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A NOTE ON GAMMA FUNCTION UNDER THE TREATMENT OF BICOMPLEX ANALYSIS

DEBASMITA DUTTA, SATAVISHA DEY, SUKALYAN SARKAR AND SANJIB KUMAR DATTA

Abstract. In complex analysis Gamma function defined by a convergent improper integral

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx,$$

z being a complex number with a positive real part. Gamma function is the commonly used extension of the factorial function to complex numbers.The analytic continuation of this integral function to a meromorphic function that is holomorphic in the whole complex plane except zero and non negative integers. In this paper our main aim is to derive the bicomplex version of Gamma function supported by relevant examples and some of its related properties, mostly with the help of idempotent representation and Ringleb decomposition of bicomplex numbers and bicomplex valued functions.

1. Introduction

The theory of bicomplex numbers is a matter of active research for quite a long time since seminal work of {cf. [12] and [3]} and in search of special algebra. the algebra of bicomplex numbers are widely use in the literature as it becomes viable commutative alternative {cf. [13]} to the non skew field of quaternions introduced by Hamilton {cf. [7]} (both are four dimensional and generalization of complex numbers).

2. Preliminaries

2.1. **The Bicomplex Numbers{cf.**[10]**}.** A bicomplex number is defined as

$$\begin{aligned} z &= x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 \\ &= (x_1 + i_1 x_2) + i_2 (x_3 + i_1 x_4) \\ &= z_1 + i_2 z_2 \end{aligned}$$

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where $x_i, i = 1, 2, 3, 4$ are all real numbers with $i_1^2 = i_2^2 = -1, i_1 i_2 = i_2 i_1, (i_1 i_2)^2 = 1$, and z_1, z_2 are complex numbers.

The set of all bicomplex numbers , complex numbers and real numbers are denoted by $\mathbb{C}_2, \mathbb{C}_1$ and $\mathbb{C}_0$ respectively.

2.2. **Algebra of Bicomplex Numbers{cf.**[10]**}.** Addition is the operation on $\mathbb{C}_2$ defined by the function $\oplus : \mathbb{C}_2 \times \mathbb{C}_2 \to \mathbb{C}_2$,

$$(x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4,\ y_1 + i_1 y_2 + i_2 y_3 + i_1 i_2 y_4) = (x_1 + y_1) + i_1 (x_2 + y_2) + i_2 (x_3 + y_3) + i_1 i_2 (x_4 + y_4).$$

Scalar multiplication is the operation on $\mathbb{C}_2$ defined by the function $\odot : \mathbb{C}_0 \times \mathbb{C}_2 \to \mathbb{C}_2$,

$$(a, x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4) = (a x_1 + i_1 a x_2 + i_2 a x_3 + i_1 i_2 a x_4).$$

The system $(\mathbb{C}_2, \oplus, \odot)$ is a linear space.

Here the norm is defined as

$$\begin{aligned} ||\ \ || &: \mathbb{C}_2 \to \mathbb{R}_{\geq 0}, \\ \|x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4\| &= \left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right)^{\frac{1}{2}}. \end{aligned}$$

So the system $(\mathbb{C}_2, \oplus, \odot, ||\ \ ||)$ is a normed linear space.

The space $\mathbb{C}_0^4$ with the Euclidean norm is known to be complete space. As $\mathbb{C}_2$ is embedded in $\mathbb{C}_0^4$ so that $x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4$ corresponds to (x_1, x_2, x_3, x_4) and for this reason the norm on $\mathbb{C}_2$ is the same as the norm of $\mathbb{C}_0^4$, then the normed linear space $(\mathbb{C}_2, \oplus, \odot, ||\ \ ||)$ is a complete Space. Hence $(\mathbb{C}_2, \oplus, \odot, ||\ \ ||)$ is a Banach Space.

The product on $\mathbb{C}_2$ is defined as

$$\otimes : \mathbb{C}_2 \times \mathbb{C}_2 \to \mathbb{C}_2,$$

$$(x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4,\ y_1 + i_1 y_2 + i_2 y_3 + i_1 i_2 y_4) = \begin{pmatrix} x_1 y_1 - x_2 y_2 - x_3 y_3 + x_4 y_4 \\ +i_1 (x_1 y_2 + x_2 y_1 - x_3 y_4 - x_4 y_3) \\ +i_2 (x_1 y_3 - x_2 y_4 + x_3 y_1 - x_4 y_2) \\ +i_1 i_2 (x_1 y_4 + x_2 y_3 + x_3 y_2 + x_4 y_1) \end{pmatrix}.$$

Since,

$$\begin{aligned} (i)\ \|z (z_1 + i_2 z_2)\| &= |z| \cdot \|z_1 + i_2 z_2\|. \\ (ii)\ \|(z_1 + i_2 z_2)(w_1 + i_2 w_2)\| &\leq \sqrt[2]{2}\, \|z_1 + i_2 z_2\| \cdot \|w_1 + i_2 w_2\|. \end{aligned}$$

where $z \in \mathbb{C}_1, (z_1 + i_2 z_2)$ and $(w_1 + i_2 w_2) \in \mathbb{C}_2$.

So, $(\mathbb{C}_2, \oplus, \odot, ||\ \ ||, \otimes)$ is a Banach Algebra.

2.3. **Idempotent Representation of Bicomplex Numbers {cf.**[10]**}.** There are four idempotent elements in $\mathbb{C}_2$. they are

$$0, 1, \frac{1 + i_1 i_2}{2}, \frac{1 - i_1 i_2}{2}.$$

We now denote two non trivial idempotent elements by

$$e_1 = \frac{1 + i_1 i_2}{2} \quad \text{and} \quad e_2 = \frac{1 - i_1 i_2}{2} \quad \text{in } \mathbb{C}_2.$$

where

$$e_1^2 = e_1, e_2^2 = e_2, e_1 e_2 = e_2 e_1 = 0, e_1 + e_2 = 1.$$

So, e_1 and e_2 are alternatively called orthogonal idempotents.

