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ON COMPOSITION OF ENTIRE FUNCTIONS WITH FINITE ITERATED ORDER

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Abstract. In this article we investigated growth of two composite entire functions of finite iterated order. We introduced finite iterated order of an entire function in terms of their maximum term. Also we proved some results on the growth of composite entire functions of finite iterated order by using maximum terms.

1. Introduction

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function. The maximum term $\mu_f(r)$ of the function $f(z)$ on $|z| = r$ is defined as $\mu_f(r) = \max_{n \geq 0} |a_n| r^n$ and the maximum modulus of $f(z)$ on $|z| = r$ is defined as $M_f(r) = \max_{|z|=r} |f(z)|$.

For two transcendental entire functions $f(z)$ and $g(z)$ it is proved that $\lim_{r\to\infty} \frac{\log T_{f\circ g}(r)}{T_f(r)} = \infty$ and $\lim_{r\to\infty} \frac{\log T_{f\circ g}(r)}{T_g(r)} = \infty$ [1]. Many results have been proved on the composition of two entire functions with finite order ([1],[3],[4],[7],[8],[13]). In 2009 [11] Tu et al. introduced the notations of iterated order for entire functions of fast growth and proved some theorems on the composition of entire functions with finite iterated order.

In this paper we investigate some results of composite entire functions on finite iterated order.

Now let $f(z)$ be a meromorphic function, by Nevanlinna theory [2], the order $\rho(f)$ and lower order $\lambda(f)$ of $f(z)$ are defined by

$$\begin{aligned} \rho(f) &= \limsup_{r\to\infty} \frac{\log T_f(r)}{\log r} \\ \lambda(f) &= \liminf_{r\to\infty} \frac{\log T_f(r)}{\log r}. \end{aligned}$$

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We use the notations $\exp_1 r = e^r, \exp_{i+1} r = \exp(\exp_i r)$ for $0 \le r < \infty$ and $i = 1, 2, ...$ Also for sufficiently large r, we use the notations $\log_1 r = \log r, \log_{i+1} r = \log(\log_i r)$.

Definition 1 In [11], Tu et al. introduced the definition of iterated p order $\rho_p(f)$ of an entire function f as

$$\rho_p(f) = \limsup_{r\to\infty} \frac{\log_{p+1} M_f(r)}{\log r} = \limsup_{r\to\infty} \frac{\log_p T_f(r)}{\log r} (p \in \mathbb{N}). \tag{1}$$

Similarly, the iterated p lower order $\lambda_p(f)$ of an entire function f as

$$\lambda_p(f) = \liminf_{r\to\infty} \frac{\log_{p+1} M_f(r)}{\log r} = \liminf_{r\to\infty} \frac{\log_p T_f(r)}{\log r} (p \in \mathbb{N}). \tag{2}$$

Definition 2 [11] The finiteness degree of the order of an entire function f is defined by

$$i(f) = \begin{cases} 0 & \text{for } f \text{ polynomial,} \\ \min\{q \in \mathbb{N} : \rho_p(f) < \infty\} & \text{for } f \text{ transcendental for which some } q \in \mathbb{N} \text{ with } \rho_q(f) < \infty \text{ exists.} \\ \infty & \text{for } f \text{ with } \rho_p(f) = \infty \text{ for all } p \in \mathbb{N}. \end{cases}$$

Then it is clear that $i(f)$ and $i(g)$ are positive integers.

2. Some preliminary lemmas

In this scetion we shall present first the known lemmas.

Lemma 1 [10] Let that $\lambda(g) < \infty$. Then for any $\varepsilon > 0$, we have for sufficiently large r

$$M_{f\circ g}\left(r^{1+\varepsilon}\right) \ge M_f\left(M_g(r)\right).$$

Lemma 2 [1] Let $f(z)$ and $g(z)$ be entire function with $g(0) = 0$. Let α satisfy $0 < \alpha < 1$ and let $c(\alpha) = \frac{(1-\alpha)^2}{4\alpha}$. Then for $r > 0$

$$M_{f\circ g}(r) \ge M_f\left(c(\alpha) M_g(\alpha r)\right). \tag{3}$$

Further if $g(z)$ is any entire function then with $\alpha = \frac{1}{2}$, for sufficiently large values of r,

$$M_{f\circ g}(r) \ge M_f\left(\frac{1}{8} M_g\left(\frac{r}{2}\right) - |g(0)|\right). \tag{4}$$

Also from the definition it is immediate consequence that

$$M_{f\circ g}(r) \le M_f\left(M_g(r)\right) \tag{5}$$

Lemma 3 [12] Suppose that $f(z)$ is a transcendental entire function of finite order. Let $r = l(u)$ be the inverse function of $u = M_f(r)$. Then, given $\varepsilon > 0$, there exists a constant $A(\varepsilon)$ such that the equation $f(z) = a$ has a root in the annulus

$$l(|a|) \le |z| \le l(|a|)^{1+\varepsilon}$$

provided that $|a| > A(\varepsilon)$.

