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CONTACT METRIC MANIFOLDS SATISFYING FISCHER-MARSDEN EQUATION

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Abstract. In this paper we study contact metric manifolds satisfying Fischer-Marsden equation. We obtain an expression of the Riemannian curvature tensor of a contact metric manifold satisfying Fischer-Marsden equation. We consider the potential function λ of the Fischer-Marsden equation as the eigenvalue of the Ricci operator Q in the direction which is orthogonal to the Reeb vector field ξ. In this case we prove that the solution of the Fischer-Marsden equation is trivial. Also, we prove the Ricci operator has only the eigenvalue 0 corresponding to the eigen vectors orthogonal to ξ. Moreover, we prove a necessary and sufficient condition of the Ricci operator to be Reeb flow invariant. Finally, we show a necessary and sufficient condition of the Ricci operator to be parallel with the Riemannian connection.

1. Introduction

Let (M^n, g) be a compact orientable manifold with Riemannian metric g and $\mathcal{M}$ be the set consisting of all Riemannian metrics on the manifold (M^n, g) of unit volume. Let g^* be any symmetric tensor field of type $(0, 2)$ on M^n. It is well known that the scalar curvature r is a non-linear function of the metric g, where the linearization $\pounds_g(g^*)$ is defined by

$$\pounds_g(g^*) = -\Delta_g(\mathrm{tr}_g\, g^*) + div(div(g^*)) - g(g^*, S_g),$$

where Δ_g is the negative Laplacian of g and S_g is its Ricci tensor. The formal L^2-adjoint $\pounds^*_g(\lambda)$ of the linearized scalar curvature operator $\pounds_g(\lambda)$ is defined by

$$\pounds^*_g(\lambda) = -(\Delta_g\lambda)g + \nabla^2_g\lambda - \lambda S_g, \tag{1}$$

where $(\nabla^2_g\lambda)(X, Y) = Hess_g\lambda(X, Y) = g(\nabla_X D\lambda, Y)$ is the Hessian of the smooth function λ on M^n, $\Delta_g\lambda = g(\nabla_{e_i} D\lambda, e_i)$ and D is the gradient of the metric g. The equation

$$\pounds^*_g(\lambda) = 0 \tag{2}$$

is referred as the Fischer-Marsden equation ([10], [12]). Obviously, if the potential function λ is a non-zero constant, then (2) becomes an Einstein metric ([3], [14]).

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In 2017, Patra et al. studied [10] the Fischer-Marsden solutions on almost CoKähler manifolds and they proved that the Fischer-Marsden equation has only trivial solution on almost CoKähler manifolds of dimension greater than 3 with ξ belonging to the (κ, μ)-nullity distribution and $\kappa < 0$. In 2018 [11], Patra and Ghosh characterize the solutions of Fischer-Marsden equation on K-contact and (k, μ)-contact manifolds and shows that a complete K-contact metric satisfying $\mathcal{L}_g^*(\lambda) = 0$ is Einstein and is isometric to the unit sphere S^{2n+1}. They also proved that a non-Sasakian (κ, μ)-contact metric satisfying the Fischer-Marsden equation is flat for dimension 1 and locally isometric to the product of a Euclidean space E^{n+1} and a sphere $S^n(4)$ of constant curvature for dimension greater than 1. Prakasha et al. in [12] studied Fischer-Marsden equation on non-Kenmotsu $(\kappa, \mu)'$-almost Kenmotsu manifolds and they demonstrated that this type of manifolds are locally isometric to the Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$ or isometric to the wrapped products $\mathbb{H}(\alpha) \times_f \mathbb{R}$, $\quad \mathbb{B}^{n+1}(\alpha') \times_{f'} \mathbb{R}^n$, where $\mathbb{H}^{n+1}(\alpha)$ is the hyperbolic space of constant curvature $\alpha = -1 - \frac{2}{n} - \frac{1}{n^2}$, tangent to the distribution $[\xi] \bigoplus [\gamma]$, $\mathbb{B}^{n+1}(\alpha')$ is a space of constant curvature $\alpha' = -1 + \frac{2}{n} - \frac{1}{n^2}$, tangent to the distribution $[\xi] \bigoplus [-\gamma]$, $f = ce^{(1-\frac{1}{n})t}$ and $f' = c'e^{(1-\frac{1}{n})t}$, with c, c' positive constants. Recently, in 2019, the work of Kar and Majhi regarding almost CoKähler manifolds satisfying Miao-Tam equation is very relevant [13], since the Fischer-Marsden equation is a particular case of the Miao-Tam equation. Then the present authors have been manifested that in a manifold of dimension greater than 3 with ξ belonging to (k, μ) nullity distribution with $k < 0$, gradient of the potential function λ is pointwise collinear with ξ and λ is constant. Finally, they have been proved that the solutions of the Miao-Tam equation on such a manifold are either trivial or Einstein.
The organization of this paper is as follows. After introduction, in section 2, we study some basic axioms and formulae of contact metric manifolds. In section 3, we classify contact metric manifolds satisfying Fischer-Marsden equation. Here we obtain an expression of Riemannian curvature tensor and prove that the solutions of this equation are trivial. Also we show the Ricci operator has only the eigenvalue 0 corresponding to any eigen vector orthogonal to ξ. In section 4, we deduce a necessary and sufficient condition for Ricci operator to be Reeb flow invariant. In the last section, we obtain a necessary and sufficient condition for Ricci operator to be parallel with respect to the Riemannian connection.