Every element $\xi : (z_1 + i_2 z_2) \in \mathbb{C}_2$ has the following unique representation

$$\begin{aligned} \xi &= (z_1 - i_1 z_2)\, e_1 + (z_1 + i_1 z_2)\, e_2 \\ &= \xi_1 e_1 + \xi_2 e_2, \text{ where } \xi_1, \xi_2 \text{ are complex numbers.} \end{aligned}$$

This is known as idempotent representation of the element $\xi : (z_1 + i_2 z_2) \in \mathbb{C}_2$.

An element $\xi : (z_1 + i_2 z_2) \in \mathbb{C}_2$ is non-singular iff $\left|z_1^2 + z_2^2\right| \neq 0$ and it is singular iff $\left|z_1^2 + z_2^2\right| = 0$. The set of all singular elements is denoted by θ_2.

If $f(z)$ be a bicomplex valued function, then f can be represented as

$$f(z) = f_1(z_1)\, e_1 + f_2(z_2)\, e_2 \quad \text{where} \quad f_1(z_1), f_2(z_2) \in \mathbb{C}_1 .$$

where $f_1(z_1), f_2(z_2)$ are both functions in $\mathbb{C}_1$.This type of decomposition is known as Ringleb decomposition in $\mathbb{C}_2\{cf.$[8]and[9]$\}$.

2.4. **Bicomplex Exponential Function {cf.**[10]**}.** If w be any bicomplex number then the sequence $\left(1 + \frac{w}{n}\right)^n$ converges to a bicomplex number denoted by $\exp w$ or e^w, called the bicomplex exponential function.

$$i.e., e^w = \lim_{n \to \infty} \left(1 + \frac{w}{n}\right)^n .$$

If $w = (z_1 + i_2 z_2)$, then we get the bicomplex version of Euler's formula

$$e^w = e^z (\cos z_2 + i_2 \sin z_2) = e^{|w|_{i_1}} (\cos \arg\ _{i_1} w + \sin \arg\ _{i_1} w)$$

where $e^w \notin \theta_2$.

2.5. **Bicomplex Logarithmic Function{cf.**[10]**}.** Let ξ be a bicomplex number and w be another bicomplex number such that $w \notin \theta_2$. If $e^{\xi} = w$, then ξ is called logarithm of w.

Let $w = (z_1 + i_2 z_2) \notin \theta_2$. i.e., if $(z_1 - i_1 z_2) \neq 0$ and $(z_1 + i_1 z_2) \neq 0$ then

$$\begin{aligned} \log(z_1 + i_2 z_2) &= \{\log|z_1 - i_1 z_2| + i_1 \arg(z_1 - i_1 z_2) + 2n_1 \pi i_1\}\, e_1 \\ &\quad + \{\log|z_1 + i_1 z_2| + i_1 \arg(z_1 + i_1 z_2) + 2n_2 \pi i_1\}\, e_2. \end{aligned}$$

where $n_1, n_2 = 0, \pm 1, \pm 2, ...$

Also we can write,

$$\log(z_1 + i_2 z_2) = \log(z_1 - i_1 z_2)\, e_1 + \log(z_1 + i_1 z_2)\, e_2.$$

2.6. **Bicomplex Holomorphic Function{cf.**[10]**}.** We start with a bicomplex valued function

$$f : \Omega \subset \mathbb{C}_2 \to \mathbb{C}_2.$$

The derivative of f at a point $\omega_0 \in \Omega$ is defined by

$$f'(\omega) = \lim_{h \to 0} \frac{f(\omega_0 + h) - f(\omega_0)}{h}$$

provided the limit exists and the domain is so chosen that

$$h = h_0 + i_1 h_1 + i_2 h_2 + i_1 i_2 h_3$$

is invertible. It is easy to prove that h is not invertible only for $h_0 = -h_3, h_1 = h_2$ or $h_0 = h_3, h_1 = -h_2 . i.e. h \notin \theta_2.$

If the bicomplex derivative of f exists at each point of its domain then in similar to complex function, f will be a bicomplex holomorphic function in Ω. Indeed if f can be expressed as

$$\begin{aligned} f(\omega) &= g_1(z_1, z_2) + i_2 g_2(z_1, z_2) \\ \omega &= z_1 + i_2 z_2 \in \Omega \end{aligned}$$

then f will be holomorphic if and only if g_1, g_2 are both complex holomorphic in z_1, z_2 and

$$\frac{\partial g_1}{\partial z_1} = \frac{\partial g_2}{\partial z_2}, \frac{\partial g_1}{\partial z_2} = -\frac{\partial g_2}{\partial z_1}.$$

Moreover,

$$f'(\omega) = \frac{\partial g_1}{\partial z_2} + i_2 \frac{\partial g_2}{\partial z_1}.$$

2.7. **Bicomplex Entire Function{cf.**[10]**}.** A function f is said to be a bicomplex entire function if f is analytic in the whole bicomplex plane $\mathbb{C}_2$.

2.8. **Bicomplex Meromorphic Function{cf.**[10]**}.** A function f is said to be bicomplex meromorphic function in an open set $\Omega \subseteq T$ if f is a quotient $\frac{g}{h}$ of two functions which are bicomplex holomorphic in Ω where $h \notin \theta_{2.}$

If $f(z)$ be a bicomplex meromorphic function, then f can be represented as

$$f(z) = f_1(z_1) e_1 + f_2(z_2) e_2 \quad \text{where} \quad f_1(z_1), f_2(z_2) \in \mathbb{C}_1 .$$

where $f_1(z_1), f_2(z_2)$ are both meromorphic functions in $\mathbb{C}_1$.

