Electronic Journal of Mathematical Analysis and Applications
Vol. 9(2) July 2021, pp. 88-98
ISSN: 2090-729X(online)
http://math-frac.org/Journals/EJMAA/

A MODERN APPROACH FOR SOLVING NONLINEAR VOLTERRA INTEGRAL EQUATIONS USING FIBONACCI WAVELETS

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Abstract. This article provides an effective technique to solve nonlinear-Volterra integral equations using Fibonacci wavelets. These equations can be reduced to a system of nonlinear algebraic equations with unknown Fibonacci coefficients, by using Fibonacci wavelets, and their operational matrix of integration and these equations can be solved by numerical methods such as Newton's method. Error estimate of the proposed method is given. Moreover, the results obtained by the method proposed are compared to exact solution with number of numerical examples to show that the method described is precise and accurate.

1. Introduction

Wavelets are mathematical functions that separate the data into various frequency components and then analyze each component with its corresponding resolution. Wavelets can be used as a mathematical device to extract information from the number of data types, which may include earthquakes, seismic waves, signal processing, music, image processing, nuclear engineering, acoustics, and astronomy.

In this article, we use Fibonacci wavelets introduced by Sedigheh Sabermahani et al. [1] in 2019. These are specific kind of wavelets which are not based on orthogonal functions. They, however, have a derivative, operational matrix of integration and so on.

Integral equations have been one of the essential tools in different areas of applied mathematics. Integral equations are extensively involved in several problems in science and technology [2, 3]. In several physical models and fields of engineering, such as, radiography, spectroscopy, image processing, cosmic radiation, etc., integral equations exist. However, analytic solutions of integral equations either do not exist or are difficult to find. Precisely because of this, several numerical methods for solutions of integral equations have been developed. Some of them are found in [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19].

2010 *Mathematics Subject Classification.* 45D05, 11B39, 65T60.
Key words and phrases. Fibonacci polynomials, Fibonacci wavelets, Volterra integral equations.
Submitted May 17, 2020. Revised Oct. 25, 2020.

In this article, an attempt is made for solving nonlinear Volterra integral equations [20] using Fibonacci wavelets. We consider the nonlinear Volterra integral equations of the form:

$$y(x) = f(x) + \int_0^x k(x,t)\mu(y(t))dt, \tag{1}$$

where $x, t \in [0,1)$, $k(x,t)$, the function of x and t, and $\mu(y(x))$ are the known functions, and $y(x)$ is the unknown that is to be determined.

This article is structured in the following way: Properties of Fibonacci polynomials, wavelets, and Fibonacci wavelets are studied in section 2. In section 3, Fibonacci wavelets operational matrix of integration is given. In section 4, a new Fibonacci wavelets operational matrix method for solving nonlinear Volterra integral equations is proposed. Error estimate of the proposed method is given in section 5. In section 6, numerical examples are presented in order to justify the efficiency of the proposed method. Ultimately, the conclusion is drawn in section 7.

2. Properties of Fibonacci polynomials, wavelets, and Fibonacci wavelets

2.1. **Fibonacci polynomials.** Fibonacci polynomials [1] are defined in general as:

$$\bar{F}_m(x) = \begin{cases} 1, & m = 0, \\ x, & m = 1, \\ x\bar{F}_{m-1}(x) + \bar{F}_{m-2}(x), & m > 1. \end{cases} \tag{2}$$

Furthermore, these polynomials [1] can be represented in the power form as follows:

$$\bar{F}_m = \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m-i}{i} x^{m-2i}, \quad m > 0. \tag{3}$$

Lemma 2.1. *If $\bar{F}_m(x)$, $m = 0, 1, ..., M-1$ are Fibonacci polynomials* [1]*, then*

$$\int_0^1 \bar{F}_m(x)\bar{F}_n(x)dx = \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{m-i}{i}\binom{n-j}{j} \frac{1}{m+n-2i-2j+1}. \tag{4}$$

2.2. **Fibonacci wavelets.** Fibonacci wavelets [1] are defined as follows:

$$\psi_{n,m}(x) = \begin{cases} 2^{\frac{k-1}{2}} \hat{F}_m(2^{k-1}x - n + 1), & \frac{n-1}{2^{k-1}} \le x < \frac{n}{2^{k-1}}, \\ 0, & otherwise, \end{cases} \tag{5}$$

in which

$$\hat{F}_m(x) = \frac{1}{\sqrt{w_m}} \bar{F}_m(x),$$

with

$$w_m = \int_0^1 \bar{F}_m^2(x)dx,$$

where, w_m, for $m = 0, 1, ..., M-1$ are obtained by equation (4), and m denotes the order of the Fibonacci polynomials and $n = 1, 2, ..., 2^{k-1}$, $k \in \mathbb{N}$. For instance, for

