2m+3}, y_{2m+2})\} \\ &= \max\{d(y_{2m+1}, y_{2m+2}), d(y_{2m+2}, y_{2m+3})\} \\ &= \max\{d(y_{n+1}, y_n), d(y_{n+1}, y_{n+2})\} \text{ and} \end{aligned}$$

$$\begin{aligned} N(x_{2m+2}, x_{2m+3}) &= \min\{d(Ax_{2m+2}, Sx_{2m+2}), d(Ax_{2m+2}, Tx_{2m+3}), d(Bx_{2m+3}, Sx_{2m+2})\} \\ &= \min\{d(y_{2m+2}, y_{2m+1}), d(y_{2m+2}, y_{2m+2}), d(y_{2m+3}, y_{2m+1})\} \\ &= 0 \end{aligned}$$

From the inequality (9), we have
$sd(y_{n+1}, y_{n+2}) \leq \beta(\max\{d(y_n, y_{n+1}), d(y_{n+1}, y_{n+2})\}) \max\{d(y_n, y_{n+1}), d(y_{n+1}, y_{n+2})\}$
If $\max\{d(y_n, y_{n+1}), d(y_{n+1}, y_{n+2}) = d(y_{n+1}, y_{n+2})$ then we get
$sd(y_{n+1}, y_{n+2}) \leq \beta(d(y_{n+1}, y_{n+2}))d(y_{n+1}, y_{n+2}) < \frac{d(y_{n+1}, y_{n+2})}{s}$,
it is a contradiction.
Therefore $\max\{d(y_n, y_{n+1}), d(y_{n+1}, y_{n+2}) = d(y_n, y_{n+1})$.
From the inequality (9), we get that

$$sd(y_{n+1}, y_{n+2}) \leq \beta(d(y_n, y_{n+1}))d(y_n, y_{n+1}) \tag{10}$$

which implies that $d(y_{n+1}, y_{n+2}) \leq d(y_n, y_{n+1})$.
Similarly, we can prove that $d(y_{n+1}, y_{n+2}) \leq d(y_n, y_{n+1})$ whenever n is even.
Therefore, $\{d(y_n, y_{n+1})\}$ is a monotone decreasing sequence which bounded below by 0.
So, there exists $r \geq 0$ such that $\lim\limits_{n\to\infty} d(y_n, y_{n+1}) = r$.
If $r > 0$, then from (10), we have
$sd(y_{n+1}, y_{n+2}) \leq \beta(d(y_n, y_{n+1}))$.
On letting limit superior as $n \to \infty$, we get
$sr \leq \limsup\limits_{n\to\infty} \beta(d(y_n, y_{n+1}))r$
implies that $\frac{1}{s} \leq 1 \leq \limsup\limits_{n\to\infty} \beta(d(y_n, y_{n+1})) \leq \frac{1}{s}$
which implies that $\limsup\limits_{n\to\infty} \beta(d(y_n, y_{n+1})) = \frac{1}{s}$.
Since $\beta \in \mathfrak{F}$, we have $\lim\limits_{n\to\infty} d(y_n, y_{n+1}) = 0$,
which is a contradiction.
Therefore $r = 0$.
We now prove that $\{y_n\}$ is b-Cauchy.
It is sufficient to show that $\{y_{2n}\}$ is b-Cauchy in X.
Otherwise, there is an $\epsilon > 0$ and there exists sequences $\{2m_k\}, \{2n_k\}$ with $2n_k > 2m_k > k$ such that

$$d(y_{2m_k}, y_{2n_k}) \geq \epsilon \text{ and } d(y_{2m_k}, y_{2n_k-2}) < \epsilon. \tag{11}$$

From the inequality (1), (11) and by b-triangular inequality, we have

$$\begin{aligned}\epsilon &\leq d(y_{2m_k}, y_{2n_k})\\ &\leq s[d(y_{2n_k}, y_{2n_k-1}) + d(y_{2n_k-1}, y_{2m_k})]\\ &= sd(y_{2n_k}, y_{2n_k-1}) + sd(Ax_{2m_k}, Bx_{2n_k-1})\\ &\leq sd(y_{2n_k}, y_{2n_k-1}) + \beta(M(x_{2m_k}, x_{2n_k-1}))M(x_{2m_k}, x_{2n_k-1}) + LN(x_{2m_k}, x_{2n_k-1})\end{aligned} \tag{12}$$

where

$$\begin{aligned}M(x_{2m_k+1}, x_{2n_k}) &= \max\{d(Sx_{2n_k}, Tx_{2m_k+1}), d(Ax_{2n_k}, Sx_{2n_k}), d(Bx_{2m_k+1}, Tx_{2m_k+1})\}\\ &= \max\{d(y_{2n_k-1}, y_{2m_k}), d(y_{2n_k}, y_{2n_k-1}), d(y_{2m_k+1}, y_{2m_k})\} \text{ and}\\ N(x_{2m_k}, x_{2n_k-1}) &= \min\{d(Ax_{2n_k}, Sx_{2n_k}), d(Ax_{2n_k}, Tx_{2m_k+1}), d(Bx_{2m_k+1}, Sx_{2n_k})\}.\end{aligned}$$

On taking limit superior as $k \to \infty$ on $M(x_{2m_k}, x_{2n_k-1})$ and $N(x_{2m_k}, x_{2n_k-1})$,we get
$\limsup\limits_{k\to\infty} M(x_{2m_k}, x_{2n_k-1}) = \limsup\limits_{k\to\infty} d(y_{2m_k}, y_{2n_k-1})$
and $\limsup\limits_{k\to\infty} N(x_{2m_k}, x_{2n_k-1}) = 0$
From the b-triangular inequality, we have
$d(y_{2m_k}, y_{2n_k-1}) \leq s[d(y_{2m_k}, y_{2n_k-2}) + d(y_{2n_k-2}, y_{2n_k-1})]$.
On letting limit superior as $k \to \infty$ and using (11) in the above inequality, we get
$\limsup\limits_{k\to\infty} d(y_{2m_k}, y_{2n_k-1}) \leq s\epsilon$.
On taking limit superior as $k \to \infty$ in (12), we get

$$\begin{aligned}\epsilon &\leq \limsup_{k\to\infty}[sd(y_{2n_k}, y_{2n_k-1}) + \beta(M(x_{2m_k}, x_{2n_k-1}))M(x_{2m_k}, x_{2n_k-1}) + LN(x_{2m_k}, x_{2n_k-1})]\\ &= \limsup_{k\to\infty} \beta(M(x_{2m_k}, x_{2n_k-1})) \limsup_{k\to\infty} M(x_{2m_k}, x_{2n_k-1}) + L \limsup_{k\to\infty} N(x_{2m_k}, x_{2n_k-1})]\\ &= \limsup_{k\to\infty} \beta(M(x_{2m_k}, x_{2n_k-1})) \limsup_{k\to\infty} M(y_{2m_k}, y_{2n_k-1}).\end{aligned}$$

Therefore $\epsilon \leq \limsup\limits_{k\to\infty} \beta(M(x_{2m_k}, x_{2n_k-1}))s\epsilon$ implies that
$\frac{1}{s} \leq \limsup\limits_{k\to\infty} \beta(M(x_{2m_k}, x_{2n_k-1})) \leq \frac{1}{s}$ which implies that
$\limsup\limits_{k\to\infty} \beta(M(x_{2m_k}, x_{2n_k-1})) = \frac{1}{s}$.
Since $\beta \in \mathfrak{F}$, we have
$\lim\limits_{k\to\infty} M(x_{2m_k}, x_{2n_k-1}) = 0$. i.e, $\lim\limits_{k\to\infty} d(y_{2m_k}, y_{2n_k-1}) = 0$.
From the inequality (11) and by using b-triangular inequality, we get
$\epsilon \leq d(y_{2m_k}, y_{2n_k}) \leq s[d(y_{2m_k}, y_{2n_k-1}) + d(y_{2n_k-1}, y_{2n_k})]$.
By taking limit superior as $k \to \infty$, we get
$0 < \epsilon \leq \limsup\limits_{k\to\infty} d(y_{2m_k}, y_{2n_k}) \leq 0$.
Therefore $\lim\limits_{k\to\infty} d(y_{2m_k}, y_{2n_k}) = 0$,
it is a contradiction.
Therefore, $\{y_n\}$ is a b-Cauchy sequence in X.

The following is the main result of this paper.