2.9. **Infinite Series of Bicomplex Numbers{cf.**[6]**}.** $_{k=0}^{\infty}\xi_k \quad \forall\ k,\ \xi_k \in \mathbb{C}_2$ is called infinite series in $\mathbb{C}_2$. Define the sequence $S : \mathbb{N} \to \mathbb{C}_2$ by

$$S_n =_{k=0}^{n} \xi_k \qquad \forall\ n \in \mathbb{N}.$$

Then the infinite sum converges iff $\lim\limits_{n\to\infty} S_n$ exists and diverges if the limit does not exists.

If $\lim\limits_{n\to\infty} S_n = \xi^*$ then ξ^* is called sum of the series and we write$_{k=0}^{\infty}\xi_k = \xi^*$.

The infinite series $_{k=0}^{\infty}\xi_k$ has the sum $\xi^* = z_1^* + i_2 z_2^*$ iff the following infinite series converge and have the sums

$$\begin{aligned} {}_{k=0}^{\infty}(z_{1k} - i_1 z_{2k}) &= z_1^* - i_1 z_2^*, \\ {}_{k=0}^{\infty}(z_{1k} + i_1 z_{2k}) &= z_1^* + i_1 z_2^*. \end{aligned}$$

2.10. **Infinite Product of Bicomplex Numbers{cf.**[6]**}.** If we multiply an infinite number of factors according to some definite law then the product so obtained is called an infinite product. Let $\{u_k\}$be the sequence of bicomplex numbers. Thus the product $u_1u_2u_3$....of infinite number of factors is denoted symbolically as $_{k=1}^{\infty}u_k$ and in case the factors be finite we write it as

$$P_n =_{k=1}^{n} u_k$$

It is also clear from above that

$$\frac{P_n}{P_{n-1}} = u_n \quad \text{and} \quad \frac{P_{n+p}}{P_n} = u_{n+1}u_{n+2}........u_{n+p}.$$

For the sake of convenience we will choose the factors to be of the form $(1 + u_k)$,

$$_{k=1}^{\infty}(1 + u_k) = (1 + u_1)(1 + u_2)(1 + u_3)......$$

The product of n factors is written as

$$_{k=1}^{n}\left(1+u_k\right)=\left(1+u_1\right)\left(1+u_2\right)\left(1+u_3\right).......\left(1+u_n\right).$$

During the year 1729, 1730 Euler introduced an analytic function which has the property to interpolate the factorial whenever the argument of the function is an integer.

{cf. [4],[1] and [2]} Let $x>0$

$$\Gamma\left(x\right)=\int_0^1\left(-\log(t)\right)^{x-1}dt.$$

By elementary changes of variables this historical definition takes the more usual forms:

$$\Gamma\left(x\right)=\int_0^\infty e^{-t}\cdot t^{x-1}dt.For\ x>0.$$

For complex numbers with a positive real part the Gamma finction is defined via a convergent improper integral:

$$\Gamma\left(z\right)=\int_0^\infty e^{-t}\cdot t^{z-1}dt,\text{ for }Re(z)>0.$$

The Gamma function is defined as the analytic continuation of the integral function to a meromorphic function that is holomorphic in the whole complex plane except the non positive integers, where the functions has simple poles.

In this paper we wish to find out the formation of Gamma function with some of its important properties under the treatment of bicomplex analysis.Further, we improve some results of usual Gamma function as derived in the complex field in the flavour of the notion of bicomplex analysis.We do not explain the standerd definitions and notatios of the theories of bicomplex valued entire function as those are available in {cf.[10], [4], [8] and [9]}.

3. Lemmas

In this section we present some relevant lemmas which will be needed in the sequel.

{cf.[6]}The necessary and sufficient condition for the convergence of infinite product $(1+a_n)$ is that the series $\log\left(1+a_n\right)$ is convergent where each logarithm has ita principle value and $(1+a_n)\notin\theta_2$, for each bicomplex number $a_i=a_i^{'}e_1+a_i^{''}e_2$.

{cf.[6]}The infinite product $(1+a_n)$ where $(1+a_n)\notin\theta_2$ is absolutely convergent iff the series $\log\left(1+a_n\right)$ is absolutely convergent i.e., iff a_n is absolutely convergent series of bicomplex numbers $a_i=a_i^{'}e_1+a_i^{''}e_2$.

Let $z=z_1e_1+z_2e_2\in\mathbb{C}_2$.

$$a_1\leq Rez_1\leq A_1,\ a_2\leq Rez_2\leq A_2\ \text{ where }0<a_1<A_1<\infty,\ 0<a_2<A_2<\infty.$$

if $a=\min\left\{a_1,a_2\right\}$ and $A=\max\left\{A_1,A_2\right\}$ then

$$a\leq Rez_1\leq A\ \text{ also }\ a\leq Rez_2\leq A\ \text{ where }\ 0<a<A<\infty.$$

Consider the set

$$S=\left\{z=z_1e_1+z_2e_2\in\mathbb{C}_2:a\leq Rez_1\leq A\ \text{ also }\ a\leq Rez_2\leq A\right\}$$

(a) for every $\epsilon > 0 \ \exists \ \ \delta > 0$ such that for all z in S

$$\left\| \int_{\alpha}^{\beta} e^{-t} \cdot t^{z-1} dt \right\| < \epsilon \ \text{ whenever } \ 0 < \alpha < \beta < \delta.$$

(b) for $\epsilon > 0$ there is a number K such that for all z in S

$$\left\| \int_{\alpha}^{\beta} e^{-t} \cdot t^{z-1} dt \right\| < \epsilon \ \text{ whenever } \ \beta > \alpha > K.$$