Lemma 4 [9] For $0 \le r < R$,

$$\mu_f(r) \le M_f(r) \le \frac{R}{R-r} \mu_f(R). \tag{6}$$

Using this result we get

$$\rho_p(f) = \limsup_{r\to\infty} \frac{\log_{p+1} \mu_f(r)}{\log r}$$

and

$$\lambda_p(f) = \liminf_{r\to\infty} \frac{\log_{p+1} \mu_f(r)}{\log r}.$$

Lemma 5 [9] Let $f(z)$ and $g(z)$ be entire functions, then for $\alpha > 1$, and $0 < r < R$,

$$\mu_{f\circ g}(r) \leq \frac{\alpha}{\alpha - 1}\mu_f\left(\frac{\alpha R}{R-r}\mu_g(r)\right).$$

In particular taking $\alpha = 2$ and $R = 2r$,

$$\mu_{f\circ g}(r) \leq 2\mu_f(4\mu_g(2r)) \tag{7}$$

Lemma 6 [9] Let $f(z)$ and $g(z)$ be entire functions with $g(0) = 0$. Let α satisfy $0 < \alpha < 1$ and let $c(\alpha) = \frac{(1-\alpha)^2}{4\alpha}$. Also let $0 < \delta < 1$ then

$$\mu_{f\circ g}(r) \geq (1-\delta)\mu_f(c(\alpha)\mu_g(\alpha\delta r)). \tag{8}$$

And if $g(z)$ is any entire function, then with $\alpha = \delta = \frac{1}{2}$, for sufficiently large values of r,

$$\mu_{f\circ g}(r) \geq \frac{1}{2}\mu_f\left(\frac{1}{8}\mu_g\left(\frac{r}{4}\right) - |g(0)|\right). \tag{9}$$

3. Main results

Using the previous lemmas and from[5], we can prove the following lemma.

Lemma 7 Suppose that $f(z)$ and $g(z)$ are entire functions of finite iterated order and put

$$\log_p M_f(r) \equiv (\log r)^{\phi_f(r)}. \tag{10}$$

Then, for any $\varepsilon > 0$,

$$\log_{p+q+1} M_{f\circ g}\left(r^{1+\varepsilon}\right) \geq \log_q\left(\phi_f(M_g(r))\log(\log M_g(r))\right) \tag{11}$$

and

$$\log_{p+q+1} M_{f\circ g}(r) \leq \log_q\left(\phi_f(M_g(r))\log(\log M_g(r))\right) \tag{12}$$

for all sufficiently large values of r.

Proof. From Lemma 3, given $\varepsilon > 0$, there exists a constant $A(\varepsilon)$ such that the equation $g(z) = a$ has a root in the annulus

$$l(|a|) \leq |z| \leq l(|a|)^{1+\varepsilon}$$

provided that $|a| > A(\varepsilon)$. We choose r_0 such that

$$M_g(r_0) > A(\varepsilon),$$

and we take $\rho = \rho_g(r) = M_g(r)$ for any $r \geq r_0$. Then, there exists an a_q such that $|a_q| = \rho$ and

$$\max_{|\omega|=\rho} |f(\omega)| = |f(a_q)|$$

and such that the equation $g(z) = a_q$ has a root in the annulus

$$r = (|a_q|) \leq |z| \leq (|a_q|)^{1+\varepsilon} = r^{1+\varepsilon}.$$

Thus, there exists a z_0 such that

$$|z_0| \leq r^{1+\varepsilon} \text{ and } g(z_0) = a_q.$$

Therefore, we have

$$M_{f\circ g}\left(r^{1+\varepsilon}\right) \geq |(f\circ g)(z_0)| = |f(a_q)| = M_f(\rho)$$

for all $r \geq r_0$, where $\rho = M_g(\rho)$. Hence we have

$$\begin{aligned} \log_{p+q} M_{f\circ g}\left(r^{1+\varepsilon}\right) &\geq \log_{p+q} M_f(\rho) \\ &= \log_q\left(\log_p M_f(\rho)\right) \\ &= \log_q (\log\rho)^{\phi_f(\rho)} \\ &= \log_{q-1}\left(\log(\log\rho)^{\phi_f(\rho)}\right) \\ &= \log_{q-1}\left(\phi_f(\rho)\log(\log\rho)\right) \\ &= \log_{q-1}\left(\phi_f(M_g(r))\log(\log M_g(r))\right) \end{aligned}$$

and

$$\log_{p+q+1} M_{f\circ g}\left(r^{1+\varepsilon}\right) \geq \log_q\left(\phi_f(M_g(r))\log(\log M_g(r))\right)$$

for all $r \geq r_0$.

Also, by the maximum modulus principle, we get

$$M_{f\circ g}(r) \leq M_f(M_g(r)).$$

Then we have

$$\begin{aligned} \log_{p+q+1} M_{f\circ g}(r) &\leq \log_{q+1}\left(\log_p\left(M_f(M_g(r))\right)\right) \\ &= \log_{q+1}\left((\log M_g(r))^{\phi_f(M_g(r))}\right) \\ &= \log_q\left(\phi_f(M_g(r))\log(\log M_g(r))\right) \end{aligned}$$

This proves the lemma.

Now we can proof the following theorems.