Theorem 2.6 *Let A, B, S and T be selfmaps on a complete b-metric space (X, d) and satisfy* (2) *and the maps are Geraghty-Berinde type contraction maps . If the pairs (A, S) and (B, T) are weakly compatible and one of the range sets $S(X), T(X), A(X)$ and $B(X)$ is closed, then for any $x_0 \in X$, the sequence $\{y_n\}$ defined by* (3) *is b-Cauchy in X and $\lim_{n\to\infty} y_n = z$(say), $z \in X$ and z is the unique common fixed point of A, B, S and T.*

Proof. By Lemma 2.5, the sequence $\{y_n\}$ is b-Cauchy in X.
Since X is b-complete, there exists $z \in X$ such that $\lim_{n\to\infty} y_n = z$.
Then

$$\begin{cases} \lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} = z \text{ and} \\ \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z. \end{cases} \tag{13}$$

We now consider the following four cases.
<u>*Case* (i)</u>. $S(X)$ is closed.
In this case $z \in S(X)$ and there exists $t \in X$ such that $z = St$.
Now we prove that $At = z$.
Suppose that $At \neq z$.
We now consider

$$sd(At, Bx_{2n+1}) \leq \beta(M(t, x_{2n+1}))M(t, x_{2n+1}) + LN(t, x_{2n+1}) \tag{14}$$

where $M(t, x_{2n+1}) = \max\{d(St, Tx_{2n+1}), d(At, St), d(Bx_{2n+1}, Tx_{2n+1})\}$
and $N(t, x_{2n+1}) = \min\{d(At, St), d(At, Tx_{2n+1}), d(Bx_{2n+1}, st)\}$
On letting limitsuperior as $n \to \infty$, using Lemma 1.11 and (13), we get

$$\limsup_{n\to\infty} M(t, x_{2n+1}) = d(At, z) \quad \text{and} \quad \limsup_{n\to\infty} N(t, x_{2n+1}) = 0. \tag{15}$$

Since, $\frac{1}{s}(sd(At, z)) \leq \liminf_{n\to\infty}(sd(At, Bx_{2n+1})) \leq \limsup_{n\to\infty}(sd(At, Bx_{2n+1})) \leq s(sd(At, Bx_{2n+1}))$
On letting limit superior as $n \to \infty$ in (14) and using (15), we get
$\frac{1}{s}(sd(At, z)) \leq \limsup_{n\to\infty}(d(At, Bx_{2n+1}))$
$\leq \limsup_{n\to\infty} \beta(M(t, x_{2n+1})) \limsup_{n\to\infty} M(t, x_{2n+1}) + L \limsup_{n\to\infty} N(t, x_{2n+1})$.
Therefore $d(At, z) \leq \limsup_{n\to\infty} \beta(M(t, x_{2n+1}))d(At, z)$
which implies that $\frac{1}{s} \leq 1 \leq \limsup_{n\to\infty} \beta(M(t, x_{2n+1})) \leq \frac{1}{s}$
implies that $\limsup_{n\to\infty} \beta(M(t, x_{2n+1})) = \frac{1}{s}$.
Since $\beta \in \mathfrak{F}$, we have $\lim_{n\to\infty} M(t, x_{2n+1}) = 0$,
which is a contradiction.
Hence $At = z$.
Therefore, $At = z = St$.
Since the pair (A, S) is weakly compatible and $At = St$, we have $ASt = SAt$.
i.e., $Az = Sz$.
Now, we prove that $Az = z$.
Suppose $Az \neq z$
we now consider

$$sd(Az, Bx_{2n+1}) \leq \beta(M(z, x_{2n+1}))M(z, x_{2n+1}) + LN(z, x_{2n+1}) \tag{16}$$

where

$$\begin{cases} M(z, x_{2n+1}) = \max\{d(Sz, Tx_{2n+1}), d(Az, Sz), d(Bx_{2n+1}, Tx_{2n+1})\} \text{ and} \\ N(z, x_{2n+1}) = \min\{d(Az, Sz), d(Az, Tx_{2n+1}), d(Bx_{2n+1}, Sz)\} \end{cases} \tag{17}$$

On taking limitsuperior as $n \to \infty$, using the inequality (17) and Lemma 1.11, we get

$$\limsup_{n\to\infty} M(z, x_{2n+1}) \le sd(Az, z) \text{ and } \limsup_{n\to\infty} N(z, x_{2n+1}) = 0 \tag{18}$$

On letting limitsuperior as $n \to \infty$ in (16) and using (18), we get

$\frac{1}{s}(sd(Az, z)) \le \limsup_{n\to\infty}(d(Az, Bx_{2n+1}))$

$\le \limsup_{n\to\infty} \beta(M(z, x_{2n+1})) \limsup_{n\to\infty} M(z, x_{2n+1}) + L \limsup_{n\to\infty} N(z, x_{2n+1})$

implies that $d(Az, z) \le \limsup_{n\to\infty} \beta(M(z, x_{2n+1}))sd(Az, z)$

which implies that $\frac{1}{s} \le \limsup_{n\to\infty} \beta(M(z, x_{2n+1})) \le \frac{1}{s}$

Therefore $\limsup_{n\to\infty} \beta(M(z, x_{2n+1})) = \frac{1}{s}$

Since $\beta \in \mathfrak{F}$, we have $\lim_{n\to\infty} M(z, x_{2n+1}) = 0$,

it is a contradiction.

Hence, $Az = z$.

Therefore $Az = Sz = z$.

Hence, z is a common fixed point of A and S.

By Lemma 2.4, we get that z is a unique common fixed point of A, B, S and T.

Case (ii). $T(X)$ is closed.

In this case $z \in T(X)$ and there exists $u \in X$ such that $z = Tu$.

Now we claim that $Bu = z$. Suppose that $Bu \neq z$.

We now consider

$$sd(Ax_{2n+2}, Bu) \le \beta(M(x_{2n+2}, u))M(x_{2n+2}, u) + LN(x_{2n+2}, u) \tag{19}$$

where

$$\begin{cases} M(x_{2n+2}, u) = \max\{d(Sx_{2n+2}, Tu), d(Ax_{2n+2}, Sx_{2n+2}), d(Bu, Tu)\} \text{ and} \\ N(x_{2n+2}, u) = \min\{d(Ax_{2n+2}, Sx_{2n+2}), d(Ax_{2n+2}, Tu), d(Bu, Sx_{2n+2})\} \end{cases} \tag{20}$$

On letting limit superior as $n \to \infty$ in (20) , using the inequality (13), we get

$$\limsup_{n\to\infty} M(x_{2n+2}, u) = d(z, Bu) \text{ and } \limsup_{n\to\infty} N(x_{2n+2}, u) = 0. \tag{21}$$

On letting limit superior as $n \to \infty$ in (19) , using the inequality (21) and Lemma 1.11, we get

$\frac{1}{s}(sd(z, Bu)) \le \limsup_{n\to\infty}(sd(Ax_{2n+2}, Bu))$

$\le \limsup_{n\to\infty} \beta(M(x_{2n+2}, u)) \limsup_{n\to\infty} M(x_{2n+2}, u) + L \limsup_{n\to\infty} N(x_{2n+2}, u)$

$= \limsup_{n\to\infty} \beta(M(x_{2n+2}, u))d(z, Bu)$

which implies that

$\frac{1}{s} \le 1 \le \limsup_{n\to\infty} \beta(M(x_{2n+2}, u)) \le \frac{1}{s}$ implies that $\limsup_{n\to\infty} \beta(M(x_{2n+2}, u)) = \frac{1}{s}$

Since $\beta \in \mathfrak{F}$, we get $\lim_{n\to\infty} M(x_{2n+2}, u) = 0$,

which is a contradiction.

Hence $Bu = z$.

Therefore, $Bu = z = Tu$.