Proof. If $o < t \leq 1$ and $z \in S$ then

$$(Re\,(z_1) - 1) \log t \leq (a-1) \log t$$

and

$$(Re\,(z_2) - 1) \log t \leq (a-1) \log t$$

Since $e^t \leq 1$

$$\left| e^{-t} \cdot t^{z_1 - 1} \right| \leq t^{Re(z_1)-1} \leq t^{a-1}$$

and

$$\left| e^{-t} \cdot t^{z_2 - 1} \right| \leq t^{Re(z_2)-1} \leq t^{a-1}$$

So, if $0 < \alpha < \beta < 1$ then

$$\begin{aligned} \left| \int_{\alpha}^{\beta} e^{-t} \cdot t^{z_1-1} dt \right| &\leq \int_{\alpha}^{\beta} t^{a-1} dt \\ &= \frac{1}{a} \left(\beta^a - \alpha^a \right). \end{aligned}$$

Similarly

$$\left| \int_{\alpha}^{\beta} e^{-t} \cdot t^{z_2-1} dt \right| \leq \frac{1}{a} \left(\beta^a - \alpha^a \right).$$

let us consider $\epsilon > 0$. For choosen ϵ, $\exists$ 0¡$\delta < 1$ such that

$$\frac{1}{a} \left(\beta^a - \alpha^a \right) < \frac{\epsilon}{\sqrt{2}}.$$

In view of Ringleb decomposition {cf..[8]} ,for all $z \in S$,

$$\begin{aligned} \left\| \int_{\alpha}^{\beta} e^{-t} \cdot t^{z-1} dt \right\| &= \left\| \left(\int_{\alpha}^{\beta} e^{-t} \cdot t^{z_1-1} dt \right) e_1 + \left(\int_{\alpha}^{\beta} e^{-t} \cdot t^{z_2-1} dt \right) e_2 \right\| \\ &\leq \left| \int_{\alpha}^{\beta} e^{-t} \cdot t^{z_1-1} dt \right| \cdot \|e_1\| + \left| \int_{\alpha}^{\beta} e^{-t} \cdot t^{z_2-1} dt \right| \cdot \|e_2\| \\ &\leq \frac{1}{a} \left(\beta^a - \alpha^a \right) \cdot \frac{\sqrt{2}}{2} + \frac{1}{a} \left(\beta^a - \alpha^a \right) \cdot \frac{\sqrt{2}}{2} \\ &= \frac{\sqrt{2} \left(\beta^a - \alpha^a \right)}{a} < \epsilon \ \text{ for } \ |\alpha - \beta| < \delta. \end{aligned}$$

This proves Part (a) of the Lemma 3.3.

To prove Part (b) we should note that for $z \in S$ and $t \geq 1$,

$$\left| t^{z_1 - 1} \right| \leq t^{A-1} \ \text{ and } \ \left| t^{z_2 - 1} \right| \leq t^{A-1}$$

Since $t^{A-1}\cdot\exp\left(-\frac{1}{2}t\right)$ is continuous on $[1,\infty)$ and converges to zero as $t\to\infty$. There is a constant C such that

$$t^{A-1}\cdot\exp\left(-\frac{1}{2}t\right)\leq C\quad\forall\ t\geq 1.$$

This gives that

$$\left|e^{-t}\cdot t^{z_1-1}\right|\leq C\cdot e^{-\frac{1}{2}t}\ \text{ and }\ \left|e^{-t}\cdot t^{z_2-1}\right|\leq C\cdot e^{-\frac{1}{2}t}$$

For all $z\in S$ and $t\geq 1$. If $\beta>\alpha>1$ then

$$\begin{aligned}\left\|\int_{\alpha}^{\beta}e^{-t}\cdot t^{z-1}dt\right\| &\leq \left|\int_{\alpha}^{\beta}e^{-t}\cdot t^{z_1-1}dt\right|\cdot\|e_1\|+\left|\int_{\alpha}^{\beta}e^{-t}\cdot t^{z_2-1}dt\right|\cdot\|e_2\|\\ &\leq c\int_{\alpha}^{\beta}e^{-\frac{1}{2}t}dt\cdot\left(\frac{\sqrt{2}}{2}\right)+c\int_{\alpha}^{\beta}e^{-\frac{1}{2}t}dt\cdot\left(\frac{\sqrt{2}}{2}\right)\\ &= \sqrt{2}c\int_{\alpha}^{\beta}e^{-\frac{1}{2}t}dt\\ &= \sqrt{2}c\left(e^{-\frac{1}{2}\alpha}-e^{-\frac{1}{2}\beta}\right).\end{aligned}$$

{cf.[1]} Again for $\epsilon>0$, $\exists$ a number $K>1$ such that

$$\sqrt{2}c\left(e^{-\frac{1}{2}\alpha}-e^{-\frac{1}{2}\beta}\right)<\epsilon\ \text{ whenever }\ \alpha,\beta>K.$$

□

Part (b) of the lemma 3.3 follows.
{cf.[2]} If $0\leq t\leq n$ then

$$0\leq e^{-t}-\left(1-\frac{t}{n}\right)^{n}\leq\frac{t^2e^{-t}}{n}.$$

4. Results

In this section is subdivided into two subsections 4.A and 4.B.

4.A : It deals with some theorems one of which is most important to derive the definition of Gamma function in bicomplex analysis with its related properties.

Theorem 4.1. *Let $a_1,a_2,a_3,$........be a given sequence of non zero bicomplex numbers such that $\frac{1}{\|a_n\|^2}<\infty$. Then if $g(z)$ is any entire function, the function*

$$f(z)=e^{g(z)}\cdot z^k\left(\underset{n=1}{\overset{\infty}{}}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}}\right)$$

is entire.

