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UNIQUENESS OF ENTIRE FUNCTIONS CONCERNING THE SHIFT DIFFERENTIAL POLYNOMIALS

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Abstract. In this paper we consider the uniqueness problem of the shift differential polynomial $(f^n(z)(f(z)-1)^m\prod_{j=1}^{s} f(z+c_j)^{\mu_j})^{(k)}$, where $f(z)$ is a transcendental entire function of finite order, $c_j(j=1,2,...,s)$ are distinct finite complex numbers and $n(\geq 1)$, $m(\geq 1)$, $k(\geq 0)$, s and $\mu_j(j=1,2,...,s)$ are integers. The results of the paper improve and extend some results given by K. Zhang and H. X. Yi [Acta Mathematica Scientis Series Manuscript, 34B(3)(2014), 719-728] and P. Sahoo and the present first author [Applied Mathematics E-Notes, 16(2016) 33-44].

1. Introduction, Definitions and Results

In this paper, a meromorphic function $f(z)$ means meromorphic in the complex plane. We shall adopt the standard notations in Nevanlinna's value distribution theory of meromorphic functions as explained in [6], [8] and [14]. For a nonconstant meromorphic function h, we denote by $T(r,h)$ the Nevanlinna characteristic of h and by $S(r,h)$ any quantity satisfying $S(r,h)=o\{T(r,h)\}$ $(r\to\infty, r\notin E)$.

Let f and g be two nonconstant meromorphic functions and $a\in\mathbb{C}\cup\{\infty\}$. If the zeros of $f-a$ and $g-a$ coincide in locations and multiplicity, we say that f and g share the value a CM (counting multiplicities). On the other hand, if the zeros of $f-a$ and $g-a$ coincide only in their locations, then we say that f and g share the value a IM (ignoring multiplicities). For a positive integer p, we denote by $N_p(r,a;f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m\leq p$ and p times if $m>p$. A meromorphic function $\alpha(\not\equiv 0,\infty)$ is called a small function with respect to f, if $T(r,\alpha)=S(r,f)$.

Recently, the topic of difference equation and difference product in the complex plane $\mathbb{C}$ has attracted many mathematicians, a large number of papers have focused on value distribution of differences and differences operator analogues of Nevanlinna theory (including [3], [4], [5], and [9]), and many people paid their attention to the uniqueness of differences and difference polynomials of meromorphic function and

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obtained many interesting results. K. Liu and L.Z. Yang [10] also considered the zeros of $f^n(z)f(z+c)-p(z)$ and $f^n\Delta_c f$, where $p(z)$ is a nonzero polynomial and obtain the following theorem.

Theorem A. *Let f be a transcendental entire function of finite order and $p(z)$ be a polynomial. If $n \geq 2$, then $f^n(z)f(z+c)-p(z)$ has infinitely many zeros. If f is not a periodic function with period c and $n \geq 2$, then $\Delta_c f = f(z+c)-f(z)$ has infinitely many zeros.*

In 2010, X.G. Qi, L.Z. Yang and K. Liu [12] proved the following uniqueness result which corresponded to Theorem A.

Theorem B. *Let f and g be two transcendental entire functions of finite order, and c be a nonzero complex constant, and let $n \geq 6$ be an integer. If $f^n(z)f(z+c)$ and $g^n(z)g(z+c)$ share the value 1 CM, then either $fg = t_1$ or $f = t_2 g$ for some constants t_1 and t_2 satisfying $t_1^{n+1} = t_2^{n+1} = 1$.*

In the same year J.L. Zhang [15] considered the zeros of one certain type of difference polynomial and obtained the following result.

Theorem C. *Let f be a transcendental entire function of finite order, $\alpha(z)(\not\equiv 0)$ be a small function with respect to f and c be a nonzero complex constant. If $n \geq 2$ is an integer, then $f^n(z)(f(z)-1)f(z+c)-\alpha(z)$ has infinitely many zeros.*

In the same paper the author also proved the following uniqueness result which corresponds to Theorem C.

Theorem D. *Let f and g be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0)$ be a small function with respect to both f and g. Suppose that c is a nonzero complex constant and $n \geq 7$ is an integer. If $f^n(z)(f(z)-1)f(z+c)$ and $g^n(z)(g(z)-1)g(z+c)$ share $\alpha(z)$ CM, then $f = g$.*

In 2012, M.R. Chen and Z.X. Chen [2] considered zeros of one certain type of difference polynomials and obtain the following theorem.