Theorem 1 If f and g are transcendental entire functions of finite iterated order with $i(f) = p$, $i(g) = q$ also $\rho_p(f) = 0$ and $0 < \rho_q(g) < \infty$, then $\rho_{p+q}(f\circ g) = \infty$ provided (a) $\lambda_q(g) > 0$ and $\limsup_{r\to\infty} \log_p \phi_f(r) = \infty$ or (b) $\lambda_q(g) = 0$ and $\lim_{r\to\infty} \log_p \phi_f(r) = \infty$, where $\phi_f(r)$ is defined by

$$\log_{p+1} M_f(r) = (\log r)^{\phi(r)} \tag{13}$$

for sufficiently large values of r

Proof. By (11), for any $\varepsilon > 0$, we have

$$\limsup_{r\to\infty} \frac{\log_{p+q+1} M_{f\circ g}\left(r^{1+\varepsilon}\right)}{\log r^{1+\varepsilon}} \geq \limsup_{r\to\infty} \frac{\log_p \phi_f(M_g(r)) \log_{q+1} M_g(r)}{(1+\varepsilon)\log r}.$$

(a) If $\lambda_q(g) > 0$ and $\limsup_{r\to\infty} \log_p \phi(r) = \infty$, then taking $\varepsilon = \frac{\lambda_q(g)}{2}$,
we see

$$\log_q M_g(r) > r^{\frac{\lambda_q(g)}{2}}$$

for all sufficiently large values of r. Thus

$$\begin{aligned} \limsup_{r\to\infty} \frac{\log_{p+q+1} M_{f\circ g}\left(r^{1+\varepsilon}\right)}{\log r^{1+\varepsilon}} &\geq \limsup_{r\to\infty} \frac{\log_p \phi_f(M_g(r)) \log r^{\frac{\lambda_q(g)}{2}}}{\left(1 + r^{\frac{\lambda_q(g)}{2}}\right)\log r} \\ &= \frac{\frac{\lambda_q(g)}{2}}{1 + \frac{\lambda_q(g)}{2}} \limsup_{r\to\infty} \log_p \phi_f(M_g(r)) \\ &= \infty \end{aligned}$$

since $M_g(r)$ is increasing, continuous and unbounded in r.

(b) $\lambda_q(g) = 0$ and $\lim_{r\to\infty} \log_p \phi_f(r) = \infty$, then for any $\varepsilon' > 0$, it holds that

$$\begin{aligned}
\limsup_{r\to\infty} \frac{\log_{p+q+1} M_{f\circ g}\left(r^{1+\varepsilon'}\right)}{\log r^{1+\varepsilon'}} &\geq \limsup_{r\to\infty} \frac{\log_p \phi_f\left(M_g(r)\right) \log_{q+1} M_g(r)}{(1+\varepsilon')\log r} \\
&\geq \liminf_{r\to\infty} \log_p \phi_f\left(M_g(r)\right) . \limsup_{r\to\infty} \frac{\log_{q+1} M_g(r)}{(1+\varepsilon')\log r} \\
&= \frac{\lambda_q}{1+\varepsilon'} \liminf_{r\to\infty} \log_p \phi_f\left(M_g(r)\right) = \infty.
\end{aligned}$$

This proves the theorem.

Theorem 2 [9]Suppose that f and g are transcendental entire functions of finite iterated order with $i(f) = p$, $i(g) = q$ and $\rho_q(g) > 0, \rho_p(f) = 0$. Let

$$\limsup_{r\to\infty} \log_p \phi_f(r) = \tau. \tag{14}$$

If $\tau < \infty$, then

$$\rho_{p+q}\left(f\circ g\right) \leq \tau \rho_q(g). \tag{15}$$

Furthermore, if $\lim_{r\to\infty} \log_p \phi_f(r) = \tau$, then the equality in 15 holds.

Proof For given any $\varepsilon > 0$, and since $g(z)$ is of order $\rho_q(g)$ and $\limsup_{r\to\infty} \log_p \phi_f(r) = \tau$ then we obtain

$$\log_q M_g(r) < r^{\rho_q(g)+\varepsilon}, \tag{16}$$

for all sufficiently large values of r and

$$\log_p \phi_f(r) < \tau + \varepsilon. \tag{17}$$

Now from (12),

$$\log_{p+q+1} M_{f\circ g}\left(r\right) \leq \log_p \phi_f\left(M_f\left(r\right)\right) \log\left(\log M_f(r)\right)$$

i.e

$$\begin{aligned}
\limsup_{r\to\infty} \frac{\log_{p+q+1} M_{f\circ g}\left(r\right)}{\log r} &\leq \limsup_{r\to\infty} \frac{\log_p \phi_f\left(M_g\left(r\right)\right) \log_{q+1} M_g(r)}{\log r} \\
&\leq \limsup_{r\to\infty} \frac{(\tau+\varepsilon)\log r^{\rho_q(g)+\varepsilon}}{\log r} \\
&= (\tau+\varepsilon)\left(\rho_q(g)+\varepsilon\right).
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, therefore we get

$$\rho_{p+q}\left(f\circ g\right) \leq \tau \rho_q(g).$$

Also it is clear that the equality is hold if $\lim_{r\to\infty} \log_p \phi_f(r) = \tau$. Hence the theorem.

Theorem 3 If f and g are transcendental entire functions of finite iterated order with $i(f) = p$, $i(g) = q$ and if (i) $\lambda_q(g) = \infty$ or (ii) $\lambda_p(f) > 0$ then $\lambda_{p+q}\left(f\circ g\right) = \infty$.

Proof. (i) Let $\lambda_q(g) = \infty$.