$k = 2$ and $M = 3$, we get

$$\left.\begin{array}{l} \psi_{1,0}(x) = \sqrt{2} \\ \psi_{1,1}(x) = 2\sqrt{6}x \\ \psi_{1,2}(x) = \sqrt{\frac{15}{14}}(1+4x^2) \end{array}\right\} \quad 0 \leq x < \frac{1}{2},$$

$$\left.\begin{array}{l} \psi_{2,0}(x) = \sqrt{2} \\ \psi_{2,1}(x) = \sqrt{6}(2x-1) \\ \psi_{2,2}(x) = \sqrt{\frac{30}{7}}(2x^2-2x+1) \end{array}\right\} \quad \frac{1}{2} \leq x < 1.$$

2.3. **Function approximation.** Suppose $f(x) \in L^2[0,1)$ is expanded in terms of the Fibonacci wavelets as:

$$f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(x). \tag{6}$$

Truncating the above infinite series, we get

$$f(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(x) = C^T\psi(x) = f_{\hat{m}}(x), \tag{7}$$

where, C and $\psi(x)$ are $\hat{m} \times 1$ ($\hat{m} = 2^{k-1}M$) matrices given by

$$C = \left[c_{1,0}, c_{1,1}, ..., c_{1,M-1}, c_{2,0}, c_{2,1}, ..., c_{2,M-1}, c_{2^{k-1},0}, c_{2^{k-1},1}, ..., c_{2^{k-1},M-1}\right], \tag{8}$$

and

$$\begin{aligned} \psi(x) = [&\psi_{1,0}(x), \psi_{1,1}(x), ..., \psi_{1,M-1}(x), \psi_{2,0}(x), \psi_{2,1}(x), ..., \psi_{2,M-1}(x), \\ &\psi_{2^{k-1},0}(x), \psi_{2^{k-1},1}(x), ..., \psi_{2^{k-1},M-1}(x)]. \end{aligned} \tag{9}$$

Remark 2.2. *If F is a $\hat{m}$-vector, then*

$$\psi(x)\psi^T(x)F = \tilde{F}\psi(x), \tag{10}$$

where, $\psi(x)$ is the Fibonacci wavelet coefficient matrix and $\tilde{F}$ is an $\hat{m} \times \hat{m}$ matrix given by

$$\tilde{F} = \psi(x)\bar{F}\psi^{-1}(x), \tag{11}$$

where $\bar{F} = \text{diag}(\psi^{-1}(x)F)$. Also, for a $\hat{m} \times \hat{m}$ matrix C,

$$\psi^T(x)C\psi(x) = \hat{C}^T\psi(x), \tag{12}$$

in which $X = \text{diag}(\psi^T(x)C\psi(x))$ is a $\hat{m}$-vector and $\hat{C}^T = X\psi^{-1}(x)$.

Remark 2.3. *If μ is a analytic function on R and $C^T\psi(x)$ be the the expansion of $f(x)$ in terms of Fibonacci wavelets, where C is given in equation (8), then*

$$\mu(f(x)) \simeq \mu(C^T)\psi(x), \tag{13}$$

where, $\mu(C^T) = [\mu(C_1), \mu(C_2), ..., \mu(C_{\hat{m}})]$.

Remark 2.4. *If μ is a analytic function on R and $C^T\psi(x)$ be the the expansion of $f(x)$ in terms of Fibonacci wavelets, where C is given in equation (8), then*

$$\mu(f(x)) \simeq \mu(\tilde{C}^T)\psi^{-1}(x)\psi(x), \tag{14}$$

where, $\tilde{C}^T = C^T\psi(x)$, $\psi(x)$ is the Fibonacci wavelets coefficient matrix given in (9) and $\mu(C^T)$ is given in Remark 2.3.