Theorem E. *Let f be transcendental entire function of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to $f(z)$, $c_j (j = 1, 2, ..., s)$, n, m, s and $\mu_j (j = 1, 2, ..., s)$ be integers. If $n \geq 2$ then $f^n(z)(f^m(z)-1)\prod_{j=1}^{s} f(z+c_j)^{\mu_j} - \alpha(z)$ has infinitely many zeros.*

In the same paper the author also proved the following uniqueness result which corresponds to Theorem E.

Theorem F. *Let f and g be two transcendental entire functions of finite order, $\alpha(z)(\not\equiv 0)$ be a common small function with respect to f and g, c be nonzero finite complex numbers. If $n \geq m + 8\sigma$, n, m, s, $\mu_j (j = 1, 2, ..., s)$ and $\sigma = \sum_{j=1}^{s} \mu_j$ are integers, and $f^n(z)(f^m(z)-1)\prod_{j=1}^{s} f(z+c_j)^{\mu_j}$ and $g^n(z)(g^m(z)-1)\prod_{j=1}^{s} g(z+c_j)^{\mu_j}$ share $\alpha(z)$ CM, then $f = tg$ where $t^m = t^{m+\sigma} = 1$.*

In 2014, K. Zhang and H.X. Yi [17] investigated the difference-differential polynomial of the form $(f^n(z)(f(z)-1)^m \prod_{j=1}^{s} f(z+c_j)^{\mu_j})^{(k)}$, where f is transcendental entire function of finite order, $c_j(j=1,2,...,s)$, n, m, s and $\mu_j(j=1,2,...,s)$ are nonnegative integers, and $\sigma = \sum_{j=1}^{s} \mu_j$ and obtained the following theorem.

Theorem G. *Let f and g be transcendental entire functions of finite order, $\alpha(z)(\not\equiv 0)$ be a common small function with respect to f and g, c_j $(j=1,2,...,s)$ be distinct finite complex numbers and n, m, s, and $\mu_j(j=1,2,...,s)$ are nonnegative integers. If $n \geq 4k-m+\sigma+9$, and the differential-difference polynomial $(f^n(z)(f(z)-1)^m \prod_{j=1}^{s} f(z+c_j)^{\mu_j})^{(k)}$ and $(g^n(z)(g(z)-1)^m \prod_{j=1}^{s} g(z+c_j)^{\mu_j})^{(k)}$ share $\alpha(z)$ CM, then $f=g$.*

An increment to uniqueness theory has been to considering weighted sharing instead of sharing IM or CM, this implies a gradual change from sharing IM to sharing CM. This notion of weighted sharing has been introduced by I. Lahiri around 2001, which measure how close a shared value is to being shared CM or to being shared IM. The definition is as follows.

Definition 1.1. *([7]) Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and k+1 times if $m > k$. If $E_k(a;f) = E_k(a;g)$, we say that f, g share the value a with weight k.*

The definition implies that if f, g share a value a with weight k, then z_0 is an a-point of f with multiplicity $m(\leq k)$ if and only if it is an a-point of g with multiplicity $m(\leq k)$ and z_0 is an a-point of f with multiplicity $m(> k)$ if and only if it is an a-point of g with multiplicity $n(> k)$, where m is not necessarily equal to n.

We write f, g share (a,k) to mean that f, g share the value a with weight k. Clearly if f, g share (a,k) then f, g share (a,p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞) respectively.

If α is a small function of f and g, then f, g share α with weight k means that $f-\alpha$, $g-\alpha$ share the value 0 with weight k.
In 2016, P. Sahoo and the present author [13] proved the following theorems.

Theorem H. *Let f and g be two transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f and g. Suppose that c is a nonzero complex constant, $n(\geq 1)$, $m(\geq 1)$ and $k(\geq 0)$ are integers satisfying $n \geq 2k+m+6$ when $m \leq k+1$ and $n \geq 4k-m+10$ when $m > k+1$. If $(f^n(z)(f(z)-1)^m f(z+c))^{(k)}$ and $(g^n(z)(g(z)-1)^m g(z+c))^{(k)}$ share $(\alpha,2)$ then either $f=g$ or f and g satisfy the algebraic equation $R(f,g)=0$ where $R(f,g)$ is given by*

$$R(w_1,w_2) = w_1^n(w_1-1)^m w_1(z+c) - w_2^n(w_2-1)^m w_2(z+c)$$

Theorem I. *Let f and g be two transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f and g. Suppose that c is* *a nonzero complex constant, $n(\geq 1)$, $m(\geq 1)$ and $k(\geq 0)$ are integers satisfying $n \geq 5k+4m+12$ when $m \leq k+1$ and $n \geq 10k-m+19$ when $m > k+1$. If $(f^n(z)(f(z)-1)^m f(z+c))^{(k)}$ and $(g^n(z)(g(z)-1)^m g(z+c))^{(k)}$ share $\alpha(z)$ IM, then the conclusions of theorem H hold.*

Now it is natural to ask the following questions which are the motivation of the paper.