From Lemma 2

$$\begin{aligned}\log_{p+q} M_{f\circ g}(r) &\geq \log_{p+q} M_f\left(\frac{1}{8}M_g\left(\frac{r}{2}\right)-|g(0)|\right)\\ &\geq \frac{\log_{p+q} M_f\left(\frac{1}{8}M_g\left(\frac{r}{2}\right)-|g(0)|\right)}{\log_q\left(\frac{1}{8}M_g\left(\frac{r}{2}\right)-|g(0)|\right)}.\log_q\left(\frac{1}{8}M_g\left(\frac{r}{2}\right)-|g(0)|\right)\\ &\geq \frac{\log_{p+q} M_f(r)}{\log_q(r)}.\left(\log_q M_g\left(\frac{r}{2}\right)+O(1)\right).\end{aligned}$$

Since $\frac{\log_{p+q} M_f(r)}{\log_q(r)}$ is an increasing function of r for large r and $\frac{1}{8}M_g\left(\frac{r}{2}\right)-|g(0)|$ is large for r, we get

$$\log_{p+q} M_{f\circ g}(r) \geq \log_q M_g\left(\frac{r}{2}\right)$$

for large r.

Hence

$$\lambda_{p+q}(f\circ g) \geq \lambda_q(g) = \infty.$$

$$\begin{aligned}T_{f\circ g}(r) &\geq \frac{1}{3}\log M_f\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right)\\ &\geq \frac{1}{3}\log M_f\left\{\left(\frac{1}{9}M_g\left(\frac{r}{4}\right)\right)^{\lambda_p(f)-\varepsilon}\right\}\\ &\geq \frac{1}{3}\exp_{p-1}\left\{c_1\exp_q\left(c_2 r^{\lambda_q(g)-\varepsilon}\right)\right\}\end{aligned}$$

where $c_1 > \rho_p(f), c_2 \geq 1$ are constants, not necessarily the same at each occurence. Then we get

$$\begin{aligned}\log_{p+q} T_{f\circ g}(r) &\geq \log_{q+1}\left\{c_1\exp_q\left(c_2 r^{\lambda_q(g)-\varepsilon}\right)\right\}+O(1)\\ &= \log\left(c_2 r^{\lambda_q(g)-\varepsilon}\right)+O(1)\\ &= (\lambda_q(g)-\varepsilon)\log r+O(1)\end{aligned}$$

i.e

$$\frac{\log_{p+q} T_{f\circ g}(r)}{\log r} \geq \lambda_q(g).$$

Hence first part of Theorem 3 is proved.

(ii) If $\lambda_p(f) > 0$ and also let $\lambda_q(g) < \infty$.

$$\begin{aligned}\lambda_{p+q}(f\circ g) &= \liminf_{r\to\infty}\frac{\log_{p+q+1} M_{f\circ g}\left(r^{1+\varepsilon}\right)}{(1+\varepsilon)\log r}\\ &\geq \liminf_{r\to\infty}\left(\frac{\log_{p+q+1} M_f(M_g(r))}{\log M_g(r)}\frac{\log M_g(r)}{(1+\varepsilon)\log r}\right).\end{aligned}$$

As $g(z)$ is transcendental, for large number $k > 0$, then for $r \geq r_0$

$$\frac{\log_{q+1} M_g(r)}{(1+\varepsilon)\log r} > k.$$

This shows that,

$$\lambda_{p+q}(f\circ g) \geq \lambda_p(f).k.$$

Since $M_g(r)$ is continuous, increasing and unbounded in r, we get

$$\lambda_{p+q}(f \circ g) = \infty,$$

since $\lambda_{p+q}(f) > 0$. Hence proved the theorem.

Theorem 4 If f and g are transcendental entire functions of finite iterated order with $i(f) = p$, $i(g) = q$ and if $\lambda_q(g) < \infty$ and $\limsup_{r\to\infty} \log_p \phi_f(r) = \tau < \infty$, then

$$\lambda_{p+q}(f \circ g) \le \tau.\lambda(g) \le \rho_{p+q}(f \circ g). \tag{18}$$

Furthermore, in the above result the first inequaity becomes equality if

$$\lim_{r\to\infty} \log_p \phi_f(r) = \tau < \infty.$$

Proof. We have from maximum modulus principle,

$$M_{f\circ g}(r) \le M_f(M_g(r)) \tag{19}$$

Hence,

$$\begin{aligned} \lambda_{p+q}(f \circ g) &= \liminf_{r\to\infty} \frac{\log_{p+q+1} M_{f\circ g}(r)}{\log r} \\ &\le \liminf_{r\to\infty} \frac{\log_{p+q+1} M_f(M_g(r))}{\log r} \\ &\le \limsup_{r\to\infty} \frac{\log_{p+q+1} M_f(M_g(r))}{\log_{q+1} M_g(r)} \liminf_{r\to\infty} \frac{\log_{q+1} M_g(r)}{\log r} \\ &= \tau.\lambda_q(g) \end{aligned}$$

which proves the first inequality of (18) .

Again by Lemma 1, we have

$$\begin{aligned} \rho_{p+q}(f \circ g) &= \limsup_{r\to\infty} \frac{\log_{p+q+1} M_{f\circ g}\left(r^{1+\varepsilon}\right)}{(1+\varepsilon)\log r} \\ &\ge \limsup_{r\to\infty} \left[\frac{\log_{p+q+1} M_f(M_g(r))}{\log_{q+1} M_g(r)}.\frac{\log_{q+1} M_g(r)}{(1+\varepsilon)\log r}\right] \\ &\ge \tau.\frac{\lambda_q(g)}{1+\varepsilon}. \end{aligned}$$

Letting $\varepsilon \to 0$, then we get

$$\tau.\lambda_q(g) \le \rho_{p+q}(f \circ g).$$

This proves the second inequality of (18).