2. Contact metric manifolds

In this section, we recall the basic definitions and formulae which are very important regarding contact metric manifolds. Full details can be found in ([1], [2], [8], [9]).
A contact manifold is an odd dimensional Riemannian manifold M^{2n+1} equipped with a global 1-form η, known as contact form for which $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. In connection of this η there exists a unique vector field ξ, called the Reeb vector field for which $\eta(\xi) = 1$ and $d\eta(\xi, .) = 0$. It is well known that a contact manifold admits an almost contact structure (ϕ, ξ, η), where ϕ is a vector field of type (1,1), called global tensor field such that

$$\phi^2 X = -X + \eta(X)\xi. \tag{3}$$

Then also

$$\phi\xi = 0, \quad \eta \circ \phi = 0. \tag{4}$$

Furthermore, an almost contact structure is said to be a contact metric structure if the following are hold:

$$d\eta(X,Y) = g(X,\phi Y), \qquad g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y). \tag{5}$$

The manifold M^{2n+1} together with the structure tensor (η,ξ,ϕ,g) is called a contact metric manifold.
On a contact metric manifold, we define two symmetric operators h and l of type $(1,1)$ by $h = \frac{1}{2}L_\xi\phi$ and $l = R(.,\xi)\xi$ satisfying the axioms

$$\operatorname{Tr} h = \operatorname{Tr} h\phi = l\xi = h\phi + \phi h = 0, \tag{6}$$

where $L_\xi\phi$ is the Lie derivative of ϕ along ξ. On a contact metric manifold we have some relations as follows [2]:

$$\nabla_X\xi = -\phi X - \phi hX, \tag{7}$$

$$\nabla_\xi\phi = 0, \tag{8}$$

$$R(\xi,X)\xi - \phi R(\xi,\phi X)\xi = 2(h^2+\phi^2)X, \tag{9}$$

$$\nabla_\xi h = \phi - \phi h^2 - \phi l, \tag{10}$$

$$S(\xi,\xi) = 2n - \operatorname{Tr} h^2, \tag{11}$$

$$R(X,Y)\xi = -(\nabla_X\phi)Y + (\nabla_Y\phi)X - (\nabla_X\phi h)Y + (\nabla_Y\phi h)X, \tag{12}$$

where ∇ is the Levi-Civita connection and R is the Riemannian curvature tensor of (M,g) is given by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - [X,Y]Z.$$

The contact metric manifold is K-contact if ξ is Killing vector field which is if and only if $h = 0$. Finally, if the Riemannian curvature tensor satisfies

$$R(X,Y)Z = \eta(Y)X - \eta(X)Y, \tag{13}$$

or, equivalently, if

$$(\nabla_X\phi)Y = g(X,Y)\xi - \eta(Y)X \tag{14}$$

holds, then the manifold is Sasakian. We note that a Sasakian manifold is always K-contact, but the converse only holds in dimension three.
Now we attend the following definitions:
Definition 2.1 On an almost contact metric manifold (M,ϕ,ξ,η,g), the Ricci operator Q is said to be Reeb flow invariant if it satisfies ([4]-[7])

$$\mathcal{L}_\xi Q = 0, \tag{15}$$

where $\mathcal{L}_\xi Q$ is the Lie derivative of the Ricci operator Q along the Reeb vector field ξ.
Definition 2.2 On an almost contact metric manifold (M,ϕ,ξ,η,g), the Ricci operator Q is said to be Ricci parallelism with respect to the Riemannian connection ∇ if the Ricci operator Q satisfies [15]

$$\nabla Q = 0. \tag{16}$$

3. Contact metric manifold satisfying Fischer-Marsden equation

This section is ardent to study contact metric manifolds satisfying Fischer-Marsden equation. Now we show the expression of the Riemannian curvature tensor of such manifold in the following theorem:

Theorem 3.1 The expression of the Riemannian curvature tensor of a contact metric manifold satisfying Fischer-Marsden equation is given by

$$\begin{aligned} R(X,Y)D\lambda &= \frac{1}{2n}[(Y\lambda)r+\lambda(Yr)]X-\frac{1}{2n}[(X\lambda)r+\lambda(Xr)]Y \\ &\quad +(X\lambda)QY-(Y\lambda)QX+\lambda[(\nabla_X Q)Y-(\nabla_Y Q)X], \end{aligned}$$

where λ is the potential function of the Fischer-Marsden equation.