Question 1.1. *Is it possible to relax in any way the nature of sharing the small function in Theorem G keeping the lower bound of n fixed ?*

Question 1.2. *What can be said if we consider the difference-differential polynomial $(f^n(z)(f(z)-1)^m \prod_{j=1}^{s} f(z+c_j)^{\mu_j})^{(k)}$, where $f(z)$ is a transcendental entire function of finite order, $c_j(j=1,2,...,s)$, $n(\geq 1)$, $m(\geq 1)$, $k(\geq 0)$, s and $\mu_j(j=1,2,...,s)$ are integers, $\sigma = \sum_{j=1}^{s} \mu_j$ in theorem H and theorem I ?*

In the paper, our main concern is to find the possible answer of the above questions. The following are the main results of the paper.

Theorem 1.1. *Let f and g be two transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f and g. Suppose that c_j $(j=1,2,...,s)$ be distinct finite complex numbers and $n(\geq 1)$, $m(\geq 1)$, s, $\mu_j(j=1,2,...,s)$ and $k(\geq 0)$ are nonnegative integers satisfying $n \geq 2k+m+\sigma+5$ when $m \leq k+1$ and $n \geq 4k-m+\sigma+9$, when $m > k+1$. If the difference-differential polynomial $(f^n(z)(f(z)-1)^m \prod_{j=1}^{s} f(z+c_j)^{\mu_j})^{(k)}$ and $(g^n(z)(g(z)-1)^m \prod_{j=1}^{s} g(z+c_j)^{\mu_j})^{(k)}$ share $(\alpha, 2)$, then either $f = g$ or f and g satisfy the algebraic equation $R(f,g)=0$, where $R(f,g)$ is given by*

$$R(w_1,w_2) = w_1^n(w_1-1)^m \prod_{j=1}^{s} w_1(z+c_j)^{\mu_j} - w_2^n(w_2-1)^m \prod_{j=1}^{s} w_2(z+c_j)^{\mu_j}$$

.

Remark 1.1. *Theorem 1.1 improves Theorem G.*

Remark 1.2. *Theorem 1.1 extends Theorem H.*

Theorem 1.2. *Let f and g be two transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f and g. Suppose that c_j $(j=1,2,...,s)$ be distinct finite complex numbers and $n(\geq 1)$, $m(\geq 1)$, s, $\mu_j(j=1,2,...,s)$ and $k(\geq 0)$ are nonnegative integers satisfying $n \geq 5k+4m+4\sigma+8$ when $m \leq k+1$ and $n \geq 10k-m+4\sigma+15$, when $m > k+1$. If the difference-differential polynomial $(f^n(z)(f(z)-1)^m \prod_{j=1}^{s} f(z+c_j)^{\mu_j})^{(k)}$ and $(g^n(z)(g(z)-1)^m \prod_{j=1}^{s} g(z+c_j)^{\mu_j})^{(k)}$ share $\alpha(z)$ IM, then the conclusions of Theorem 1.1 hold.*

Remark 1.3. *Theorem 1.2 extends Theorem I.*

2. Lemmas

Let F and G be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. We denote by H the function as follows:

$$H = \left(\frac{F''}{G'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right).$$

Lemma 2.1. [11] *Let f be a meromorphic function of finite order ρ and let $c(\neq 0)$ be a fixed nonzero complex constant. Then*

$$N(r,1,f(z+c)) \leq N(r,1,f) + S(r,f),$$

outside a possible exceptional set of finite logarithmic measure.