3. Fibonacci wavelets operational matrix of integration

The Fibonacci wavelets vector is given in 9. The operational matrix of integration of Fibonacci wavelets P [1] is defined as follows:

$$\int_0^x \psi(t)dt = P\psi(x), \tag{15}$$

where P is given in general in [1]. For instance, for $k = 2$ and $M = 3$, we get P as:

$$P = \begin{pmatrix} 0 & \frac{1}{2\sqrt{3}} & 0 & \frac{1}{2} & 0 & 0 \\ -\frac{\sqrt{3}}{4} & 0 & \frac{1}{2}\sqrt{\frac{7}{5}} & \frac{\sqrt{3}}{4} & 0 & 0 \\ -\frac{29}{16\sqrt{105}} & \frac{1}{\sqrt{35}} & \frac{1}{4} & \sqrt{\frac{5}{21}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2\sqrt{3}} & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{3}}{4} & 0 & \frac{1}{2}\sqrt{\frac{7}{5}} \\ 0 & 0 & 0 & -\frac{29}{16\sqrt{105}} & \frac{1}{\sqrt{35}} & \frac{1}{4} \end{pmatrix}_{6\times 6}.$$

4. Fibonacci wavelets operational matrix method for solving nonlinear Volterra integral equations

Let us consider equation (1). Approximating $f(x)$, $y(x)$, and $k(x,t)$ with respect to Fibonacci wavelets as follows:

$$y(x) \simeq C^T\psi(x) = C\psi^T(x), \tag{16}$$

where C is given in equation (8) and is the unknown to be determined.

$$f(x) \simeq F^T\psi(x) = F\psi^T(x), \tag{17}$$

$$k(x,t) \simeq F^T\psi(x) = \psi^T(t)K\psi(x), \tag{18}$$

where C and F are Fibonacci wavelet coefficient vectors, and K is the Fibonacci wavelet matrix. Substituting equations (16), (17), and (18) in equation (1), we get

$$C^T\psi(x) \simeq F^T\psi(x) + \psi^T(x)K\left(\int_0^x \psi(t)\mu\left(C^T\psi(t)\right)dt\right). \tag{19}$$

Now, by using Remark 2.4, equation (19) can be rewritten as,

$$C^T\psi(x) \simeq F^T\psi(x) + \psi^T(x)K\left(\int_0^x \psi(t)\psi^T(t)Xdt\right), \tag{20}$$

where, $X^T = \mu(\tilde{C}^T)\psi^{-1}(x)$, where $\tilde{C}^T = C^T\psi(x)$. Using equation (20) and Remark 2.2, we get

$$C^T\psi(x) \simeq F^T\psi(x) + \psi^T(x)K\left(\int_0^x \tilde{X}\psi(t)dt\right), \tag{21}$$

where, $\tilde{X}$ is a $\hat{m} \times \hat{m}$ matrix described in Remark 2.2. Applying the OMI of Fibonacci wavelets [1] described in section 2, equation (21) reduces to:

$$C^T\psi(x) \simeq F^T\psi(x) + \psi^T(x)K\tilde{X}P\psi(x). \tag{22}$$

Let us assume that $\delta = K\tilde{X}P$. Again using Remark 2.2, equation (22) reduces to:

$$C^T\psi(x) - \hat{\delta}\psi(x) \simeq F^T\psi(x), \tag{23}$$

where, $\hat{\delta}$ is a $\hat{m}$-vector containing a nonlinear combination of elements of C. Equation (23) holds for all $x \in [0, 1)$, and hence replacing equation $\simeq$ by $=$, equation (23) reduces to a nonlinear system of equations $C^T - \hat{\delta} \simeq F^T$. Solving this nonlinear system, we get the unknown vector C. Substituting this obtained vector in equation (16), we obtain the solution of equation (1).

5. Error estimate

We compare the approximate solution and exact solution of equation (1) at the some selected points via the definition of absolute error defined as,

$$e(x) = |y(x) - y^*(x)|, \tag{24}$$

where, $y(x)$ and $y^*(x)$ denote the exact and approximate solution of equation (1).

6. Computational Experiments

Test problem 6.1. *Let us consider the nonlinear Volterra integral equation* [20]*:*

$$y(x) = \frac{1}{2}(x - x^2) + \cos(x) - \sin(x) - \frac{1}{4}\sin(2x) + \int_0^x (x-t)y^2(t)dt. \tag{25}$$

Exact solution of (25) is found to be $y(x) = \cos(x) - \sin(x)$. Table 1 shows the comparison of exact, approximate solutions, and absolute errors of test problem 6.1 obtained by using the method described in section 4 for $\hat{m} = 6$ and figure 1 shows the graph of exact and approximate solutions of test problem 6.1 for $\hat{m} = 6$.