Proof. The Fischer-Marsden equation can also be exhibited as

$$-(\Delta_g\lambda)g(X,Y)+g(\nabla_X D\lambda,Y)-\lambda S_g(X,Y)=0, \tag{17}$$

from which it follows that

$$\nabla_X D\lambda=(\Delta_g\lambda)X+\lambda QX. \tag{18}$$

Now, tracing the equation (17) we obtained $\Delta_g\lambda=-\frac{\lambda r}{2n}$ and using this in the last equation we have

$$\nabla_X D\lambda=-\frac{\lambda r}{2n}X+\lambda QX, \tag{19}$$

for any smooth vector field X.

Taking covariant differentiation of (19) with respect to an arbitrary vector field Y we get

$$\begin{aligned} \nabla_Y\nabla_X D\lambda &= -\frac{1}{2n}[(Y\lambda)r+\lambda(Yr)]X-\frac{\lambda r}{2n}\nabla_Y X \\ &\quad +(Y\lambda)QX+\lambda(\nabla_Y QX). \end{aligned} \tag{20}$$

With the help of the equations (19) and (20), we deduce

$$\begin{aligned} R(X,Y)D\lambda &= \frac{1}{2n}[(Y\lambda)r+\lambda(Yr)]X-\frac{1}{2n}[(X\lambda)r+\lambda(Xr)]Y \\ &\quad +(X\lambda)QY-(Y\lambda)QX+\lambda[(\nabla_X Q)Y-(\nabla_Y Q)X], \end{aligned} \tag{21}$$

which is the expression of the curvature tensor and hence completes the proof.

In the following theorem we show the nature of the solution of a Fischer-Marsden equation on a contact metric manifold.

Theorem 3.2 On a contact metric manifold, the solution of a Fischer-Marsden equation with the potential function is an eigenvalue of the Ricci operator corresponding to any smooth vector field orthogonal to ξ is trivial.

Proof. Let us assume that λ is the eigenvalue of the Ricci operator Q in the direction orthogonal to the Reeb vector field ξ, that is, $QX=\lambda X,\ \forall\ X\perp\xi$ and then we find

$$(\nabla_Y Q)X=(Y\lambda)X \tag{22}$$

and

$$r=(2n+1)\lambda. \tag{23}$$

Using (22) and (23) in (21) yields

$$R(X,Y)D\lambda = \frac{1}{2n}[2\lambda(2n+1)(Y\lambda)]X - \frac{1}{2n}[2\lambda(2n+1)(X\lambda)]Y + 2\lambda[(X\lambda)Y - (Y\lambda)X], \quad (24)$$

from which it follows that

$$R(X,Y)D\lambda = \frac{2n+1}{n}\lambda[(Y\lambda)X - (X\lambda)Y] - 2\lambda[(Y\lambda)X - (X\lambda)Y], \quad (25)$$

and hence

$$R(X,Y)D\lambda = \frac{1}{n}\lambda[(Y\lambda)X - (X\lambda)Y]. \quad (26)$$

Taking inner product of (26) with Z we find that

$$g(R(X,Y)D\lambda, Z) = \frac{1}{n}\lambda[(Y\lambda)g(X,Z) - (X\lambda)g(Y,Z)]. \quad (27)$$

On contraction of X and Z in the last equation shows that

$$S(Y, D\lambda) = -2\lambda(Y\lambda), \quad (28)$$

which implies that

$$QD\lambda = -2\lambda D\lambda. \quad (29)$$

In light of $QX = \lambda X$, from (29) we have

$$\lambda D\lambda = 0 \quad (30)$$

and hence either $\lambda = 0$ or $\lambda \neq 0$.

Case I: If $\lambda = 0$, then the solutions of the Fischer-Marsden equation are trivial.

Case II: If $\lambda \neq 0$, then $D\lambda = 0$ accordingly the Fischer-Marsden equation returns $r = 0$ and hence $\lambda = 0$, which leads to the contradiction that $\lambda \neq 0$. Hence the second case is invalid which completes the proof.

Our next theorem is given as follows:

Theorem 3.3 On a contact metric manifold satisfying Fischer-Marsden equation with the potential function is an eigenvalue of the Ricci operator corresponding to any smooth vector field orthogonal to ξ, the Ricci operator has only the eigenvalue 0 corresponding to any eigen vector orthogonal to ξ.