Lemma 2.2. [17] *Let f be an entire function of finite order and $(f^n(z)(f(z)-1)^m \prod_{j=1}^{s} f(z+c_j)^{\mu_j})^{(k)}$. Then*

$$T(r,F) = (n+m+\sigma)T(r,f) + S(r,f).$$

Lemma 2.3. [16] *Let f be a nonconstant meromorphic function, and p, k be two positive integers. Then*

$$N_p\left(r,0;f^{(k)}\right) \leq T\left(r,f^{(k)}\right) - T(r,f) + N_{p+k}(r,0;f) + S(r,f). \tag{2.1}$$

and

$$N_p\left(r,0;f^{(k)}\right) \leq k\overline{N}(r,\infty;f) + N_{p+k}(r,0;f) + S(r,f). \tag{2.2}$$

Lemma 2.4. [7] *Let f and g be two nonconstant meromorphic functions sharing $(1,2)$. Then one of the following three cases hold:*
(i) $T(r) \leq N_2(r,0;f) + N_2(r,0;g) + N_2(r,\infty;f) + N_2(r,\infty;g) + S(r)$,
(ii) $f = g$,
(iii) $fg = 1$,
Where $T(r) = \max\{T(r,f),T(r,g)\}$ and $S(r) = o\{T(r)\}$.

Lemma 2.5. [1] *Let F and G be two nonconstant meromorphic functions sharing the value 1 IM and $H \not\equiv 0$. Then*
$T(r,F) \leq N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + 2\overline{N}(r,0;F) + \overline{N}(r,0;G) + 2\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + S(r,F) + S(r,G)$,
and the same inequality holds for $T(r,G)$.

Lemma 2.6. *Let f and g be two entire functions and $n(\geq 1)$, $m(\geq 1)$, $k(\geq 0)$, be integers, and let $F = (f^n(z)(f(z)-1)^m \prod_{j=1}^{s} f(z+c_j)^{\mu_j})^{(k)}$, $G = (g^n(z)(g(z)-1)^m \prod_{j=1}^{s} g(z+c_j)^{\mu_j})^{(k)}$. If there exists nonzero constants c_1 and c_2 such that $\overline{N}(r,c_1;F) = \overline{N}(r;0;G)$ and $\overline{N}(r,c_2;G) = \overline{N}(r,0;F)$, then $n \leq 2k+m+\sigma+2$ for $m \leq k+1$ and $n \leq 4k-m+\sigma+4$ for $m > k+1$.*

Proof. We put $F_1 = f^n(z)(f(z)-1)^m \prod_{j=1}^{s} f(z+c_j)^{\mu_j}$, $G_1 = g^n(z)(g(z)-1)^m \prod_{j=1}^{s} g(z+c_j)^{\mu_j}$. By the second fundamental theorem of Nevanlinna we have

$$\begin{aligned} T(r,F) &\leq \overline{N}(r,0;F) + \overline{N}(r,c_1;F) + S(r,F) \\ &\leq \overline{N}(r,0;F) + \overline{N}(r,0;G) + S(r,F). \end{aligned} \tag{2.3}$$

Using (2.3), Lemmas 2.2 and 2.3, we obtain

$$\begin{aligned} (n+m+\sigma)T(r,f) &\leq T(r,F) - \overline{N}(r,0;F) + N_{k+1}(r,0;F_1) + S(r,f) \\ &\leq \overline{N}(r,0;G) + N_{k+1}(r,0;F_1) + S(r,f) \\ &\leq N_{k+1}(r,0;F_1) + N_{k+1}(r,0;G_1) + S(r,f) + S(r,g). \end{aligned} \tag{2.4}$$

If $m \leq k+1$, we deduce from (2.4) that

$$(n+m+\sigma)T(r,f) \leq (k+m+\sigma+1)(T(r,f)+T(r,g)) + S(r,f) + S(r,g). \tag{2.5}$$

Similarly,

$$(n+m+\sigma)T(r,g) \leq (k+m+\sigma+1)(T(r,f)+T(r,g)) + S(r,f) + S(r,g). \tag{2.6}$$

Combining (2.5) and (2.6) we obtain

$$(n-2k-m-\sigma-2)(T(r,f)+T(r,g)) \leq S(r,f) + S(r,g),$$

which gives

$$n \leq 2k+m+\sigma+2.$$

If $m > k+1$, we deduce from (2.4) that

$$(n+m+\sigma)T(r,f) \leq (2k+\sigma+2)(T(r,f)+T(r,g)) + S(r,f) + S(r,g). \tag{2.7}$$

Similarly,

$$(n+m+\sigma)T(r,g) \leq (2k+\sigma+2)(T(r,f)+T(r,g)) + S(r,f) + S(r,g). \tag{2.8}$$