Since the pair (B,T) is weakly compatible and $Bu = Tu$, we have $BTu = TBu$. i.e., $Bz = Tz$.
We now prove that $Bz = z$. Suppose that $Bz \neq z$.
We now consider

$$sd(Ax_{2n+2}, Bz) \leq \beta(M(x_{2n+2}, z))M(x_{2n+2}, z) + LN(x_{2n+2}, z) \tag{22}$$

where

$$\begin{cases} M(x_{2n+2}, z) = \max\{d(Sx_{2n+2}, Tz), d(Ax_{2n+2}, Sx_{2n+2}), d(Bz, Tz)\} \text{ and} \\ N(x_{2n+2}, z) = \min\{d(Ax_{2n+2}, Bx_{2n+2}), d(Ax_{2n+2}, Tz), d(Bz, Sx_{2n+2})\} \end{cases} \tag{23}$$

On taking limit superior as $n \to \infty$ in (23) , using the Lemma 1.11 and the inequality (13), we get

$$\limsup_{n\to\infty} M(x_{2n+2}, z) \leq sd(z, Bz) \text{ and } \limsup_{n\to\infty} N(x_{2n+2}, z) = 0 \tag{24}$$

On letting limit superior as $n \to \infty$ in (22) , using the inequality (13) and Lemma 1.11, we get

$$\begin{aligned}\tfrac{1}{s}(sd(z, Bz)) &\leq \limsup_{n\to\infty}(sd(Ax_{2n+2}, Bz)) \\ &\leq \limsup_{n\to\infty} \beta(M(x_{2n+2}, z)) \limsup_{n\to\infty} M(x_{2n+2}, z) + L \limsup_{n\to\infty} N(x_{2n+2}, z) \\ &\leq \limsup_{n\to\infty} \beta(M(x_{2n+2}, z))zd(z, Bz),\end{aligned}$$

which implies that
$\frac{1}{s} \leq \limsup_{n\to\infty} \beta(M(x_{2n+2}, z)) \leq \frac{1}{s}$ implies that $\limsup_{n\to\infty} \beta(M(x_{2n+2}, z)) = \frac{1}{s}$.
Since $\beta \in \mathfrak{F}$, we get $\lim_{n\to\infty}(M(x_{2n+2}, z)) = 0$,
which is a contradiction.
Hence, $Bz = z = Tz$.
Therefore, z is a common fixed point of B and T.
By Lemma 2.4, we get that z is the unique common fixed point of A, B, S and T.
Case (iii). $A(X)$ is closed.
Since $z \in A(X) \subseteq T(X)$, there exists $p \in X$ such that $z = Tp$.
Now we show that $Bp = z$.
If $Bp \neq z$, then we consider

$$sd(Ax_{2n+2}, Bp) \leq \beta(M(x_{2n+2}, p))M(x_{2n+2}, p) + LN(x_{2n+2}, p) \tag{25}$$

where

$$\begin{cases} M(x_{2n+2}, p) = \max\{d(Sx_{2n+2}, Tp), d(Ax_{2n+2}, Sx_{2n+2}), d(Bp, Tp)\} \text{ and} \\ N(x_{2n+2}, z) = \min\{d(Ax_{2n+2}, Sx_{2n+2}), d(Ax_{2n+2}, Tp), d(Bp, Sx_{2n+2})\}. \end{cases} \tag{26}$$

On taking limit superior as $n \to \infty$ in (26) , using the Lemma 1.11 and the inequality (13), we get

$$\limsup_{n\to\infty} M(x_{2n+2}, p) = d(z, Bp) \text{ and } \limsup_{n\to\infty} N(x_{2n+2}, p) = 0. \tag{27}$$

On letting limit superior as $n \to \infty$ in (25) , using the inequality (27) and Lemma 1.11, we get

$$\begin{aligned}\tfrac{1}{s}(sd(z, Bp)) &\leq \limsup_{n\to\infty}(sd(Ax_{2n+2}, Bp)) \\ &\leq \limsup_{n\to\infty} \beta(M(x_{2n+2}, p)) \limsup_{n\to\infty} M(x_{2n+2}, p) + L \limsup_{n\to\infty} N(x_{2n+2}, p) \\ &= \limsup_{n\to\infty} \beta(M(x_{2n+2}, p))d(z, Bp),\end{aligned}$$

which implies that

$\frac{1}{s} \leq 1 \leq \limsup_{n\to\infty} \beta(M(x_{2n+2}, p)) \leq \frac{1}{s}$ implies that $\limsup_{n\to\infty} \beta(M(x_{2n+2}, p)) = \frac{1}{s}$.
Since $\beta \in \mathfrak{F}$, we get $\lim_{n\to\infty}(M(x_{2n+2}, p)) = 0$,
which is a contradiction.
Hence $Bp = z = Tp$.
Now, by Case (ii), the conclusion of the theorem follows.
Case (iv). $B(X)$ is closed.
Since $z \in B(X) \subseteq S(X)$, there exists $r \in X$ such that $z = Sr$.
Now we show that $Ar = z$.
If $Ar \neq z$, then we consider

$$sd(Ar, Bx_{2n+1}) \leq \beta(M(r, x_{2n+1}))M(r, x_{2n+1}) + LN(r, x_{2n+1}) \tag{28}$$

where
$M(r, x_{2n+1}) = \max\{d(Sr, Tx_{2n+1}), d(Bx_{2n+1}, Tx_{2n+1}), d(Ar, Sr)\}$
and $N(r, x_{2n+1}) = \min\{d(Ar, Sr), d(Ar, Tx_{2n+1}), d(Bx_{2n+1}, Sr)\}$
On letting limit superior as $n \to \infty$, using the Lemma 1.11 and the inequality (13), we get

$$\limsup_{n\to\infty} M(r, x_{2n+1}) = d(Ar, z) \text{ and } \limsup_{n\to\infty} N(r, x_{2n+1}) = 0. \tag{29}$$

On taking limit superior as $n \to \infty$ in (28) , using the Lemma 1.11 and inequality (29) , we get

$$\begin{aligned}
\frac{1}{s}(sd(Ar, z)) &\leq \limsup_{n\to\infty}(sd(Ar, Bx_{2n+1})) \\
&\leq \limsup_{n\to\infty} \beta(M(r, x_{2n+1})) \limsup_{n\to\infty} M(r, x_{2n+1}) + L \limsup_{n\to\infty} N(r, x_{2n+1}) \\
&= \limsup_{n\to\infty} \beta(M(r, x_{2n+1}))d(Ar, z),
\end{aligned}$$

Therefore $\frac{1}{s} \leq 1 \leq \limsup_{n\to\infty} \beta(M(r, x_{2n+1})) \leq \frac{1}{s}$
which implies that $\limsup_{n\to\infty} \beta(M(r, x_{2n+1})) = \frac{1}{s}$.
Since $\beta \in \mathfrak{F}$, we get $\lim_{n\to\infty}(M(r, x_{2n+1})) = 0$,
it is a contradiction.
Therefore $Ar = z = Sr$.
Therefore by Case *(i)*, the conclusion of the theorem follows.

Theorem 2.7 *Let A, B, S and T be selfmaps on a b-metric space (X, d) and satisfy the inequality* (2) and are Geraghty-Berinde type contraction maps.*If the pairs (A, S) and (B, T) are weakly compatible and either one of the set $(S(X), d)$, $(T(X), d)$, $(A(X), d)$ (or) $(B(X), d)$ is complete, then for any $x_0 \in X$, the sequence $\{y_n\}$ defined by* (3) *is b- Cauchy in X and $\lim_{n\to\infty} y_n = z$(say), $z \in X$ and z is the unique common fixed point of A, B, S and T.*

Proof. By Lemma 2.5, the sequence $\{y_n\}$ is b-Cauchy in X.
Since $S(X)$ is complete, there exists $z \in S(X)$ such that $\lim_{n\to\infty} y_n = z$.
Thus,

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} = z \text{ and } \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z. \tag{30}$$

Since $z \in S(X)$, there exists $u \in X$ such that $z = Su$.
We now prove that $Au = z$. Suppose that $Au \neq z$. We now consider

$$sd(Au, Bx_{2n+1}) \leq \beta(M(u, x_{2n+1}))M(u, x_{2n+1}) + LN(u, x_{2n+1}) \tag{31}$$

where $M(u, x_{2n+1}) = \max\{d(Su, Tx_{2n+1}), d(Au, Su), d(Bx_{2n+1}, Tx_{2n+1})\}$
and $N(u, x_{2n+1}) = \min\{d(Au, Su), d(Au, Tx_{2n+1}), d(Bx_{2n+1}, Su)\}$
On letting limit superior as $n \to \infty$, using Lemma 1.11 and the inequalityv(30), we get

$$\limsup_{n\to\infty} M(u, x_{2n+1}) = d(Au, z) \text{ and } \limsup_{n\to\infty} N(u, x_{2n+1}) = 0. \tag{32}$$

Since, $\frac{1}{s}(sd(Au, z)) \leq \liminf_{n\to\infty}(sd(Au, Bx_{2n+1})) \leq \limsup_{n\to\infty}(sd(Au, Bx_{2n+1})) \leq s(s(d(Au, z)))$.
On letting limit superior as $n \to \infty$ in (31) and using Lemma 1.11 and the inequality (32), we get

$$\begin{aligned}\frac{1}{s}(sd(Au, z)) &\leq \limsup_{n\to\infty}(sd(Au, Bx_{2n+1}))\\ &\leq \limsup_{n\to\infty} \beta(M(u, x_{2n+1})) \limsup_{n\to\infty} M(u, x_{2n+1}) + L \limsup_{n\to\infty} N(u, x_{2n+1})\\ &= \limsup_{n\to\infty} \beta(M(u, x_{2n+1}))d(Au, z).\end{aligned}$$