Proof. Furthermore, $\lambda = 0$ implies that

$$QX = 0, \qquad \text{for all } \; X \perp \xi. \quad (31)$$

and hence the proof of the theorem completes.

4. Contact metric manifolds satisfying Fischer-Marsden equation with the Ricci operator is Reeb flow invariant

This section is devoted to study contact metric manifolds with Ricci invariant along the Reeb vector field ξ satisfying Fischer-Marsden equation.

We are going to prove the following:

Theorem 4.1 On a contact metric manifold satisfying Fischer-Marsden equation with the potential function is an eigenvalue of the Ricci operator corresponding to

any smooth vector field orthogonal to ξ, the Ricci operator is Reeb flow invariant if and only if the eigenvalues of the Ricci operator are constant with respect to the Reeb vector field ξ.
Proof. In light of (15) we have

$$(\pounds_\xi Q)X = 0, \tag{32}$$

for any smooth vector field X.
Now we derive that

$$\begin{aligned}(\pounds_\xi Q)X &= \pounds_\xi QX - Q(\pounds_\xi X)\\ &= [\xi, QX] - \lambda \pounds_\xi X\\ &= [\xi, \lambda X] - \lambda[\xi, X]\\ &= \lambda[\xi, X] + (\xi\lambda)X - \lambda[\xi, X]\\ &= (\xi\lambda)X.\end{aligned} \tag{33}$$

By the virtue of (32) and (33) we get

$$(\xi\lambda)X = 0. \tag{34}$$

Taking inner product of (34) with Y we infer

$$(\xi\lambda)g(X, Y) = 0. \tag{35}$$

On contraction of X in the above equation (35) we see that

$$\xi\lambda = 0, \tag{36}$$

from which we can realize that λ is constant with respect to the Reeb vector field ξ.
If λ is constant with respect to the Reeb vector field ξ, then the Ricci operator is Reeb flow invariant. This completes the proof.
Therefore, with respect to the Reeb vector field ξ we are in a position to state the following:
Theorem 4.2 On a contact metric manifold satisfying Fischer-Marsden equation with the potential function is an eigenvalue of the Ricci operator corresponding to any smooth vector field orthogonal to ξ, the Ricci operator is Reeb flow invariant if and only if the scalar curvature is constant.
Proof. By the virtue of (23) we see that r is constant, as λ is constant. Also we can easily check that if r is constant, then the Ricci operator is Reeb flow invariant and hence the proof of the theorem.

5. Ricci parallel contact metric manifolds satisfying Fischer-Marsden equation

This section deals with the study of contact metric manifolds with Ricci parallelism with respect to the Riemannian connection satisfying Fischer-Marsden equation.
We prove our next theorem given as follows:
Theorem 5.1 On a contact metric manifold satisfying Fischer-Marsden equation with the potential function is an eigenvalue of the Ricci operator corresponding to any smooth vector field orthogonal to ξ, the Ricci operator is parallel with respect to the Riemannian connection if and only if the eigenvalues of the Ricci operator

orthogonal to ξ are constant.
Proof. With the help of (16) we find

$$(\nabla_X Q)Y = 0, \tag{37}$$

for any smooth vector fields X and Y.
Now using (32) we deduce that

$$\begin{aligned}(\nabla_X Q)Y &= \nabla_X QY - Q(\nabla_X Y)\\ &= \nabla_(\lambda Y) - \lambda\nabla_X Y\\ &= (X\lambda)Y.\end{aligned} \tag{38}$$

Then applying (37) we get

$$(X\lambda)Y = 0. \tag{39}$$

Taking inner product of the last equation with respect to a smooth vector field Z we get

$$X\lambda = 0, \tag{40}$$

from which it is clear that λ is constant.
It is also quit easy to check that if λ is constant, then the Ricci operator is parallel with respect to the Riemannian connection and hence the theorem is established.
Now we are in a position to state the following:
Theorem 5.2 On a contact metric manifold satisfying Fischer-Marsden equation with the potential function is an eigenvalue of the Ricci operator corresponding to any smooth vector field orthogonal to ξ, the Ricci operator is parallel with respect to the Riemannian connection if and only if the scalar curvature is constant.
Proof. From (23) we see that r is constant, as λ is constant. Also if r is constant, then the Ricci operator is Reeb flow invariant. Hence completes the proof.
Remark 5.3 We recall that the results are proved here with the assumption $QX = \lambda X$, $\forall\ X \perp \xi$ and where Q is the Ricci operator and λ is the potential function of the Fischer-Marsden equation. But it is now an open problem is that what will be the situation in the direction of ξ.

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