Proof. Since $a_1,a_2,a_3,$........be a given sequence of non zero bicomplex numbers. So,

$$a_i=a_i^{'}e_1+a_i^{''}e_2,\quad\text{ where } a_i^{'},a_i^{''}\in\mathbb{C}_1.$$

Since $\frac{1}{\|a_n\|^2}<\infty$ and $\|a_i\|=\sqrt{\left|a_i^{'}\right|^2+\left|a_i^{''}\right|^2}$. So,

$$\frac{1}{\left\|a_i^{'}\right\|^2}<\infty\quad\text{and}\qquad\frac{1}{\left\|a_i^{''}\right\|^2}<\infty.$$

$g(z)$ is any bicomplex entire function.

$$Therefore, g(z) = g_1(z_1)e_1 + g_2(z_2)e_2 \text{ where } g_1(z_1), g_2(z_2) \in \mathbb{C}_1.$$

$Since, g_1(z_1)$ is entire function and $\frac{1}{\|a_i^{'}\|^2} < \infty$. {cf.[4]}So, $\exists$ $f_1(z_1) \in \mathbb{C}_1$ such that

$$f_1(z_1) = e^{g_1(z_1)} \cdot z_1^k \left({}_{n=1}^{\infty} \left(1 - \frac{z_1}{a_n^{'}}\right) e^{\frac{z_1}{a_n^{'}}} \right).$$

$Since, g_2(z_2)$ is entire function and $\frac{1}{\|a_i^{''}\|^2} < \infty$. {cf.[4]}So, $\exists$ $f_2(z_2) \in \mathbb{C}_1$ such that

$$f_2(z_2) = e^{g_2(z_2)} \cdot z_2^k \left({}_{n=1}^{\infty} \left(1 - \frac{z_2}{a_n^{''}}\right) e^{\frac{z_2}{a_n^{''}}} \right)$$

both are entire functions.

Hence

$$\begin{aligned} f(z) &= \left[e^{g_1(z_1)} \cdot z_1^k \left({}_{n=1}^{\infty} \left(1 - \frac{z_1}{a_n^{'}}\right) e^{\frac{z_1}{a_n^{'}}} \right) \right] e_1 \\ &\quad + \left[e^{g_2(z_2)} \cdot z_2^k \left({}_{n=1}^{\infty} \left(1 - \frac{z_2}{a_n^{''}}\right) e^{\frac{z_2}{a_n^{''}}} \right) \right] e_2 \\ &= e^{g(z)} \cdot z^k \left({}_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n}} \right) \end{aligned}$$

is an entire function.This proves the Theorem. □

Remark 1 :The following example ensures the conclusion of Theorem 4.1.

$$\sin z = z_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right) \quad \text{where } z = z_1e_1 + z_2e_2, \quad z_1, z_2 \in \mathbb{C}_1.$$

$z = z_1e_1 + z_2e_2, \ z_1, z_2 \in \mathbb{C}_1.$

we can write

$$\begin{aligned} \sin z_1 &= z_{1}{}_{n=1}^{\infty} \left(1 - \frac{z_1^2}{n^2\pi^2}\right) \text{ \{cf.[4]\}} \\ \sin z_2 &= z_{2}{}_{n=1}^{\infty} \left(1 - \frac{z_2^2}{n^2\pi^2}\right) \{cf.[4]\} \end{aligned}$$

$$(\sin z_1)e_1 + (\sin z_2)e_2 = \left[z_{1}{}_{n=1}^{\infty} \left(1 - \frac{z_1^2}{n^2\pi^2}\right) \right] e_1 + \left[z_{2}{}_{n=1}^{\infty} \left(1 - \frac{z_2^2}{n^2\pi^2}\right) \right] e_2$$

$$\sin z = z_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right) \quad \text{where } z \in \mathbb{C}_2.$$

Theorem 4.2. *Let*

$$G(z) = {}_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}, \textit{where } z \in \mathbb{C}_2 \tag{1}$$

Then $G(z)$ is an entire function of bicomplex variable with simple zeros at $-1, -2, -3, \ldots\ldots\ldots.$ Further G satisfies the identity

$$zG(z) \cdot G(-z) = \frac{\sin \pi z}{\pi}. \tag{2}$$

Further Let

$$H(z) = G(z-1) \tag{3}$$

Then the function $H(z)$ *has zeros at* $0, -1, -2, ...$*and*

$$H\left(z\right) = e^{g(z)} \cdot z_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}} = ze^{g(z)} \cdot G\left(z\right), \textit{where } z \in \mathbb{C}_2 \tag{4}$$

Proof. In Theorem 4.1 with $a_n = -n$ we have the assertion that G is entire with simple zeros at $-1, -2, -3, ...$and in view of example 1 we get that,

$$\begin{aligned} zG\left(z\right)\cdot G\left(-z\right) &= z \cdot_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}} \cdot_{n=1}^{\infty}\left(1-\frac{z}{n}\right)e^{\frac{z}{n}} \\ &= z \cdot_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right) \\ &= \frac{\sin \pi z}{\pi}. \end{aligned}$$

Let

$$\begin{aligned} H\left(z\right) &= G\left(z-1\right) \\ &= {}_{n=1}^{\infty}\left(1+\frac{z-1}{n}\right)e^{-\frac{(z-1)}{n}} \end{aligned}$$

is entire by Theorem 4.1 and zeros at $0, -1, -2, -3, ...$

Now using Lemma 3.2

$$\log H\left(z\right) = \log z + g\left(z\right) +_{n=1}^{\infty}\left(\log\left(1+\frac{z}{n}\right)-\frac{z}{n}\right)$$

converges being uniform on closed discuss, term by term differentiation is allowed.