Combining (2.7) and (2.8) we obtain

$$(n-4k+m-\sigma-4)(T(r,f)+T(r,g)) \leq S(r,f) + S(r,g),$$

which gives $n \leq 4k-m+\sigma+4$. This proves the lemma. □

3. Proof of the Theorems

Proof of Theorem 1.1. Let $F_1 = f^n(z)(f(z)-1)^m \prod_{j=1}^{s} f(z+c_j)^{\mu_j}$, $G_1 = g^n(z)(g(z)-1)^m \prod_{j=1}^{s} g(z+c_j)^{\mu_j}$, $F = \frac{F_1^{(k)}}{\alpha(z)}$ and $G = \frac{G_1^{(k)}}{\alpha(z)}$. Then F and G are transcendental meromorphic functions that share $(1,2)$ except the zeros and poles of $\alpha(z)$. Using (2.1) and Lemma 2.2 we get

$$\begin{aligned} N_2(r,0;F) &\leq N_2(r,0;(F_1)^{(k)}) + S(r,f) \\ &\leq T(r,(F_1)^{(k)}) - (n+m+\sigma)T(r,f) + N_{k+2}(r,0;F_1) + S(r,f) \\ &\leq T(r,F) - (n+m+\sigma)T(r,f) + N_{k+2}(r,0;F_1) + S(r,f). \end{aligned}$$

From this we get

$$(n+m+\sigma)T(r,f) \leq T(r,F) + N_{k+2}(r,0;F_1) - N_2(r,0;F) + S(r,f). \tag{3.1}$$

Again by (2.2) we have

$$\begin{aligned} N_2(r,0;F) &\leq N_2(r,0;F_1^{(k)}) + S(r,f) \\ &\leq N_{k+2}(r,0;F_1) + S(r,f). \end{aligned} \tag{3.2}$$

Suppose, if possible, that (i) of Lemma 2.4 holds. Then using (3.2) we obtain from (3.1)

$$\begin{aligned} (n+m+\sigma)T(r,f) &\leq N_2(r,0;G) + N_2(r,1;F) + N_2(r,1;G) + N_{k+2}(r,0;F_1) \\ &\quad +S(r,f) + S(r,g) \\ &\leq N_{k+2}(r,0;F_1) + N_{k+2}(r,0;G_1) + S(r,f) + S(r,g). \end{aligned} \tag{3.3}$$

If $m \leq k+1$ we deduce from (3.3) that

$$(n+m+\sigma)T(r,f) \leq (k+m+\sigma+2)\{T(r,f)+T(r,g)\} + S(r,f) + S(r,g). \tag{3.4}$$

In a similar manner we obtain

$$(n+m+\sigma)T(r,g) \leq (k+m+\sigma+2)\{T(r,f)+T(r,g)\} + S(r,f) + S(r,g). \tag{3.5}$$

(3.4) and (3.5) together give

$$(n-2k-m-\sigma-4)\{T(r,f)+T(r,g)\} \leq S(r,f) + S(r,g),$$

contradicting with the fact that

$$n \leq 2k+m+\sigma+5.$$

If $m > k+1$ we deduce from (3.3) that

$$(n+m+\sigma)T(r,f) \leq (2k+\sigma+4)\{T(r,f)+T(r,g)\} + S(r,f) + S(r,g). \tag{3.6}$$

In a similar manner we obtain

$$(n+m+\sigma)T(r,g) \leq (2k+\sigma+4)\{T(r,f)+T(r,g)\} + S(r,f) + S(r,g). \tag{3.7}$$

(3.6) and (3.7) together give

$$(n-4k+m-\sigma-8)\{T(r,f)+T(r,g)\} \leq S(r,f) + S(r,g),$$

contradicting with the fact that

$$n \leq 4k-m+\sigma+9.$$

Therefore, by Lemma 2.4 we have either $FG = 1$ or $F = G$. Let $FG = 1$. Then

$$(f^n(z)(f(z)-1)^m \prod_{j=1}^{s} f(z+c_j)^{\mu_j})^{(k)}$$

$$(g^n(z)(g(z)-1)^m \prod_{j=1}^{s} g(z+c_j)^{\mu_j})^{(k)} = \alpha^2$$

It can be easily viewed from above that $N(r,0;f) = S(r,f)$ and $N(r,1;f) = S(r,f)$. Thus we obtain

$$\delta(0,f) + \delta(1,f) + \delta(1,f) = 3,$$

which is not possible. Therefore, we must have $F = G$, and then

$$(f^n(z)(f(z)-1)^m \prod_{j=1}^{s} f(z+c_j)^{\mu_j})^{(k)} = (g^n(z)(g(z)-1)^m \prod_{j=1}^{s} g(z+c_j)^{\mu_j})^{(k)},$$