Table 1. Comparison of exact and approximate for test problem 6.1 for $\hat{m} = 6$.

x	Exact	Approximate	Absolute error
0.5/6	0.913293	0.912235	1.0577e-03
1.5/6	0.721508	0.721473	3.5367e-05
2.5/6	0.509729	0.510515	7.8662e-04
3.5/6	0.283822	0.283276	5.4614e-04
4.5/6	0.0500501	0.0497601	2.8998e-04
5.5/6	-0.185109	-0.18555	4.4093e-04

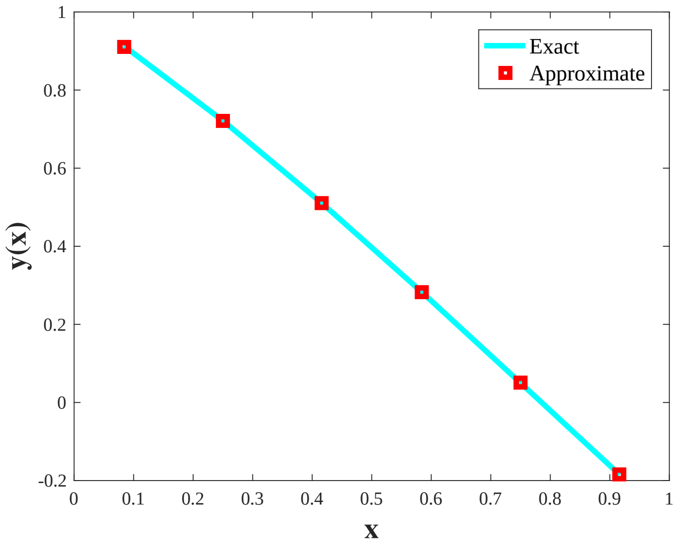

FIGURE 1. Graph of exact and approximate solutions of test problem 6.1 for $\hat{m} = 6$.

Test problem 6.2. *Let us consider the nonlinear Volterra integral equation* [20]*:*

$$y(x) = 1 + \frac{1}{2}x^2 - \frac{1}{6}x^4 - \frac{1}{30}x^6 + \int_0^x (x-t)y^2(t)dt. \tag{26}$$

Exact solution of (26) is found to be $y(x) = 1 + x^2$. Table 2 shows the comparison of exact, approximate solutions, and absolute errors of test problem 6.2 obtained by using the method described in section 4 for $\hat{m} = 6$ and figure 2 shows the graph of exact and approximate solutions of test problem 6.2 for $\hat{m} = 6$.

TABLE 2. Comparison of exact and approximate for test problem 6.2 for $\hat{m} = 6$.

x	Exact	Approximate	Absolute error
0.5/6	1.0069	1.007	7.8382e-05
1.5/6	1.0625	1.0622	2.6965e-04
2.5/6	1.1736	1.1725	1.1290e-03
3.5/6	1.3403	1.3419	1.5883e-03
4.5/6	1.5625	1.5622	2.9861e-04
5.5/6	1.8403	1.837	3.3179e-03

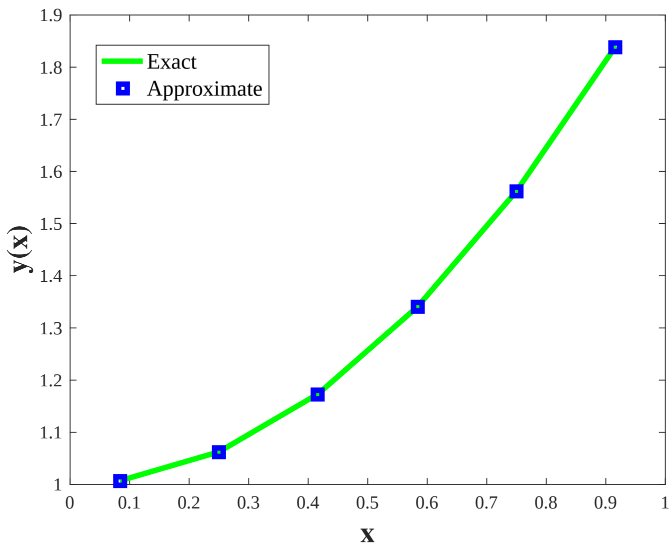

FIGURE 2. Graph of exact and approximate solutions of test problem 6.2 for $\hat{m} = 6$.