Finally, if the limit

$$\lim_{r\to\infty} \log_p \phi_f(r) = \tau$$

exists, then we have

$$\begin{aligned} \lambda_{p+q}(f \circ g) &\ge \liminf_{r\to\infty} \left[\frac{\log_{p+q+1} M_f(M_g(r))}{\log_{q+1} M_g(r)}.\frac{\log_{q+1} M_g(r)}{(1+\varepsilon)\log r}\right] \\ &= \tau\frac{\lambda_q(g)}{1+\varepsilon} \end{aligned}$$

which gives

$$\lambda_{p+q}(f \circ g) = \tau.\lambda_q(g).$$

Hence the theorem is proved.

Remark 1 If $\lambda_q(g) = \infty$, then by Theorem 3, $\rho_{p+q}(f \circ g) = \lambda_{p+q}(f \circ g) = \infty$, and the inequalities in (18) become trivial. If $\lambda_q(g) > 0$ and $\tau = \infty$, then by Theorem 1, $\rho_{p+q}(f \circ g) = \infty$. Hence the inequality are trivial.

Theorem 5 Suppose that $\lambda_p(f) = \lambda_q(g) = 0$ and that

$$\liminf_{r\to\infty} \frac{\log\log M_g(r)}{(\log r)^{\alpha}} = a > 0, \liminf_{r\to\infty} \frac{\phi_f(r)}{(\log\log r)^{\beta}} = b > 0$$

for any positive numbers α and β with $\alpha < 1$ and $\alpha(\beta+1) > 1$. Then $\lambda(f \circ g) = \infty$.

Proof. Proof of this theorem is same as the previous theorem.

Theorem 6 Suppse that $\lambda_p(f) = \lambda_q(g) = 0$ and that

$$\liminf_{r\to\infty} \frac{\log_{k+1} M_g(r)}{[\log_k(r)]^{\alpha}} = a > 0, \quad \liminf_{r\to\infty} \frac{\log_{k-1}(\phi_f(r))}{\left[\log_{k+1}(r)\right]^{\beta}} = b > 0$$

for any positive integer $k \geq q+1$ and any positive numbers α and β with $\max(\alpha, \alpha\beta) > 1$. Then $\lambda_{p+q}(f \circ g) = \infty$.

Proof. For $0 < \varepsilon < \min(a, b)$

$$\log_{p+q+1} M_{f\circ g}\left(r^{1+\varepsilon}\right) \geq \phi_f(M_g(r)) \log_{p+q+1} M_g(r)$$

i.e,

$$\begin{aligned}
\liminf_{r\to\infty} \frac{\log_{p+q+1} M_{f\circ g}\left(r^{1+\varepsilon}\right)}{\log r^{1+\varepsilon}} &\geq \liminf_{r\to\infty} \left[\phi_f(M_g(r)) \frac{\log_{p+q+1} M_g(r)}{(1+\varepsilon)\log r}\right] \\
&\geq \liminf_{r\to\infty} \frac{\exp_{k-1}\left[(b-\varepsilon)(a-\varepsilon)^{\beta}(\log_k(r))^{\alpha\beta}\right] \log_q\left[(a-\varepsilon)(\log_k(r))^{\alpha}\right]}{(1+\varepsilon)\log r}.
\end{aligned}$$

Putting $\log_k(r) = x$, $(b-\varepsilon)(a-\varepsilon)^{\beta} = d_1$ and $(a-\varepsilon) = d_2$, thus we have

$$\lambda_{p+q}(f \circ g) \geq \liminf_{r\to\infty} \frac{\exp_{k-1}\left(d_1 x^{\alpha\beta}\right) . \log_k\left(d_2 x^{\alpha}\right)}{(1+\varepsilon)\exp_{k-1}(x)} = \infty,$$

since $\max(\alpha, \alpha\beta) > 1$.

This completes the proof.

Theorem 7 Suppose that $\lambda_p(f) = \lambda_q(g) = 0$ and that one of the following conditions (I) and (II) is satisfied:

$$(I) \quad \liminf_{r\to\infty} \frac{\log_{k+1} M_g(r)}{(\log_k r)^{\alpha_1}} = a_1 < \infty, \limsup_{r\to\infty} \frac{\log_{k-1}\phi(r)}{\left(\log_{k+1} r\right)^{\beta_1}} = b_1 < \infty$$

for any positive integer $k \geq q+1$ and for any positive numbers α_1 and β_1 with $\alpha_1(\beta_1+1) < 1$;

$$(II) \quad \liminf_{r\to\infty} \frac{\log_{k+1} M_g(r)}{\left(\log_{k+1} r\right)^{\alpha_2}} = a_2 < \infty, \limsup_{r\to\infty} \frac{\log_k \phi(r)}{\left(\log_{k+1} r\right)^{\beta_2}} = b_2 < \infty$$

for any positive numbers α_2 and β_2 with $\alpha_2\beta_2 < 1$.

Then $\lambda_{p+q}(f \circ g) = 0$.

Proof The proof of this theorem is also same as those in the previous theorem.

Theorem 8 Let $f(z)$ and $g(z)$ be two entire functions of finite iterated order with $i(f) = p$, $i(g) = q$ and $\rho_q(g) < \lambda_p(f) < \rho_p(f)$, then

$$\limsup_{r\to\infty} \frac{\log_{p+q+1} \mu_{f\circ g}(r)}{\log_p \mu_f(r)} = 0.$$

Proof. From the definition of $\rho_p(f)$ and $\lambda_p(f)$ we get

$$\log_p \mu_f(r) < r^{\rho_p(f)+\varepsilon} \tag{20}$$

for large r and

$$\log_p \mu_f(r) > r^{\lambda_p(f)-\varepsilon} \tag{21}$$

for large r.