Therefore, $\frac{1}{s} \leq 1 \leq \limsup_{n\to\infty} \beta(M(u, x_{2n+1})) \leq \frac{1}{s}$ which implies that $\limsup_{n\to\infty} \beta(M(u, x_{2n+1})) = \frac{1}{s}$.
Since $\beta \in \mathfrak{F}$, we have $\lim_{n\to\infty} M(u, x_{2n+1}) = 0$,
which is a contradiction.
Hence, $Au = z = Su$. Since the pair (A, S) is weakly compatible and $Au = Su$, we have
$ASu = SAu$. i.e., $Az = Sz$.
Now we prove that $Az = z$. If $Az \neq z$, then

$$sd(Az, Bx_{2n+1}) \leq \beta(M(z, x_{2n+1}))M(z, x_{2n+1}) + LN(z, x_{2n+1}) \tag{33}$$

where $M(z, x_{2n+1}) = \max\{d(Sz, Tx_{2n+1}), d(Az, Sz), d(Bx_{2n+1}, Tx_{2n+1})\}$
and $N(z, x_{2n+1}) = \min\{d(Az, Sz), d(Az, Tx_{2n+1}), d(Bx_{2n+1}, Sz)\}$
On leting limit superior as $n \to \infty$, using the inequality (30) and Lemma 1.11, we get

$$\limsup_{n\to\infty} M(z, x_{2n+1}) \leq sd(Az, z) \text{ and } \limsup_{n\to\infty} N(z, x_{2n+1}) = 0. \tag{34}$$

On taking limit superior as $n \to \infty$ in (33) and using (34) and by Lemma 1.11, we get

$$\begin{aligned}\frac{1}{s}(sd(Az, z)) &\leq \limsup_{n\to\infty}(sd(Az, Bx_{2n+1}))\\ &\leq \limsup_{n\to\infty} \beta(M(z, x_{2n+1})) \limsup_{n\to\infty} M(z, x_{2n+1}) + L \limsup_{n\to\infty} N(z, x_{2n+1})\\ &\leq \limsup_{n\to\infty} \beta(M(z, x_{2n+1}))sd(Az, z).\end{aligned}$$

Therefore, $\frac{1}{s} \leq \limsup_{n\to\infty} \beta(M(z, x_{2n+1})) \leq \frac{1}{s}$
which implies that $\limsup_{n\to\infty} \beta(M(z, x_{2n+1})) = \frac{1}{s}$.
Since $\beta \in \mathfrak{F}$, we have $\lim_{n\to\infty} M(z, x_{2n+1}) = 0$, it is a contradiction. Hence, $Az = z$.
Therefore $Az = Sz = z$. Hence, z is a common fixed point of A and S.
By Lemma 2.4, we get that z is a unique common fixed point of A, B, S and T.

In a similar way, it is easy to see that z is the unique common fixed point of A, B, S and T whenever $T(X)$ or $A(X)$ or $B(X)$ is complete.

Theorem 2.8 Let A, B, S and T be selfmaps on a complete b-metric space (X, d) and satisfy (2) and they are Geraghty-Berinde type contraction maps. Further, assume that either

(i) the pair (A, S) is reciprocally continuous and compatible,and (B, T) is a pair of weakly compatible maps

(ii) the pair (B, T) is reciprocally continuous and compatible,and (A, S) is a pair of weakly compatible maps.

Then A, B, S and T have a unique common fixed point in X.

Proof. By Lemma 2.5, for each $x_0 \in X$, the sequence $\{y_n\}$ defined by (3) is b-Cauchy in X.
Since X is b-complete, then there exists $z \in X$ such that $\lim_{n\to\infty} y_n = z$.
Consequently, the subsequences $\{y_{2n}\}$ and $\{y_{2n+1}\}$ are also converges to $z \in X$, we have

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} = z \text{ and } \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z. \tag{35}$$

First, we assume that (i) holds.
Since (A, S) is reciprocally continuous, it follows that
$\lim_{n\to\infty} ASx_{2n+2} = Az$ and $\lim_{n\to\infty} SAx_{2n+2} = Sz$.
Since (A, S) is compatible, we have
$\lim_{n\to\infty} d(ASx_{2n+2}, SAx_{2n+2}) = 0$
i.e, $\limsup_{n\to\infty} d(ASx_{2n+2}, SAx_{2n+2}) = \liminf_{n\to\infty} d(ASx_{2n+2}, SAx_{2n+2}) = 0$
By Lemma 1.11, we have

$$\begin{aligned}\frac{1}{s^2} d(Az, Sz) &\leq \liminf_{n\to\infty} d(ASx_{2n+2}, SAx_{2n+2})\\ &\leq \limsup_{n\to\infty} d(ASx_{2n+1}, SAx_{2n+2}) \leq s^2 d(Az, Sz)\end{aligned}$$

which implies that $\frac{1}{s^2} d(Az, Sz) = 0$ which gives that $Az = Sz$.
Since $A(X) \subseteq T(X)$, there exists $u \in X$ such that $Az = Tu$.
Therefore, $Az = Sz = Tu$.
Now, we prove that $Az = Bu$. Suppose that $Az \neq Bu$.
We now consider

$$sd(Az, Bu) \leq \beta(M(z, u))M(z, u) + LN(z, u) \tag{36}$$

where $M(z, u) = \max\{d(Sz, Tu), d(Az, Sz), d(Bu, Tu)\} = d(Az, Bu)$ and
$N(z, u) = \min\{d(Az, Sz), d(Az, Tu), d(Bu, Sz)\} = 0$
Therefore from the inequality (36),we get
$sd(Az, Bu) \leq \beta(d(Az, Bu))d(Az, Bu) < d(Az, Bu)$,
which is a contradiction.
Therefore, $Az = Bu = Sz = Tu$.
Since the pair (A, S) is weakly compatible and $Az = Sz$, we have $ASz = SAz$.
i.e., $AAz = SAz$.
Now we prove that $AAz = Az$.
If $AAz \neq Az$, then
We consider

$$sd(AAz, Az) = sd(AAz, Bu) \leq \beta(M(Az, u))M(Az, u) + LN(Az, u) \tag{37}$$

where $M(Az, u) = \max\{d(SAz, Tu), d(AAz, SAz), d(Bu, Tu)\} = d(AAz, Az)$ and
$N(Az, u) = \min\{d(AAz, SAz), d(AAz, Tu), d(Bu, SAz)\} = 0$

Therefore from the inequality (37), we get
$sd(AAz, Az) \leq \beta(d(AAz, Az))d(AAz, Az) < d(AAz, Az)$,
it is a contradiction.
This implies that $AAz = Az = SAz$.
Therefore Az is a common fixed point of A and S.
Since (B, T) is weakly compatible and $Bu = Tu$,
we have $BTu = TBu$. i.e , $BAz = TAz$.
We now prove that $BAz = Az$.
Suppose that $BAz \neq Az$.
Now, we consider

$$sd(Az, BAz) \leq \beta(M(z, Az))M(z, Az) + LN(z, Az) \tag{38}$$

where $M(z, Az) = \max\{d(Sz, TAz), d(Az, Sz), d(BAz, TAz)\} = d(Az, BAz)$
and $N(z, Az) = \min\{d(Az, Sz), d(Az, BAz), d(BAz, Sz)\} = 0$.
From the inequality (38), we have
$sd(Az, BAz) \leq \beta(d(Az, BAz))d(Az, BAz) < d(Az, BAz)$,
which is a contradiction.
Therefore $BAz = TAz = Az = AAz = SAz$.
Hence Az is a common fixed point of $A,\ B,\ S$ and T.
We now show that $Az = z$. Suppose that $Az \neq z$.
We now consider

$$sd(Az, Bx_{2n+1}) \leq \beta(M(z, x_{2n+1}))M(z, x_{2n+1}) + LN(z, x_{2n+1}) \tag{39}$$

where $M(z, x_{2n+1}) = \max\{d(Sz, Tx_{2n+1}), d(Az, Sz), d(Bx_{2n+1}, Tx_{2n+1})\}$ and
$N(z, x_{2n+1}) = \min\{d(Az, Sz), d(Az, Tx_{2n+1}), d(Bx_{2n+1}, Sz)\}$.
On letting limit superior as $n \to \infty$, we get
$\limsup_{n\to\infty} M(z, x_{2n+1}) \leq sd(Az, z)$ and $\limsup_{n\to\infty} N(z, x_{2n+1}) = 0$.
On taking limit superior as $n \to \infty$ in (39), using the Lemma 1.11 and the inequality (35), we get

$$\begin{aligned} s(\tfrac{1}{s}d(Az, z)) &\leq s(\limsup_{n\to\infty} d(Az, Bx_{2n+1})) \\ &\leq \limsup_{n\to\infty} \beta(M(z, x_{2n+1})) \limsup_{n\to\infty} M(z, x_{2n+1}) + L \limsup_{n\to\infty} N(z, x_{2n+1}) \\ &\leq \limsup_{n\to\infty} \beta(M(z, x_{2n+1}))sd(Az, z) \end{aligned}$$

which implies that
$\frac{1}{s} \leq \limsup_{n\to\infty} \beta(M(z, x_{2n+1})) \leq \frac{1}{s}$ implies that $\limsup_{n\to\infty} \beta(M(z, x_{2n+1})) = \frac{1}{s}$.
Since $\beta \in \mathfrak{F}$, we have $\lim_{n\to\infty} M(z, x_{2n+1}) = 0$,
which is a contradiction.
Therefore $Az = z$.
Therefore z is a common fixed point of A, B, S and T.