$$\begin{aligned} \frac{d}{dz}\left(\log H\left(z\right)\right) &= \frac{1}{z} + g^{'}\left(z\right) +_{n=1}^{\infty}\left(\frac{1}{z-1+n}-\frac{1}{n}\right) = \frac{1}{z} - 1. \qquad (5) \\ &= {}_{n=2}^{\infty}\left(\frac{1}{z+1-n}-\frac{1}{n}\right) \\ &= \frac{1}{z} - 1 +_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n+1}\right) \\ &= \frac{1}{z} - 1 +_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n}\right) +_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right) \\ &= \frac{1}{2} +_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n}\right). \qquad (6) \end{aligned}$$

Comparing (5) and (6) and using (3) we see that $g^{'}\left(z\right) = 0$ and $g\left(z\right)$ is constant say $\gamma.\{cf.[4]\}$

Thus

$$G\left(z-1\right) = ze^{\gamma} \cdot G\left(z\right). \tag{7}$$

This proves the theorem. □

Theorem 4.3.

$$\begin{aligned} \Gamma\left(z\right) &= \left[ze^{\gamma z} \cdot G\left(z\right)\right]^{-1} \\ &= \left[ze^{\gamma_z}_{n=1}{}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}\right]^{-1} \end{aligned}$$

is a meromorphic functions with simple poles at $0, -1, -2, -3, ...$

Proof. Since G(z) is entire function with simple zeros at negative integers $-1, -2, -3, ...$

Thus $\Gamma\left(z\right)$ is a meromorphic functions with simple poles at $0, -1, -2, -3, ...$

□

This completes the theorem.

Now we are in a position to define Gamma function in bicomplex field and to derive some of its properties.

4.B : Gamma Function.

The Gamma function $\Gamma(z)$ is a meromorphic function on $\mathbb{C}_2$ with simple poles $0, -1, -2, -3, ...$ defined by

$$\Gamma(z) = \frac{e^{-\gamma_z}}{z} {}_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{\frac{z}{n}}, \text{where } z \in \mathbb{C}_2.$$

where γ is constant chosen so that $\Gamma(1) = 1$, γ is called Euler's constant.

Now in view of Lemma 3.2 we would like to find γ :

Since {cf.[4]} $\Gamma(1) = 1$

$$\begin{aligned} e^{\gamma} &= {}_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{-1} e^{\frac{1}{n}} \\ &\Rightarrow \gamma = {}_{k=1}^{\infty} \left[\log \left(1 + \frac{1}{k}\right)^{-1} e^{\frac{1}{k}}\right] \\ &= {}_{k=1}^{\infty} \left[\frac{1}{k} - \log(k+1) + \log k\right] \\ &= \lim_{n\to\infty} {}_{k=1}^{n} \left[\frac{1}{n} - \log(k+1) + \log k\right] \\ &= \lim_{n\to\infty} \left[\left(1 + \frac{1}{2} + ... + \frac{1}{n}\right) - \log(n+1)\right]. \end{aligned}$$

Adding and substracting to each term of the sequence and using the fact

$$\lim_{n\to\infty} \log\left(\frac{n+1}{n}\right) = 0$$

yeilds

$$\gamma = \lim_{n\to\infty} \left[\left(1 + \frac{1}{2} + ... + \frac{1}{n}\right) - \log n\right].$$

In the next sequel we deduce some properties of Gamma function following the course of bicomplex analysis.

Example I $\Gamma(z+1) = z \cdot \Gamma(z)$, $z \neq 0, -1, -2, ...$where $z \in \mathbb{C}_2$

Proof. In view of (7) of Theorem 4.2

$$\begin{aligned} \Gamma(z+1) &= \left[(z+1) \cdot e^{\gamma(z+1)} \cdot G(z+1)\right]^{-1} \\ &= [(z+1) \cdot e^{\gamma} \cdot G(z+1) \cdot e^{\gamma_z}]^{-1} \\ &\quad [G(z) \cdot e^{\gamma_z}]^{-1} \\ &= z \cdot \Gamma(z). \end{aligned}$$

This completes the theorem. □

Example II $\Gamma(n+1) = n!$

Proof. We have $\Gamma(1) = 1$

$$\text{Since, } \Gamma(z) = \left[z \cdot e^{\gamma_z} G(z)\right]^{-1}$$

and

$$G(1) = e^{-\gamma}$$

$$\begin{aligned} i.e, \Gamma(2) &= 2 \cdot 1 = 2! \\ i.e, \Gamma(3) &= 3 \cdot 2 \cdot 1 = 3! \end{aligned}$$

In this way

$$i.e, \Gamma(n+1) = n!.$$

This completes the proof. □

Example III $\Gamma(z) \cdot \Gamma(1-z) = \frac{\pi}{\sin \pi z}$,where $z \in \mathbb{C}_2$

Proof. In view of Equation (2) and Theorem 4.2{cf.[4]}

$$\frac{1}{zG(z) \cdot G(-z)} = \frac{\pi}{\sin \pi z}$$

but

$$\frac{1}{zG(z)} = e^{\gamma_z} \Gamma(z) \quad \text{and}$$

$$\frac{1}{G(-z)} = -e^{-\gamma_z} z\Gamma(-z)$$

Thus

$$\frac{1}{zG(z) \cdot G(-z)} = \frac{\pi}{\sin \pi z}$$

$$i.e, -z \cdot G(z) \cdot \Gamma(-z) = \frac{\pi}{\sin \pi z}.$$

In view of Property I ,completes the proof. □

Example IV

$$\Gamma(z) = \lim_{n \to \infty} \frac{n! n^z}{z(z+1) \cdot \cdots \cdot (z+n)}$$

for $z \neq 0, -2$, where $z \in \mathbb{C}_2$.