From (7),

$$\begin{aligned}\log_{p+q+1} \mu_{f\circ g}(r) &\leq \log_{p+q+1}\left[2\mu_f\left(4\mu_g\left(2r\right)\right)\right] \\ &\leq \log_{p+q+1}\left[\mu_f\left(4\mu_g\left(2r\right)\right)\right] + O\left(1\right).\end{aligned}$$

Using (20) we have,

$$\begin{aligned}\log_{p+q+1} \mu_{f\circ g}(r) &\leq \log_{q+1}\left[\left\{4\mu_g\left(2r\right)\right\}^{\rho_p(f)+\varepsilon}\right] + O\left(1\right) \\ &\leq \log_q\left(\rho_p(f)+\varepsilon\right)\log\left\{4\mu_g\left(2r\right)\right\} + O\left(1\right) \\ &\leq \log_q\left(\rho_p(f)+\varepsilon\right)\log\left\{\mu_g\left(2r\right)\right\} + O\left(1\right) \\ &\leq \log_q\left(\rho_p(f)+\varepsilon\right)\exp_{q-1}\left(2r\right)^{\rho_q(g)+\varepsilon}. \end{aligned} \tag{22}$$

From (21) and (22) we get,

$$\frac{\log_{p+q+1} \mu_{f\circ g}(r)}{\log_p \mu_f(r)} \leq \frac{\log_q\left(\rho_p(f)+\varepsilon\right)\exp_{q-1}\left(2r\right)^{\rho_q(g)+\varepsilon}}{r^{\lambda_p(f)-\varepsilon}}$$

Since $\rho_q(g) < \lambda_p(f)$, now we choose $\varepsilon > 0$ such that

$$\rho_q(g) + \varepsilon < \lambda_p(f) - \varepsilon.$$

Therefore we have

$$\limsup_{r\to\infty} \frac{\log_{p+q+1} \mu_{f\circ g}(r)}{\log_p \mu_f(r)} = 0.$$

Theorem 9 Let $f(z)$ and $g(z)$ be entire functions of finite iterated order p and q respectively. If $\rho_q(g) > \rho_p(f)$ then

$$\limsup_{r\to\infty} \frac{\log_{p+q+1} \mu_{f\circ g}(r)}{\log_p \mu_f(r)} = \infty.$$

Proof. From Lemma 6 we get for large r

$$\begin{aligned}\log_{p+q+1}\left(\mu_{f\circ g}(r)\right) &\geq \log_{p+q+1}\left[\frac{1}{2}\mu_f\left(\frac{1}{8}\mu_g\left(\frac{r}{4}\right) - |g(0)|\right)\right] \\ &\geq \log_{p+q+1}\left[\mu_f\left(\frac{1}{8}\mu_g\left(\frac{r}{4}\right) - |g(0)|\right)\right] + O(1) \\ &\geq \log_{q+1}\left(\frac{1}{8}\mu_g\left(\frac{r}{4}\right)\right)^{\lambda_p(f)-\varepsilon} + O(1) \\ &> \log_q\left(\lambda_p(f)-\varepsilon\right)\log\left(\frac{1}{8}\mu_g\left(\frac{r}{4}\right)\right) + O(1) \\ &> \log_q\left(\lambda_p(f)-\varepsilon\right)\log\left(\mu_g\left(\frac{r}{4}\right)\right) + O(1) \\ &> \log_q\left(\lambda_p(f)-\varepsilon\right)\exp_{q-1}\left(\frac{r}{4}\right)^{\rho_q(g)-\varepsilon} + O(1).\end{aligned}$$

Thus for sufficiently large r, there exists a sequence $r = r_n$

$$\log_{p+q+1}\left(\mu_{f\circ g}\left(r_n\right)\right) > \log_q\left(\lambda_p(f)-\varepsilon\right)\exp_{q-1}\left(\frac{r_n}{4}\right)^{\rho_q(g)-\varepsilon} + O(1). \tag{23}$$

Also for large r,

$$\log_p \mu\left(r, f\right) < r^{\rho_p(f)+\varepsilon}.$$

So for the sequence $r = r_n$, sufficiently large

$$\frac{\log_{p+q+1}\left(\mu_{f\circ g}\left(r_n\right)\right)}{\log_p \mu\left(r_n, f\right)} > \frac{\log_q\left(\lambda_p(f)-\varepsilon\right)\exp_{q-1}\left(\frac{r_n}{4}\right)^{\rho_q(g)-\varepsilon}}{r_n^{\rho_p(f)+\varepsilon}}.$$

Since $\rho_q\left(g\right) > \rho_p\left(f\right)$, we choose $\varepsilon > 0$ such that $\rho_q\left(g\right) - \varepsilon > \rho_p\left(f\right) + \varepsilon$.

So we have

$$\limsup_{r\to\infty}\frac{\log_{p+q+1}\mu_{f\circ g}\left(r\right)}{\log_p \mu_f\left(r\right)} = \infty.$$

Corollary 1 Let $f(z)$ and $g(z)$ be transcendental entire functions of finite iterated order p and q respectively and let $\rho_q(g) > \rho_p(f)$. Then $f\circ g$ is of infinite order.