In a similar way, under the assumption (ii), we obtain the existence of common fixed point of A, B, S and T.

Uniqueness of common fixed point follows from the inequality.

Theorem 2.9 Let (X, d) be a b-metric space with coefficient $s \geq 1$. Assumr that $A, B, S, T : X \to X$ are Geraghty-Berinde type contraction maps and satisfy (2). Suppose that one of the pairs (A, S) and (B, T) satisfies the b-(E.A)-property and that one of the subspace $A(X), B(X), S(X)$ and $T(X)$ is b-closed in X. Then the pairs (A, S) and (B, T) have a point of coincidence in X. Moreover, if the pairs (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point in X.

Proof. We first assume that the pair (A, S) satisfies the b-(E.A)-property. So there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = q \text{ for some } q \in X \tag{40}$$

Since $A(X) \subseteq T(X)$, there exists a sequence $\{y_n\}$ in X such that $Ax_n = Ty_n$, and hence

$$\lim_{n\to\infty} Ty_n = q. \tag{41}$$

Now we show that $\lim\limits_{n\to\infty} By_n = q$. Suppose that $\lim\limits_{n\to\infty} By_n \neq q$.
From the inequality (1), we have

$$sd(d(Ax_n, By_n) \leq \beta(M(x_n, y_n))M(x_n, y_n) + LN(x_n, y_n) \tag{42}$$

where $M(x_n, y_n) = \max\{d(Sx_n, Ty_n), d(Ax_n, Sx_n), d(By_n, Ty_n)\}$ and
$N(x_n, y_n) = \min\{d(Ax_n, Sx_n), d(Ax_n, Ty_n), d(By_n, Sx_n)\}$.
We have $\frac{1}{s}d(q, By_n) \leq \liminf\limits_{n\to\infty} M(x_n, y_n) \leq \limsup\limits_{n\to\infty} M(x_n, y_n) \leq sd(q, By_n)$
and $\limsup\limits_{n\to\infty} N(x_n, y_n) = 0$.
By taking limit superior as $n \to \infty$ in (42), and using (40) and (41), we obtain
$\limsup\limits_{n\to\infty} sd(Ax_n, By_n) \leq \limsup\limits_{n\to\infty}(\beta(M(x_n, y_n))M(x_n, y_n)) + L \limsup\limits_{n\to\infty} N(x_n, y_n)$
which implies that
$s\frac{1}{s}d(q, By_n) \leq s \limsup\limits_{n\to\infty} d(Ax_n, By_n) \leq \limsup\limits_{n\to\infty}(\beta(M(x_n, y_n))sd(q, By_n)$.
Therefore $\frac{1}{s} \leq \limsup\limits_{n\to\infty}(\beta(M(x_n, y_n)) \leq \frac{1}{s}$
which implies that $\limsup\limits_{n\to\infty}(\beta(M(x_n, y_n)) = \frac{1}{s}$.
Since $\beta \in \mathfrak{F}$, we have $\lim\limits_{n\to\infty} M(x_n, y_n) = 0$,
it is a contradiction.

$$\lim_{n\to\infty} By_n = q. \tag{43}$$

Case (i). Assume that $T(X)$ is a b-closed in X.
In this case $q \in T(X)$, we can choose $r \in X$ such that $Tr = q$.
We now prove that $Br = q$. Suppose that $d(Br, q) > 0$.
From the inequality (1), we have

$$sd(d(Ax_{2n+2}, Br) \leq \beta(M(x_{2n+2}, r))M(x_{2n+2}, r) + LN(x_{2n+2}, r) \tag{44}$$

where $M(x_{2n+2}, r) = \max\{d(Sx_{2n+2}, Tr), d(Ax_{2n+2}, Sx_{2n+2}), d(Br, Tr)\}$ and
$N(x_{2n+2}, r) = \min\{d(Ax_{2n+2}, Sx_{2n+2}), d(Ax_{2n+2}, Tr), d(Br, Sx_{2n+2})\}$.
We have that $\frac{1}{s}d(q, Br) \leq \liminf\limits_{n\to\infty} M(x_{2n+2}, r) \leq \limsup\limits_{n\to\infty} M(x_{2n+2}, r) \leq sd(q, Br)$
and $\limsup\limits_{n\to\infty} N(x_{2n+2}, r) = 0$.
On letting limit superior as $n \to \infty$ in the inequality (44), using (40), (41), (43) and Lemma 1.11, we have

$s\frac{1}{s}d(q, Br) \leq \limsup\limits_{n\to\infty} d(Ax_{2n+2}, Br) \leq \limsup\limits_{n\to\infty} \beta(M(x_{2n+2}, r))d(q, Br)$
which implies that $\frac{1}{s} \leq 1 \leq \limsup\limits_{n\to\infty} \beta(M(x_{2n+2}, r)) \leq \frac{1}{s}$ implies that
$\limsup\limits_{n\to\infty} \beta(M(x_{2n+2}, r)) = \frac{1}{s}$.

Since $\beta \in \mathfrak{F}$, we have $\lim\limits_{n\to\infty} M(x_{2n+2}, r) = 0$,
it is a contradiction.
Therefore $Br = q$. Hence $Br = Tr = q$, so that q is a coincidence point of B and T.
Since $B(X) \subseteq S(X)$, we have $q \in S(X)$, there exists $z \in X$ such that $Sz = q = Br$.
Now we show that $Az = q$. Suppose $Az \neq q$.
From the inequality (1), we have

$$sd(Az, q) = sd(Az, Br) \leq \beta(M(z, r))M(z, r) + LN(z, r) \tag{45}$$

where $M(z, r) = \max\{d(Sz, Tr), d(Az, Sz), d(Br, Tr)\} = d(Az, q)$ and
$N(z, r) = \min\{d(Az, Sz), d(Az, Tr), d(Br, Sz)\} = 0$.
From the inequality (45), we have
$sd(Az, q) \leq \beta(d(Az, q))d(Az, q) < \frac{d(Az,q)}{s} < d(Az, q)$,
it is a contradiction.
Therefore $Az = Sz = q$ so that z is a coincidence point of A and S.
Since the pairs (A, S) and (B, T) are weakly compatible, we have $Aq = Sq$ and $Bq = Tq$.
Therefore q is also a coincidence point of the pairs (A, S) and (B, T).
We now show that q is a common fixed point of A, B, S and T.
Suppose $Aq \neq q$.
From the inequality (1), we have

$$sd(Aq, q) = sd(Aq, Br) \leq \beta(M(q, r))M(q, r) + LN(q, r) \tag{46}$$

where $M(q, r) = \max\{d(Sq, Tr), d(Aq, Sq), d(Br, Tr)\} = d(Aq, q)$ and
$N(q, r) = \min\{d(Aq, Sq), d(Aq, Tr), d(Br, Sq)\} = 0$.
From the inequality (46), we have
$sd(Aq, q) \leq \beta(d(Aq, q))d(Aq, q) < \frac{d(Aq,q)}{s} < d(Aq, q)$,
it is a contradiction.
Hence $Aq = q$. Therefore $Aq = Sq = q$ so that q is a common fixed point of A and S.
By Lemma 2.4, we get that q is a unique common fixed point of A, B, S and T.
Case (ii). Suppose $A(X)$ is b-closed.
In this case, we have $q \in A(X)$ and since $A(X) \subseteq T(X)$, we choose $r \in X$ such that $q = Tr$.
The proof follows as in Case (i).
Case (iii). Suppose $S(X)$ is b-closed.
We follow the argument similar as Case (i) and we get conclusion.
Case (iv). Suppose $B(X)$ is b-closed.
As in Case (ii), we get the conclusion.

For the case of (B, T) satisfies the b-(E.A)-property, we follow the argument similar to the case (A, S) satisfies the b-(E.A)-property.

3. Examples and corollaries

In this section, we draw some corollaries from the main results of Section 2 and provide examples in support of our results.

The importance of the class of Geraghty-Berinde type contraction maps is that this class properly includes the class of Geraghty contraction type maps.so that the class of Geraghty-Berinde type contraction maps is larger than the class of Geraghty contraction type maps, which illustrated in Example 3.1, Example 3.2 and Example 3.3.