Integrating above we obtain

$$(f^n(z)(f(z)-1)^m\prod_{j=1}^{s}f(z+c_j)^{\mu_j})^{(k-1)}=(g^n(z)(g(z)-1)^m\prod_{j=1}^{s}g(z+c_j)^{\mu_j})^{(k-1)}+c_{k-1},$$

where c_{k-1} is a constant. If $c_{k-1}\neq 0$, using Lemma 2.6 it follows that $n\leq 2k+m+\sigma+2$, when $m\leq k+1$ and $n\leq 4k-m+\sigma+4$ when $n>k+1$, a contradiction. Hence $c_{k-1}=0$. Repeating the process k-times, we deduce that

$$f^n(z)(f(z)-1)^m\prod_{j=1}^{s}f(z+c_j)^{\mu_j}=g^n(z)(g(z)-1)^m\prod_{j=1}^{s}g(z+c_j)^{\mu_j}, \tag{3.8}$$

Set $h=\frac{f}{g}$. If h is a constant, then substituting $f=gh$ in (3.8), we deduce that

$$\prod_{j=1}^{s}g(z+c_j)^{\mu_j}[g^m(h^{n+m+\sigma}-1)-^m C_1g^{m-1}(h^{n+m+\sigma-1}-1)+...+(-1)^m(h^{n+\sigma}-1)]=0.$$

Since g is a transcendental entire function, we have $\prod_{j=1}^{s}g(z+c_j)^{\mu_j}\neq 0$. So from above we obtain

$$g^m(h^{n+m+\sigma}-1)-^m C_1g^{m-1}(h^{n+m+\sigma-1}-1)+...+(-1)^m(h^{n+\sigma}-1)=0,$$

which implies $h=1$ and hence $f=g$. If h is not a constant, then it follows from (3.8) that f and g satisfy the algebraic equation $R(f,g)=0$ where $R(f,g)$ is given by

$$R(w_1,w_2)=w_1^n(w_1-1)^m\prod_{j=1}^{s}w_1(z+c_j)^{\mu_j}-w_2^n(w_2-1)^m\prod_{j=1}^{s}w_2(z+c_j)^{\mu_j}.$$

This proves Theorem 1.1. □

Proof of Theorem 1.2. Let F, G, F_1 and G_1 be defined as in the proof of Theorem 1.1. Then F and G are transcendental meromorphic functions that share the value 1 IM except the zeros and poles of $\alpha(z)$. We assume, if possible, that $H\not\equiv 0$. Using Lemma 2.5 and (3.2) we obtain from (3.1)

$$\begin{aligned}(n+m+\sigma)T(r,f) &\leq N_2(r,0;G)+N_2(r,1;F)+N_2(r,1;G)+2\overline{N}(r,0;F)+\overline{N}(r,0;G)\\ &\quad +N_{k+2}(r,0;F_1)+2\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+S(r,f)+S(r,g)\\ &\leq N_{k+2}(r,0;F_1)+N_{k+2}(r;0;G_1)+2N_{k+1}(r,0;F_1)\\ &\quad +N_{k+1}(r,0;G_1)+S(r,f)+S(r,g)\end{aligned} \tag{3.9}$$

If $m\leq k+1$ we deduce from (3.9) that

$$\begin{aligned}(n+m+\sigma)T(r,f) &\leq (3k+3m+3\sigma+4)T(r,f)+(2k+2m+2\sigma+3)T(r,g)\\ &\quad +S(r,f)+S(r,g)\\ &\leq (5k+5m+5\sigma+7)T(r)+S(r).\end{aligned} \tag{3.10}$$

In a similar manner we obtain

$$(n+m+\sigma)T(r,f)\leq(5k+5m+5\sigma+7)T(r)+S(r). \tag{3.11}$$

(3.10) and (3.11) together give

$$(n-5k-4m-4\sigma-7)T(r)\leq S(r),$$

contradicting with the fact that

$$n \leq 5k + 4m + 4\sigma + 8.$$

If $m > k+1$ we deduce from (3.9) that

$$\begin{aligned}(n+m+\sigma)T(r,f) &\leq (6k+3\sigma+8)T(r,f) + (4k+2\sigma+6)T(r,g) + S(r,f) + S(r,g)\\ &\leq (10k+5\sigma+14)T(r) + S(r). \qquad (3.12)\end{aligned}$$