Proof.

$$\begin{aligned}\limsup_{r\to\infty}\frac{\log_{p+q+1}\mu_{f\circ g}\left(r\right)}{\log r} &= \limsup_{r\to\infty}\left[\frac{\log_{p+q+1}\mu_{f\circ g}\left(r\right)}{\log_p \mu_f\left(r\right)}.\frac{\log_p \mu_f\left(r\right)}{\log r}\right]\\ &\geq \limsup_{r\to\infty}\frac{\log_{p+q+1}\mu_{f\circ g}\left(r\right)}{\log_p \mu_f\left(r\right)}.\liminf_{r\to\infty}\frac{\log_p \mu_f\left(r\right)}{\log r}\end{aligned}$$

since for any transcendental entire function,

$$\liminf_{r\to\infty}\frac{\log_p \mu_f\left(r\right)}{\log r} = \infty.$$

From the previous theorem the result follows.

Theorem 10 Let $f(z)$ and $g(z)$ be transcendental entire functions of finite iterated order p and q respectively with $\rho_q\left(g\right) > 0$. Then

$$\limsup_{r\to\infty}\frac{\log_{p+q+1}\mu_{f\circ g}\left(r\right)}{\log_{q+1}\mu_g\left(r\right)} = \infty.$$

Proof. For a sequence $r = r_n$, sufficiently large, from (23),

$$\log_{p+q+1}\left(\mu_{f\circ g}\left(r_n\right)\right) > \log_q\left(\lambda_p(f)-\varepsilon\right)\exp_{q-1}\left(\frac{r_n}{4}\right)^{\rho_q(g)-\varepsilon} + O(1).$$

Also using the definition of $\rho_q(g)$ for the entire function g, we get

$$\log_{q+1}\mu_g(r) < \left(\rho_q(g)+\varepsilon\right)\log r$$

for large r.

Thus for a sequence $r = r_n$, sufficiently large, we obtain,

$$\frac{\log_{p+q+1}\mu_{f\circ g}\left(r_n\right)}{\log_{q+1}\mu_g(r_n)} > \frac{\log_q\left(\lambda_p(f)-\varepsilon\right)\exp_{q-1}\left(\frac{r_n}{4}\right)^{\rho_q(g)-\varepsilon}}{\left(\rho_q(g)+\varepsilon\right)\log r_n}$$

since $\rho_q(g) > 0$ and so we can choose $\varepsilon > 0$ such that $\rho_q(g) - \varepsilon > 0$.

Hence

$$\limsup_{r\to\infty}\frac{\log_{p+q+1}\mu_{f\circ g}\left(r\right)}{\log_{q+1}\mu_g\left(r\right)} = \infty.$$

Remark 2 In particular, $\lambda_q(g) > 0$, which implies that $\rho_q(g) > 0$, therefore we have

$$\limsup_{r\to\infty} \frac{\log_{p+q+1} \mu_{f\circ g}(r)}{\log_{q+1} \mu_g(r)} = \infty.$$

Theorem 11 Let $f(z)$ and $g(z)$ be transcendental entire function of finite iterated order p and q and let $\lambda_q(g) > 0$, then

$$\limsup_{r\to\infty} \frac{\log_{p+q+2} \mu_{f\circ g}(r)}{\log_{q+1} \mu_g(r)} \le \frac{\rho_q(g)}{\lambda_q(g)}.$$

Proof. From (22) it easily follows that

$$\log_{p+q+1} \mu_{f\circ g}(r) \le \log_q (\rho_p(f) + \varepsilon) \exp_{q-1} (2r)^{\rho_q(g)+\varepsilon}$$

for large r.

So for sufficiently large r

$$\log_{p+q+2} \mu_{f\circ g}(r) \le \log_q \{\log(\rho_p(f) + \varepsilon) + (\rho_q(g) + \varepsilon) \log r\} + O(1).$$

Again we have for sufficiently large r

$$\log_{q+1} \mu_g(r) > (\lambda_q(g) - \varepsilon) \log r.$$

Hence

$$\limsup_{r\to\infty} \frac{\log_{p+q+2} \mu_{f\circ g}(r)}{\log_{q+1} \mu_g(r)} \le \frac{\rho_q(g)}{\lambda_q(g)}.$$

Remark 3 Note that the result in Theorem 11 is sharp in the sense that there exists transcendental entire functions f and g with finite iterated order p and q such that

$$\limsup_{r\to\infty} \frac{\log_{p+q+2} \mu_{f\circ g}(r)}{\log_{q+1} \mu_g(r)} = \frac{\rho_q(g)}{\lambda_q(g)}.$$

Theorem 12 Let $h(z)$ and $f(z)$ be two entire functions of finite iterated order such that $\rho_s(h) < \lambda_p(f)$ then

$$\lim_{r\to\infty} \frac{\log_{q+s} \mu_{h\circ g}(r)}{\log_{p+q} \mu_{f\circ g}(r)} = 0$$

for any nonconstant entire function $g(z)$ of finite iterated order $\rho_q(g)$.

Proof. We have from Niino [6],

$$\begin{aligned}\mu_{f\circ g}(r) &\ge \frac{r-r'}{r} M_{f\circ g}(r') \\ &= \frac{r-r+r^{-\beta}}{r} M_{f\circ g}\left(r-r^{-\beta}\right) \\ &= \frac{1}{r^{\beta+1}} M_{f\circ g}\left(r-r^{-\beta}\right).\end{aligned}$$

Now, from the definition of $\lambda_p(f)$ we get,

$$r^{\lambda_p(f)-\varepsilon} < \log_p M_f(r)$$

for large r and also from the definition of $\rho_s(h)$ we have,

$$r^{\rho_s(h)+\varepsilon} \ge \log_s M_h(r)$$

for large r.