The following is an example in support of Theorem 2.6.

Example 3.1 Let $X = [0, \infty)$ and let $d : X \times X \to [0, \infty)$ defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 4 & \text{if } x, y \in (0,1), \\ \frac{9}{2} + \frac{1}{x+y} & \text{if } x, y \in [1, \infty), \\ \frac{12}{5} & \text{otherwise.} \end{cases}$$

Then clearly (X, d) is a complete b-metric space with coefficient $s = \frac{25}{24}$.
We define $A, B, S, T : X \to X$ by

$A(x) = 1$ if $x \in [0, \infty)$, $B(x) = \begin{cases} \frac{5}{3} & \text{if } x \in [0,1) \\ 2x - 1 & \text{if } x \in [1, \infty) \end{cases}$

$S(x) = \begin{cases} x & \text{if } x \in [0,1) \\ \frac{1+x}{2} & \text{if } x \in [1, \infty) \end{cases}$ and $T(x) = \begin{cases} \frac{x^2}{2} & \text{if } x \in [0,1) \\ x^2 & \text{if } x \in [1, \infty). \end{cases}$

We define $\beta : [0, \infty) \to [0, \frac{1}{s})$ by $\beta(t) = \frac{24}{25}e^{-t}$. Then $\beta \in \mathfrak{F}$.
Clearly $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$. Also clear that $A(X)$ is closed.
The pairs (A, S) and (B, T) are weakly compatible.
We now verify the inequality (1). For this purpose we consider the following cases.

Case (i). Let $x, y \in [0, 1)$.
$d(Ax, By) = \frac{9}{2} + \frac{1}{x+y}, d(Ax, Sx) = \frac{12}{5}, d(By, Ty) = \frac{12}{5}$,
$d(Ax, Ty) = \frac{12}{5}, d(By, Sx) = \frac{12}{5}$,
$M(x, y) = \max\{d(Ax, By), d(Ax, Sx), d(By, Ty)\} = \frac{9}{2} + \frac{1}{x+y}$ and
$N(x, y) = \min\{d(Ax, Sx), d(Ax, Ty), d(By, Sx)\} = \frac{12}{5}$.
We now consider

$$\begin{aligned} sd(Ax, By) = \frac{25}{24}\left(\frac{9}{2} + \frac{1}{x+y}\right) &\leq \frac{24}{25}e^{-(\frac{9}{2}+\frac{1}{x+y})}\left(\frac{9}{2} + \frac{1}{x+y}\right) + L\left(\frac{12}{5}\right) \\ &\leq \beta(M(x, y))M(x, y) + LN(x, y). \end{aligned}$$

Case (ii). $x, y \in [1, \infty)$.
$d(Ax, By) = \frac{9}{2} + \frac{1}{x+y}, d(Ax, Sx) = \frac{9}{2} + \frac{1}{x+y}, d(By, Ty) = \frac{9}{2} + \frac{1}{x+y}$,
$d(Ax, Ty) = \frac{9}{2} + \frac{1}{x+y}, d(By, Sx) = \frac{9}{2} + \frac{1}{x+y}$,
$M(x, y) = \max\{d(Ax, By), d(Ax, Sx), d(By, Ty)\} = \frac{9}{2} + \frac{1}{x+y}$ and
$N(x, y) = \min\{d(Ax, Sx), d(Ax, Ty), d(By, Sx)\} = \frac{9}{2} + \frac{1}{x+y}$.
We now consider

$$\begin{aligned} sd(Ax, By) = \frac{25}{24}\left(\frac{9}{2} + \frac{1}{x+y}\right) &\leq \frac{24}{25}e^{-(\frac{9}{2}+\frac{1}{x+y})}\left(\frac{9}{2} + \frac{1}{x+y}\right) + L\left(\frac{9}{2} + \frac{1}{x+y}\right) \\ &\leq \beta(M(x, y))M(x, y) + LN(x, y). \end{aligned}$$

Case (iii). $x \in [0, 1), y \in [1, \infty)$.
$d(Ax, By) = \frac{9}{2} + \frac{1}{x+y}, d(Ax, Sx) = \frac{12}{5}, d(By, Ty) = \frac{9}{2} + \frac{1}{x+y}$,
$d(Ax, Ty) = \frac{9}{2} + \frac{1}{x+y}, d(By, Sx) = \frac{12}{5}$,
$M(x, y) = \max\{d(Ax, By), d(Ax, Sx), d(By, Ty)\} = \frac{9}{2} + \frac{1}{x+y}$ and
$N(x, y) = \min\{d(Ax, Sx), d(Ax, Ty), d(By, Sx)\} = \frac{12}{5}$.
We now consider

$$\begin{aligned} sd(Ax, By) = \frac{25}{24}\left(\frac{9}{2} + \frac{1}{x+y}\right) &\leq \frac{24}{25}e^{-(\frac{9}{2}+\frac{1}{x+y})}\left(\frac{9}{2} + \frac{1}{x+y}\right) + L\left(\frac{12}{5}\right) \\ &\leq \beta(M(x, y))M(x, y) + LN(x, y). \end{aligned}$$

Case (iv). $x \in [1,\infty), y \in [0,1)$.
$d(Ax,By) = \frac{9}{2} + \frac{1}{x+y}, d(Ax,Sx) = \frac{9}{2} + \frac{1}{x+y}, d(By,Ty) = \frac{12}{5}$,
$d(Ax,Ty) = \frac{12}{5}, d(By,Sx) = \frac{9}{2} + \frac{1}{x+y}$,
$M(x,y) = \max\{d(Ax,By), d(Ax,Sx), d(By,Ty)\} = \frac{9}{2} + \frac{1}{x+y}$ and
$N(x,y) = \min\{d(Ax,Sx), d(Ax,Ty), d(By,Sx)\} = \frac{12}{5}$.
We now consider

$$\begin{aligned} sd(Ax,By) = \tfrac{25}{24}(\tfrac{9}{2} + \tfrac{1}{x+y}) &\leq \tfrac{24}{25}e^{-(\frac{9}{2}+\frac{1}{x+y})}(\tfrac{9}{2} + \tfrac{1}{x+y}) + L(\tfrac{12}{5}) \\ &\leq \beta(M(x,y))M(x,y) + LN(x,y). \end{aligned}$$

From all the above cases, A, B, S and T are Geraghty-Berinde type contraction maps with $L = 3$.
Threfore A, B, S and T satisfy all the hypotheses of Theorem 2.6 and 1 is the unique common fixed point of A, B, S and T.

Here we observe that if $L = 0$ then the inequality (1) fails to hold.
For, by choosing $x = 0$ and $y = 2$ we have
$d(Ax,By) = 5, d(Ax,Sx) = \frac{12}{5}, d(By,Ty) = 5$,
$M(x,y) = \max\{d(Ax,By), d(Ax,Sx), d(By,Ty)\} = 5$.
Here we note that
$d(fx,fy) = 5 \not\leq \beta(5)5 = \beta(M(x,y))M(x,y)$ for any $\beta \in \mathfrak{F}$.

The following is an example in support of Theorem 2.8.

Example 3.2 Let $X = [0,2]$ and let $d : X \times X \to [0,\infty)$ defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ \frac{5}{4} & \text{if } x, y \in [0,1], \\ \frac{25}{14} + \frac{1}{2x+3y} & \text{if } x, y \in (1,2), \\ \frac{9}{10} & \text{otherwise.} \end{cases}$$

Then clearly (X,d) is a b-metric space with coefficient $s = \frac{11}{9}$.
We define selfmaps A, B, S, T on X by

$$A(x) = \begin{cases} \frac{2}{3} & \text{if } 0 \leq x \leq 1 \\ \frac{3}{4} & \text{if } 1 < x \leq 2, \end{cases}, \; B(x) = \frac{2}{3} \text{ if } x \in [0,2],$$

$$S(x) = \begin{cases} \frac{4}{3} - x & \text{if } 0 \leq x \leq 1 \\ \frac{4}{3} & \text{if } 1 < x \leq 2. \end{cases} \text{ and } T(x) = \begin{cases} 1 - \frac{x}{2} & \text{if } 0 \leq x \leq 1 \\ \frac{5}{4} & \text{if } 1 < x \leq 2. \end{cases}$$

We define $\beta : [0,\infty) \to [0, \frac{1}{s})$ by $\beta(t) = \frac{24}{25}e^{-t}$.
Then $\beta \in \mathfrak{F}$.
Let $\{x_n\} \subseteq [0,1]$.
Then $\lim\limits_{n\to\infty} Ax_n = \lim\limits_{n\to\infty} Sx_n$.
$\Rightarrow \frac{2}{3} = \lim\limits_{n\to\infty} (\frac{4}{3} - x_n)$.
$\Rightarrow \lim\limits_{n\to\infty} x_n = \frac{2}{3}$.
Therefore for any $\{x_n\} \subseteq [0,1]$ with $\lim\limits_{n\to\infty} x_n = \frac{2}{3}$, we have
$\lim\limits_{n\to\infty} Ax_n = \lim\limits_{n\to\infty} Sx_n = \frac{2}{3}$.
Now, $\lim\limits_{n\to\infty} SAx_n = \lim\limits_{n\to\infty} S(\frac{2}{3}) = \frac{2}{3} = S(\frac{2}{3})$
and $\lim\limits_{n\to\infty} ASx_n = \lim\limits_{n\to\infty} A(\frac{4}{3} - x_n), 0 \leq x_n \leq 1$.
Case (a): $\frac{1}{3} \leq \frac{4}{3} - x_n \leq 1$.
In this case, $A(\frac{4}{3} - x_n) = \frac{2}{3}$.