Test problem 6.3. *Let us consider the nonlinear Volterra integral equation* [20]*:*

$$y(x) = 1 + 3x - \frac{1}{2}x^2 - x^3 - \frac{3}{4}x^4 + \int_0^x (x-t)y^2(t)dt. \tag{27}$$

Exact solution of (27) is found to be $y(x) = 1 + 3x$. Table 3 shows the comparison of exact, approximate solutions, and absolute errors of test problem 6.3 obtained by using the method described in section 4 for $\hat{m} = 6$ and figure 3 shows the graph of exact and approximate solutions of test problem 6.3 for $\hat{m} = 6$.

TABLE 3. Comparison of exact and approximate for test problem 6.3 for $\hat{m} = 6$.

x	Exact	Approximate	Absolute error
0.5/6	1.25	1.2538	3.8310e-03
1.5/6	1.75	1.7494	6.4548e-04
2.5/6	2.25	2.243	7.0081e-03
3.5/6	2.75	2.7585	8.4652e-03
4.5/6	3.25	3.2495	5.2499e-04
5.5/6	3.75	3.7386	1.1424e-02

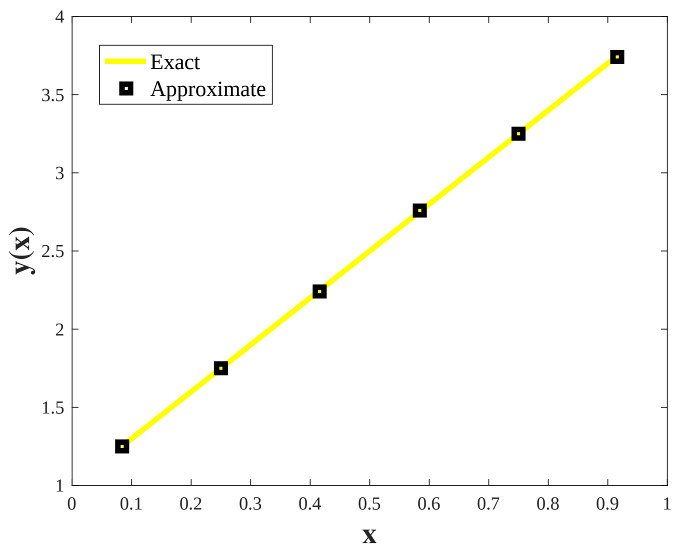

FIGURE 3. Graph of exact and approximate solutions of test problem 6.3 for $\hat{m} = 6$.

Test problem 6.4. *Let us consider the nonlinear Volterra integral equation* [20]*:*

$$y(x) = \frac{1}{9} + \frac{3}{4}x + \exp(x) - \frac{1}{9}\exp(3x) + \int_0^x (x-t)y^3(t)dt. \tag{28}$$

Exact solution of (28) is found to be $y(x) = \exp(x)$. Table 3 shows the comparison of exact, approximate solutions, and absolute errors of test problem 6.4 obtained by using the method described in section 4 for $\hat{m} = 6$ and figure 4 shows the graph of exact and approximate solutions of test problem 6.4 for $\hat{m} = 6$.

TABLE 4. Comparison of exact and approximate for test problem 6.4 for $\hat{m} = 6$.

x	Exact	Approximate	Absolute error
0.5/6	1.0869	1.0885	1.6078e-03
1.5/6	1.284	1.2835	5.3428e-04
2.5/6	1.5169	1.5122	4.7052e-03
3.5/6	1.792	1.8	8.0140e-03
4.5/6	2.117	2.1157	1.2998e-03
5.5/6	2.5009	2.4813	1.9641e-02