In a similar manner we obtain

$$(n+m+\sigma)T(r,f) \leq (10k+5\sigma+14)T(r) + S(r). \qquad (3.13)$$

(3.12) and (3.13) together give

$$(n - 10k + m - 4\sigma - 14)T(r) \leq S(r),$$

contradicting with the fact that

$$n \leq 10k - m + 4\sigma + 15.$$

We now assume that $H \equiv 0$. Then

$$\left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right) = 0.$$

Integrating both sides of the above equality twice we get

$$\frac{1}{F-1} = \frac{A}{G-1} + B, \qquad (3.14)$$

where $A(\neq 0)$ and B are constants. From (3.14) it is obvious that F, G share the value 1 CM and hence they share (1, 2). Therefore $n \geq 2k + m + \sigma + 5$ when $m \leq k+1$ and $n \geq 4k - m + \sigma + 9$ when $m > k+1$. We now discuss the following three cases separately.

Case 1. Suppose that $B \neq 0$ and $A = B$. Then from (3.14) we obtain

$$\frac{1}{F-1} = \frac{BG}{G-1}. \qquad (3.15)$$

If $B = -1$, then from (3.15) we obtain

$$FG = 1,$$

which is a contradiction as in the proof of Theorem 1.1.

If $B \neq -1$, from (3.15), we have $\frac{1}{F} = \frac{BG}{(1+B)G-1}$ and so $\overline{N}(r, \frac{1}{1+B}; G) = \overline{N}(r, 0; F)$. Using (2.1), (2.2) and the second fundamental theorem of Nevanlinna, we deduce that

$$\begin{aligned}T(r,G) &\leq \overline{N}(r,0;G) + \overline{N}\left(r, \frac{1}{B+1}; G\right) + \overline{N}(r,\infty;F) + S(r,G)\\ &\leq \overline{N}(r,0;F) + \overline{N}(r,0;G) + \overline{N}(r,\infty;G) + S(r,G)\\ &\leq N_{k+1}(r,0;F_1) + T(r,G) + N_{k+1}(r,0;G_1)\\ &\quad -(n+m+\sigma)T(r,g) + S(r,g). \qquad (3.16)\end{aligned}$$

If $m \leq k+1$ we deduce from (3.16) that

$$(n+m+\sigma)T(r,g) \leq (k+m+\sigma+1)\{T(r,f) + T(r,g)\} + S(r,g).$$

Thus we obtain

$$(n - 2k - m - \sigma - 2)\{T(r,f) + T(r,g)\} \leq S(r,f) + S(r,g),$$

which is a contradiction as $n \geq 2k+m+\sigma+5$.
If $m > k+1$ we deduce from (3.16) that

$$(n+m+\sigma)T(r,g) \quad \leq \quad (2k+\sigma+2)\{T(r,f)+T(r,g)\}+S(r,g).$$

Thus we obtain

$$(n-4k+m-\sigma-4)\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g),$$

which is a contradiction as $n \geq 4k-m+\sigma+9$.
Case 2. Let $B \neq 0$ and $A \neq B$. Then from (3.14) we get $F = \frac{(B+1)G-(B-A+1)}{BG(A-B)}$ and so $\overline{N}(r, \frac{B-A+1}{B+1}; G) = \overline{N}(r,0;F)$. Proceeding in a manner similar to case 1 we can arrive at a contradiction.
Case 3. Let $B = 0$ and $A \neq 0$. Then from (3.14) we get $F = \frac{G+A-1}{A}$ and $G = AF-(A-1)$. If $A \neq 1$, it follows that $\overline{N}(r, \frac{A-1}{A}; F) = \overline{N}(r,0;G)$ and $\overline{N}(r, 1-A; G) = \overline{N}(r,0;F)$. Now applying Lemma 6 it can be shown that $n \leq 2k+m+\sigma+2$ for $m \leq k+1$ and $n \leq 4k-m+\sigma+4$ for $m > k+1$, which is a contradiction. Thus $A=1$ and then $F=G$. Now the result follows from the proof of Theorem 1.1. This completes the proof of Theorem 1.2. □

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