Hence

$$\begin{aligned}
\mu_{f\circ g}(r) &\geq \frac{1}{r^{\beta+1}}\exp_p\log_p M_{f\circ g}\left(r-r^{-\beta}\right) \\
&\geq \frac{1}{r^{\beta+1}}\exp_p\log_p M_f\left(\frac{M_g\left(r-r^{-\alpha}\right)-|g(0)|}{5r^{2(\alpha+1)}}-|g(0)|\right) \\
&> \frac{1}{r^{\beta+1}}\exp_p\left(\frac{M_g\left(r-r^{-\alpha}\right)-|g(0)|}{5r^{2(\alpha+1)}}-|g(0)|\right)^{\lambda_p(f)-\varepsilon} \\
&> \frac{1}{r^{\beta+1}}\exp_p\left(\frac{M_g\left(r-r^{-\alpha}\right)}{6r^{2(\alpha+1)}}\right)^{\lambda_p(f)-\varepsilon} \\
&> \exp_p\left(M_g(r,g)\right)^{\lambda_p(f)-\varepsilon}. \qquad (24)
\end{aligned}$$

On the other hand, for large r,

$$\begin{aligned}
\mu_{h\circ g}(r) &\leq M_{h\circ g}(r) \\
&\leq M_h\left(M_g(r)\right) \\
&= \exp_s\log_s M_h\left(M_g(r)\right) \\
&= \exp_s\left(M_g(r)\right)^{\rho_s(h)+\varepsilon}. \qquad (25)
\end{aligned}$$

Choose $\varepsilon > 0$ such that $\rho_s(h)+\varepsilon < \lambda_p(f)-\varepsilon$, thus from (24) and (25) we get,

$$\frac{\mu_{h\circ g}(r)}{\mu_{f\circ g}(r)} < \frac{\exp_s\left(M_g(r)\right)^{\rho_s(h)+\varepsilon}}{\exp_p\left(M_g(r)\right)^{\lambda_p(f)-\varepsilon}}.$$

Therefore

$$\lim_{r\to\infty}\frac{\mu_{h\circ g}(r)}{\mu_{f\circ g}(r)} = 0.$$

Theorem 13 Let f and g be entire functions of finite iterated order such that $0 < \lambda_p(f) < \infty$ and $0 < \lambda_q(g) < \infty$. If the entire functions h and k with finite iterated order s and t respectively satisfy

$$0 < \liminf_{r\to\infty}\frac{\log_{p+s}\mu_{f\circ h}(r)}{\log_{q+t}\mu_{g\circ k}(r)} \leq \limsup_{r\to\infty}\frac{\log_{p+s}\mu_{f\circ h}(r)}{\log_{q+t}\mu_{g\circ k}(r)} < \infty$$

then

$$\rho_s(h) = \rho_t(k).$$

Proof. First suppose that $\rho_s(h) = \rho_t(k)$.
Now for sufficiently large r, using (9) we obtain,

$$\begin{aligned}
\log_{p+s}\left[\mu_{f\circ h}(r)\right] &\geq \log_{p+s}\left[\frac{1}{2}\mu_f\left(\frac{1}{8}\mu_h\left(\frac{r}{4}\right)+O(1)\right)\right] \\
&\geq \left(\frac{1}{8}\mu_h\left(\frac{r}{4}\right)+\circ(1)\right)^{\lambda_p(f)-\varepsilon}+O(1) \\
&\geq \left(\frac{1}{9}\mu_h\left(\frac{r}{4}\right)\right)^{\lambda_p(f)-\varepsilon}+O(1).
\end{aligned}$$

And so for a sequence $r = r_n$ with $r_n \geq r_0$,

$$\begin{aligned}\log_{p+s}\left[\mu_{f\circ h}\left(r_n\right)\right] &\geq \exp_s\left(\log_s\left(\frac{1}{9}\mu_h\left(\frac{r_n}{4}\right)\right)^{\lambda_p(f)-\varepsilon}\right) + O(1)\\ &\geq \exp_s\left(\lambda_p(f)-\varepsilon\right)\left(\frac{r_n}{4}\right)^{\rho_s(h)-\varepsilon}. \end{aligned} \tag{26}$$

On the other hand,

$$\begin{aligned}\log_{q+t}\left[\mu_{g\circ k}\left(r\right)\right] &\leq \log_q M_{g\circ k}\left(r\right)\\ &\leq \log_q M_g\left(M_k(r)\right)\\ &\leq \left[M_k(r)\right]^{\rho_q(g)+\varepsilon}\\ &\leq \exp_t \log_t\left[M_k(r)\right]^{\rho_q(g)+\varepsilon}\\ &= \exp_t\left(\rho_q(g)+\varepsilon\right) r^{\rho_t(k)+\varepsilon} \end{aligned} \tag{27}$$

We choose ε such that $\rho_s(h) - \varepsilon < \rho_t(k) + \varepsilon$, then from (26) and (27) it follows that as $r \to \infty$

$$\frac{\log_{p+s}\mu_{f\circ h}\left(r\right)}{\log_{q+t}\mu_{g\circ k}\left(r\right)} = \infty.$$

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