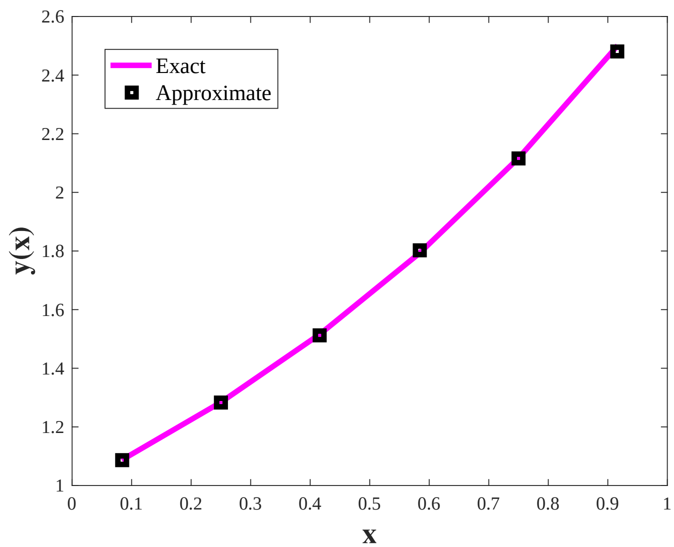

FIGURE 4. Graph of exact and approximate solutions of test problem 6.4 for $\hat{m} = 6$.

Test problem 6.5. *Let us consider the nonlinear Volterra integral equation* [21]*:*

$$y(x) = x + \frac{1}{5}x^5 - \int_0^x t y^3(t)dt. \tag{29}$$

Exact solution of (29) is found to be $y(x) = x$. *Table 3 shows the comparison of exact, approximate solutions, and absolute errors of test problem 6.5 obtained by using the method described in section 4 for* $\hat{m} = 6$ *and figure 5 shows the graph of exact and approximate solutions of test problem 6.5 for* $\hat{m} = 6$.

TABLE 5. Comparison of exact and approximate for test problem 6.5 for $\hat{m} = 6$.

x	Exact	Approximate	Absolute error
0.5/6	0.083333	0.08334	6.9153e-06
1.5/6	0.25	0.24995	5.4236e-05
2.5/6	0.41667	0.41618	4.8342e-04
3.5/6	0.58333	0.58436	1.0284e-03
4.5/6	0.75	0.74969	3.1162e-04
5.5/6	0.91667	0.91435	2.3187e-03

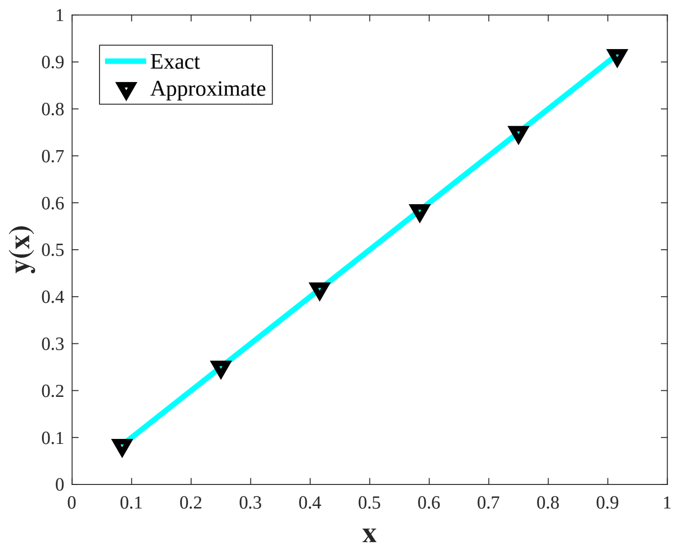

FIGURE 5. Graph of exact and approximate solutions of test problem 6.5 for $\hat{m} = 6$.

7. Conclusion

In this article we have provided an effective technique to solve nonlinear Volterra integral equations using Fibonacci wavelets. Nonlinear-Volterra integral equations are reduced to a system of nonlinear algebraic equations with unknown Fibonacci coefficients, by using Fibonacci wavelets, and their operational matrix of integration and these equations are solved by Newton's method. Error estimate of the proposed method is given. Moreover, the results obtained are in good agreement with that of exact solution and hence we conclude that the method described is precise and accurate.

Acknowledgement

We would like to thank the editor and the referees for their positive comments, which have strengthened this manuscript significantly.

S. C. Shiralashetti thank University Grants Commission (UGC), New Delhi, through UGC-SAP DRS-III for 2016-2021: F.510/3/DRS-III/2016 (SAP-I) for supporting this work partially.

Lata Lamani thank Karnatak University Dharwad (KUD) under University Research Studentship(URS) for 2016-2019: K. U. 40 (SC/ST)URS/2018-19/32/3/841 Dated: 07/07/2018 for supporting this work.

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