Therefore $\lim\limits_{n\to\infty} ASx_n = \lim\limits_{n\to\infty} A(\frac{4}{3} - x_n) = \frac{2}{3} = A(\frac{2}{3})$.
Case (b): $1 \leq x_n \leq \frac{4}{3}$.
This case doesn't arise, since we are considering the sequence $\{x_n\}$ with $\lim\limits_{n\to\infty} x_n = \frac{2}{3}$.
Therefore the pair (A, S) is reciprocally continuous and compatible with the sequence $\{\frac{2}{3}\} \subseteq X$. Clearly the pair (B, T) is weakly compatible and $B(X)$ is closed.
We now verify the inequality (1). With out loss of generality, we assume that $x \geq y$.
Case (i): $x, y \in [0, 1]$
$sd(Ax, By) = 0 \leq \beta(M(x, y))M(x, y) + LN(x, y)$.
In this case, the inequality (1) trivially holds.
Case (ii): $x, y \in (1, 2]$
$d(Ax, By) = \frac{5}{4}, d(Sx, Ty) = \frac{25}{14} + \frac{1}{2x+3y}, d(Ax, Sx) = \frac{9}{10}, d(By, Ty) = \frac{9}{10}$,
$d(Ax, Ty) = \frac{9}{10}, d(By, Sx) = \frac{9}{10}$.
$M(x, y) = \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty)\} = \frac{25}{14} + \frac{1}{2x+3y}$ and
$N(x, y) = \min\{d(Ax, Sx), d(Ax, Ty), d(By, Sx)\} = \frac{9}{10}$.
We now consider

$$\begin{aligned} sd(Ax, By) = \tfrac{11}{9}(\tfrac{5}{4}) &\leq \tfrac{9}{11} e^{-(\frac{25}{14} + \frac{1}{2x+3y})}(\tfrac{25}{14} + \tfrac{1}{2x+3y}) + L\tfrac{9}{10} \\ &\leq \beta(M(x, y))M(x, y) + LN(x, y). \end{aligned}$$

Case (iii): $x \in (1, 2], y \in [0, 1]$
$d(Ax, By) = \frac{5}{4}, d(Sx, Ty) = \frac{9}{10}, d(Ax, Sx) = \frac{9}{10}, d(By, Ty) = \frac{5}{4}, d(Ax, Ty) = \frac{5}{4}$,
$d(By, Sx) = \frac{9}{10}$.
$M(x, y) = \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty)\} = \frac{5}{4}$ and
$N(x, y) = \min\{d(Ax, Sx), d(Ax, Ty), d(By, Sx)\} = \frac{9}{10}$.
We now consider
$sd(Ax, By) = \frac{11}{9}(\frac{5}{4}) \leq \frac{9}{11} e^{-(\frac{5}{4})}(\frac{5}{4}) + L\frac{9}{10} \leq \beta(M(x, y))M(x, y) + LN(x, y)$.
Hence from the above cases the selfmaps A, B, S, T are Geraghty-Berinde type contraction maps with $L = 3$. Therefore A, B, S and T satisfy all the hypotheses of Theorem 2.8 and $\frac{2}{3}$ is a unique common fixed point of A, B, S and T in X.

Here we observe that if $L = 0$ then the inequality (1) fails to hold.
For, by choosing $x = 2$ and $y = 0$ we have
$d(Ax, By) = \frac{5}{4}, d(Ax, Sx) = \frac{12}{5}, d(By, Ty) = 5$,
$M(x, y) = \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty)\} = \frac{5}{4}$.
Here we note that
$sd(Ax, By) = \frac{11}{9}(\frac{5}{4}) \not\leq \beta(\frac{5}{4})\frac{5}{4} = \beta(M(x, y))M(x, y)$ for any $\beta \in \mathfrak{F}$.

The following is an example in support of Theorem 2.9.

Example 3.3 Let $X = [0, 1]$ and let $d : X \times X \to \mathbb{R}^+$ defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ \frac{11}{15} + \frac{x}{23} & \text{if } x, y \in (0, \frac{2}{3}), \\ \frac{4}{5} + \frac{x+y}{10} & \text{if } x, y \in [\frac{2}{3}, 1], \\ \frac{12}{25} & \text{otherwise.} \end{cases}$$

Then clearly (X, d) is a complete b-metric space with coefficient $s = \frac{52}{49}$.
We define $A, B, S, T : X \to X$ by

$$A(x) = \tfrac{2}{3} \text{ if } x \in [0, 1],\ B(x) = \begin{cases} \frac{3}{4} & \text{if } x \in [0, \frac{2}{3}) \\ 1 - \frac{x}{2} & \text{if } x \in [\frac{2}{3}, 1], \end{cases}$$

$S(x) = \begin{cases} \frac{2}{3} + \frac{x}{9} & \text{if } x \in [0, \frac{2}{3}) \\ \frac{2+5x}{8} & \text{if } x \in [\frac{2}{3}, 1], \end{cases}$ and $T(x) = \begin{cases} \frac{3}{4} + \frac{\sqrt{x}}{5} & \text{if } x \in [0, \frac{2}{3}) \\ x & \text{if } x \in [\frac{2}{3}, 1]. \end{cases}$

We define $\beta : [0, \infty) \to [0, \frac{1}{s})$ by $\beta(t) = \frac{24}{25} e^{-t}$. Then $\beta \in \mathfrak{F}$.

Clearly $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$. $A(X) = \{\frac{2}{3}\}$ is b-closed.

We choose a sequence $\{x_n\}$ with $\{x_n\} = \frac{2}{3} + \frac{1}{n}, n \geq 4$ with $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \frac{2}{3}$, hence the pair (A, S) satisfies the b-(E.A)-property.

Clearly the pairs (A, S) and (B, T) are weakly compatible.

Case (i). $x, y \in (0, \frac{2}{3})$.

$d(Ax, By) = \frac{4}{5} + \frac{x+y}{10}, d(Sx, Ty) = \frac{4}{5} + \frac{x+y}{10}, d(Ax, Sx) = \frac{4}{5} + \frac{x+y}{10}$,
$d(By, Ty) = \frac{4}{5} + \frac{x+y}{10}, d(Ax, Ty) = \frac{4}{5} + \frac{x+y}{10}, d(By, Sx) = \frac{4}{5} + \frac{x+y}{10}$,
$M(x, y) = \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty)\} = \frac{4}{5} + \frac{x+y}{10}$ and
$N(x, y) = \min\{d(Ax, Sx), d(Ax, Ty), d(By, Sx)\} = \frac{4}{5} + \frac{x+y}{10}$.

We now consider

$$\begin{aligned} sd(Ax, By) = \tfrac{52}{49}(\tfrac{4}{5} + \tfrac{x+y}{10}) &\leq (\tfrac{49}{52}) e^{-(\frac{4}{5} + \frac{x+y}{10})} (\tfrac{4}{5} + \tfrac{x+y}{10}) + L(\tfrac{4}{5} + \tfrac{x+y}{10}) \\ &\leq \beta(M(x, y)) M(x, y) + LN(x, y). \end{aligned}$$

Case (ii). $x, y \in (\frac{2}{3}, 1]$.

$d(Ax, By) = \frac{12}{25}, d(Sx, Ty) = \frac{4}{5} + \frac{x+y}{10}, d(Ax, Sx) = \frac{4}{5} + \frac{x+y}{10}, d(By, Ty) = \frac{12}{25}$,
$d(Ax, Ty) = \frac{4}{5} + \frac{x+y}{10}, d(By, Sx) = \frac{12}{25}$,
$M(x, y) = \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty)\} = \frac{4}{5} + \frac{x+y}{10}$ and
$N(x, y) = \min\{d(Ax, Sx), d(Ax, Ty), d(By, Sx)\} = \frac{12}{25}$.