Proof. By definition of $\Gamma(z)$ we can write,

$$\begin{aligned} \frac{1}{\Gamma(z)} &= ze^{\gamma z}_{k=1}{}^{\infty} \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}} \\ &= \lim_{n\to\infty} ze^{\gamma z}_{k=1}{}^{\infty} \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}} \\ &= \lim_{n\to\infty} z \exp\left\{z \left({}^{n}_{k=1} \frac{1}{k} - \log n\right)\right\}^{n}_{k=1} \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}} \\ &= \lim_{n\to\infty} \left[z \exp\left(z^{n}_{k=1} \frac{1}{k}\right) \exp(-z \log n) \cdot^{n}_{k=1} \left(1 + \frac{z}{k}\right) \exp\left(-z^{n}_{k=1} \frac{1}{k}\right)\right] \\ &= \lim_{n\to\infty} ze^{-z \log n n}_{k=1} \left(1 + \frac{z}{k}\right) \\ &= \lim_{n\to\infty} \left[zn^{-z}(1+z)\left(1 + \frac{z}{2}\right) \cdots \left(1 + \frac{z}{n}\right)\right] \\ &= \lim_{n\to\infty} \frac{z(z+1) \cdot \cdots \cdot (z+n)}{n^z n!}. \end{aligned}$$

Thus

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^z}{z(z+1)\cdot\cdots\cdot(z+n)}.$$

□

Remark 2 : This Property is analogues to Gauss′s Formula in $\mathbb{C}_1$.

Example V {cf.[4]} For any fixed positive integer $n \geq 2$,

$$\Gamma(z)\cdot\Gamma\left(z+\frac{1}{n}\right)\cdots\cdots\Gamma\left(z+\frac{n-1}{n}\right) = (2\pi)^{\frac{n-1}{2}}\cdot n^{\frac{1}{2}-nz}\cdot\Gamma(nz), where\ z\in\mathbb{C}_2. \quad (8)$$

Proof. We have

$$\begin{aligned}\Gamma(z) &= \lim_{m\to\infty}\frac{m!m^z}{z(z+1)....(z+m)} = \lim_{m\to\infty}\frac{(m-1)!m^z}{z(z+1)....(z+m-1)} \\ &= \lim_{m\to\infty}\frac{(mn-1)!(mn)^z}{z(z+1)....(z+mn-1)}.\end{aligned}$$

We define $f(z)$ as follows:

$$f(z) = \frac{n^{nz}\Gamma(z)\Gamma\left(z+\frac{1}{n}\right)\cdot\cdots\cdot\Gamma\left(z+\frac{n-1}{n}\right)}{n\Gamma(nz)} \quad (9)$$

$$\begin{aligned} &= n^{nz-1}\frac{{}_{k=0}^{n-1}\lim_{m\to\infty}\frac{\{(m-1)!\}^n m^z\cdots m^{\left(z+\frac{n-1}{n}\right)}}{\left(z+\frac{k}{n}\right)\left(z+\frac{k}{n}+1\right)\cdots\left(z+\frac{k}{n}+m-1\right)}}{\lim_{m\to\infty}\frac{(mn-1)!(mn)^{nz}}{nz(nz+1)\cdots(nz+mn-1)}} \\ &= \lim_{m\to\infty}\frac{\{(m-1)!\}^n m^{\frac{n-1}{2}} n^{mn-1}(nz)(nz+1)\cdots(nz+mn-1)}{(mn-1)!_{k=0}^{n-1}(nz+k)(nz+k+n)\cdots(nz+k+mn-n)} \\ &= \lim_{m\to\infty}\frac{\{(m-1)!\}^n m^{\frac{n-1}{2}} n^{mn-1}}{(mn-1)!}\end{aligned}$$

This shows that f is constant. Setting $z=\frac{1}{n}$, we get

$$f(z) = \Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\cdots\cdot\Gamma\left(\frac{n-1}{n}\right) > 0$$

and so

$$[f(z)]^2 = \frac{\pi^{n-1}}{\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\cdots\sin\left(\frac{n-1}{n}\pi\right)}$$

From the fact that

$$\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\cdots\sin\left(\frac{n-1}{n}\pi\right) = \frac{\pi}{2^{n-1}};\ n=2,3,......$$

which follows from the fact that the product can be written as $\frac{1}{2^{n-1}}$ times the product of the non-zero roots of polynomial $(1-z)^n-1$, we have

$$[f(z)]^2 = \frac{(2\pi)^{n-1}}{n}.$$

Since $f(z)>0$, $f(z) = \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}}$.

Thus by (9)

$$\Gamma(z)\cdot\Gamma\left(z+\frac{1}{n}\right)\cdot\cdots\cdots\cdot\Gamma\left(z+\frac{n-1}{n}\right) = (2\pi)^{\frac{n-1}{2}}\cdot n^{\frac{1}{2}-nz}\cdot\Gamma(nz),\quad n\geq 2.$$

This completes the proof. □

Example VI $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$.

Proof. Put $z=\frac{1}{2}$ in Property V we can write

$$\Gamma\left(\frac{1}{2}\right)\Gamma(1) = 2\sqrt{\pi}\sqrt{2}$$

$$\Rightarrow \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}.$$

$$\Gamma(z)\cdot\Gamma(1-z)=\frac{\pi}{\sin\pi z}.$$

This completes the proof. □

Now we want to find the Gamma function in terms of integral of bicomplex function.

Theorem 4.4. *Let* $z=z_1e_1+z_2e_2\in\mathbb{C}_2$, $Re(z_1)>0$ *and* $Re(z_2)>0$ *then*

$$\Gamma(z)=\int_0^\infty e^{-t}\cdot t^{z-1}dt.$$

Proof. We know that

$$\lim_{n\to\infty}\left(1-\frac{t}{n}\right)^n=e^{-t}.$$

We have,

$$\Gamma(z)=\lim_{n\to\infty}\int_0^n\left(1-\frac{t}{n}\right)^n\cdot t^{z-1}dt.$$

Let $f(z)=\int_0^\infty e^{-t}\cdot t^{z-1}dt$ from Lemma 3.3, we can say that this integral converges.

And further $\int_1^\infty e^{-t}\cdot t^{\alpha}dt$ and $\int_0^1 t^p dt$ converges for $P>-1$ (*By* comparision test in $\mathbb{R}$)

Now,

$$\begin{aligned} f(z)-\Gamma(z) &= \lim_{n\to\infty}\left[\int_0^n\left\{e^{-t}-\left(1-\frac{t}{n}\right)^n\right\}t^{z-1}dt+\int_n^\infty e^{-t}\cdot t^{z-1}dt\right] \\ &= \left(\lim_{n\to\infty}\left[\int_0^n\left\{e^{-t}-\left(1-\frac{t}{n}\right)^n\right\}t^{z_1-1}dt+\int_n^\infty e^{-t}\cdot t^{z_1-1}dt\right]\right)e_1 \\ &\quad+\left(\lim_{n\to\infty}\left[\int_0^n\left\{e^{-t}-\left(1-\frac{t}{n}\right)^n\right\}t^{z_2-1}dt+\int_n^\infty e^{-t}\cdot t^{z_2-1}dt\right]\right)e_2 \end{aligned} \quad (10)$$

First note that $\int_n^\infty e^{-t}\cdot t^{z_1-1}dt\to 0$ as $n\to\infty$ and $\int_n^\infty e^{-t}\cdot t^{z_2-1}dt\to 0$ as $n\to\infty$.

In fact if $t>1$ then

$$\left|e^{-t}\cdot t^{z_1-1}\right|\le e^{-t}\cdot t^m \text{ where m is an integersuch that } m\ge Re(z_1)>0.$$

And also,

$$\left|e^{-t}\cdot t^{z_2-1}\right|\le e^{-t}\cdot t^k \text{ where m is an integersuch that } k\ge Re(z_2)>0.$$

Using integration by parts it can be shown that

$$\int_0^\infty e^{-t}\cdot t^m dt<\infty \quad\text{and}\quad \int_0^\infty e^{-t}\cdot t^k dt<\infty.$$

So,

$$\int_0^\infty e^{-t}\cdot t^m dt \to 0 \quad \text{as} \quad n\to\infty \text{ and } \int_0^\infty e^{-t}\cdot t^k dt \to 0 \quad \text{as} \quad n\to\infty.$$

The only thing which we shall have to show now is that

$$\int_0^n \left\{e^{-t}-\left(1-\frac{t}{n}\right)^n\right\} t^{z_1-1}dt \to 0 \quad \text{as} \quad n\to\infty$$

and

$$\int_0^n \left\{e^{-t}-\left(1-\frac{t}{n}\right)^n\right\} t^{z_2-1}dt \to 0 \quad \text{as} \quad n\to\infty.$$

Now by Lemma 3.4,

$$\begin{aligned}\left|\int_0^n \left\{e^{-t}-\left(1-\frac{t}{n}\right)^n\right\} t^{z_1-1}dt\right| &\leq \int_0^n \frac{e^{-t}\cdot t^{Rez_1+1}}{n} \\ &\leq \frac{1}{n}\int_0^\infty e^{-t}\cdot t^{Rez_1+1}dt \to 0 \quad \text{as} \quad n\to\infty.\end{aligned}$$

Similarly,

$$\begin{aligned}\left|\int_0^n \left\{e^{-t}-\left(1-\frac{t}{n}\right)^n\right\} t^{z_2-1}dt\right| &\leq \int_0^n \frac{e^{-t}\cdot t^{Rez_2+1}}{n} \\ &\leq \frac{1}{n}\int_0^\infty e^{-t}\cdot t^{Rez_2+1}dt \to 0 \quad \text{as} \quad n\to\infty.\end{aligned}$$

From (10) we can write for $Re\,(z_1)>0$ and $Re\,(z_2)>0$,

$$\Gamma(z)=\int_0^\infty e^{-t}\cdot t^{z-1}dt.$$

□

Thus the theorem is established.

We can state that $\Gamma(z)$ as, where z is a bicomplex number $z=z_1e_1+z_2e_2$, $z_1\in\mathbb{C}_1$ and $z_2\in\mathbb{C}_1$ and $Re\,(z_1)>0$ and $Re\,(z_2)>0$

$$\Gamma(z)=\int_0^\infty e^{-t}\cdot t^{z-1}dt.$$

5. Future Prospect

In the line of the works as carried out in the paper one may think of the analytic continuation of bicomplex valued Gamma function. As a consequence the derivation of relevant results in this area may be an active area of research.

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Debasmita Dutta, Department of Mathematics, Lady Brabourne College, P-1/2 Suhrawardy Avenue, Beniapukur, Dist: Kolkata, PIN: 700017, West Bengal, India.
Email address: debasmita.dut@gmail.com

Satavisha Dey, Department of Mathematics, Bijoy Krishna Girls' College, M.G. Road, Dist: Howrah, PIN: 711101,West Bengal, India.
Email address: itzmesata@gmail.com

Sukalyan Sarkar, Department of Mathematics, Dukhulal Nibaran Chandra College, P.O.: Aurangabad, Dist: Murshidabad, PIN: 742201, West Bengal, India.
Email address: sukalyanmath.knc@gmail.com

Sanjib Kumar Datta, Department of Mathematics, University of Kalyani, P.O.: Kalyani, Dist: Nadia, PIN: 741235, West Bengal, India.
Email address: sanjibdatta05@gmail.com