We now consider

$$\begin{aligned} sd(Ax, By) = \tfrac{52}{49}(\tfrac{12}{25}) &\leq (\tfrac{49}{52}) e^{-(\frac{4}{5} + \frac{x+y}{10})} (\tfrac{4}{5} + \tfrac{x+y}{10}) + L(\tfrac{12}{25}) \\ &\leq \beta(M(x, y)) M(x, y) + LN(x, y). \end{aligned}$$

Case (iii). $x \in (\frac{2}{3}, 1], y \in (0, \frac{2}{3})$.

$d(Ax, By) = \frac{4}{5} + \frac{x+y}{10}, d(Sx, Ty) = \frac{4}{5} + \frac{x+y}{10}, d(Ax, Sx) = \frac{4}{5} + \frac{x+y}{10}$,
$d(By, Ty) = \frac{4}{5} + \frac{x+y}{10}, d(Ax, Ty) = \frac{4}{5} + \frac{x+y}{10}, d(By, Sx) = \frac{4}{5} + \frac{x+y}{10}$,
$M(x, y) = \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty)\} = \frac{4}{5} + \frac{x+y}{10}$ and
$N(x, y) = \min\{d(Ax, Sx), d(Ax, Ty), d(By, Sx)\} = \frac{4}{5} + \frac{x+y}{10}$.

We now consider

$$\begin{aligned} sd(Ax, By) = \tfrac{52}{49}(\tfrac{4}{5} + \tfrac{x+y}{10}) &\leq (\tfrac{49}{52}) e^{-(\frac{4}{5} + \frac{x+y}{10})} (\tfrac{4}{5} + \tfrac{x+y}{10}) + L(\tfrac{4}{5} + \tfrac{x+y}{10}) \\ &\leq \beta(M(x, y)) M(x, y) + LN(x, y). \end{aligned}$$

Case (iv). $x = \frac{2}{3}, y \in (0, \frac{2}{3})$.

$d(Ax, By) = \frac{4}{5} + \frac{x+y}{10}, d(Sx, Ty) = \frac{4}{5} + \frac{x+y}{10}, d(Ax, Sx) = \frac{4}{5} + \frac{x+y}{10}$,
$d(By, Ty) = \frac{4}{5} + \frac{x+y}{10}, d(Ax, Ty) = \frac{4}{5} + \frac{x+y}{10}, d(By, Sx) = \frac{4}{5} + \frac{x+y}{10}$,
$M(x, y) = \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty)\} = \frac{4}{5} + \frac{x+y}{10}$ and
$N(x, y) = \min\{d(Ax, Sx), d(Ax, Ty), d(By, Sx)\} = \frac{4}{5} + \frac{x+y}{10}$.

We now consider

$$\begin{aligned} sd(Ax, By) = \tfrac{52}{49}(\tfrac{4}{5} + \tfrac{x+y}{10}) &\leq (\tfrac{49}{52}) e^{-(\frac{4}{5} + \frac{x+y}{10})} (\tfrac{4}{5} + \tfrac{x+y}{10}) + L(\tfrac{4}{5} + \tfrac{x+y}{10}) \\ &\leq \beta(M(x, y)) M(x, y) + LN(x, y). \end{aligned}$$

Case (v). $y \in (\frac{2}{3}, 1], x \in (0, \frac{2}{3})$.

$d(Ax, By) = \frac{12}{25}, d(Sx, Ty) = \frac{4}{5} + \frac{x+y}{10}, d(Ax, Sx) = \frac{4}{5} + \frac{x+y}{10}, d(By, Ty) = \frac{12}{25}$,
$d(Ax, Ty) = \frac{4}{5} + \frac{x+y}{10}, d(By, Sx) = \frac{12}{25}$,
$M(x, y) = \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty)\} = \frac{4}{5} + \frac{x+y}{10}$ and
$N(x, y) = \min\{d(Ax, Sx), d(Ax, Ty), d(By, Sx)\} = \frac{12}{25}$.

We now consider

$$sd(Ax, By) = \tfrac{52}{49}(\tfrac{12}{25}) \leq (\tfrac{49}{52}) e^{-(\frac{4}{5} + \frac{x+y}{10})} (\tfrac{4}{5} + \tfrac{x+y}{10}) + L(\tfrac{12}{25})$$

$$\leq \beta(M(x,y))M(x,y) + LN(x,y).$$

Case (vi). $y = \frac{2}{3}, x \in (0, \frac{2}{3})$.
$d(Ax, By) = 0$. In this case the inequality (1) trivially holds.
Therefore A, B, S and T satisfy all the hypotheses of Theorem 2.9 and $\frac{2}{3}$ is the unique common fixed point in X.

Corollary 3.4 Let $\{A_n\}_{n=1}^{\infty}, S$ and T be selfmaps on a complete b-metric space (X, d) satisfying $A_1 \subseteq S(X)$ and $A_1 \subseteq T(X)$. Assume that there exists $\beta \in \mathfrak{F}$ and $L \geq 0$ such that

$$sd(A_1x, A_jy) \leq \beta(M(x,y))M(x,y) + LN(x,y) \tag{47}$$

for all $x, y \in X$ and $j = 1, 2, 3, \ldots$., where
$M(x,y) = \max\{d(Sx,Ty), d(A_1x,Sx), d(A_jy,Ty)\}$ and
$N(x,y) = \min\{d(A_1x,Sx), d(A_1x,Ty), d(A_jy,Sx)\}$. If the pairs (A_1, S) and (A_1, T) are weakly compatible and one of the range sets $A_1(X), S(X)$ and $T(X)$ is closed, then $\{A_n\}_{n=1}^{\infty}, S$ and T have a unique common fixed point in X.
Proof. Under the assumptions on A_1, S and T, the existence of common fixed point z of A_1, S and T follows by choosing $A = B = A_1$ in Theorem 2.6.
Therefore $A_1z = Sz = Tz = z$.
Now, let $j \in \mathbb{N}$ with $j \neq 1$.
We now consider

$$sd(z, A_jz) = sd(A_1z, A_jz) \leq \beta(M(z,z))M(z,z) + LN(z,z) \tag{48}$$

where $M(z,z) = \max\{d(Sz,Tz), d(A_1z,Sz), d(A_jz,Tz)\} = d(A_jz, z)$ and
$N(z,z) = \min\{d(A_1z,Sz), d(A_1z,Tz), d(A_jz,Sz)\} = 0$.
From the inequality (48), we have
$sd(z, A_z) \leq \beta(d(z,A_jz))d(z,A_jz) < \frac{d(z,A_jz)}{s}$,
which is a contradiction.
Therefore $d(z, A_jz) \leq 0$ which implies that $A_jz = z$ for $j = 1, 2, 3, \ldots$ and uniqueness of common fixed point follows from the inequality (47).
Hence, $\{A_n\}_{n=1}^{\infty}, S$ and T have a unique common fixed point in X.

Corollary 3.5 Let $\{A_n\}_{n=1}^{\infty}, S$ and T be selfmaps on a complete b- metric space (X, d) satisfy the conditions $A_1 \subseteq S(X), A_1 \subseteq T(X)$ and (47). If the pairs (A_1, S) and (A_1, T) are weakly compatible and either $(A_1(X), d), (S(X), d)$ or $(T(X), d)$ is b-complete, then $\{A_n\}_{n=1}^{\infty}, S$ and T have a unique common fixed point in X.
Proof. Under the assumptions on A_1, S and T, the existence of common fixed point z of A_1, S and T follows by choosing $A = B = A_1$ in Theorem 2.7.
Therefore $A_1z = Sz = Tz = z$.
Now, let $j \in \mathbb{N}$ with $j \neq 1$.
We now consider

$$sd(z, A_jz) = sd(A_1z, A_jz) \leq \beta(M(z,z))M(z,z) + LN(z,z) \tag{49}$$

where $M(z,z) = \max\{d(Sz,Tz), d(A_1z,Sz), d(A_jz,Tz)\} = d(A_jz, z)$ and
$N(z,z) = \min\{d(A_1z,Sz), d(A_1z,Tz), d(A_jz,Sz)\} = 0$.
From the inequality (49), we have
$sd(z, A_z) \leq \beta(d(z,A_jz))d(z,A_jz) < \frac{d(z,A_jz)}{s}$,
which is a contradiction.

Therefore $d(z, A_j z) \leq 0$ which implies that $A_j z = z$ for $j = 1, 2, 3, \ldots$ and uniqueness of common fixed point follows from the inequality (47).
Hence, $\{A_n\}_{n=1}^{\infty}, S$ and T have a unique common fixed point in X.

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