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Transient Solution of Two –Class Priority Queuing System with Balking and Catastrophes

Ahmed Tarabia[1], Rehab G ElQady[1] and Ghadeer Alshreef[1]

[1]Department of Mathematics, Faculty of Science, Damietta University.

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[*] Corresponding author's E-mail: rehab.alqady@gmail.com

Abstract

This paper scrutinizes the transient solution for two-class priority queuing system model with both balking and catastrophes with infinite system capacity. Moreover, some special cases are also presented.

Keywords: Transient Solution, Generating Function, Balking, Catastrophes.

Introduction

Our daily life events full of priority queues, which occur in many fields, for example telecommunications field, medical care field etc. In any computer system arrangement if there are many data say for simplicity (customers) arranged in the system, a newcomer data may not able to subjoin the arrangement. Also, if a data hitting with virus or damage, it may destroy or relocate to other spare processors, these types of systems can be represented and modeled as queuing models with balking and catastrophes. Therefore, these queuing models have extensive applications in telecommunication orderliness, computer science and manufacturing suits.

The mechanisms of priority are a valuable scheduling method that allows messages from various categories to extradite various fineness of service. Thus, the priority queue has received great importance in the literature. The first who consider a queuing model was Cobham (1957), he proposed the non-preemptive priority for queues with Poisson arrival and exponential distribution for the holding time and studied the model in two cases, for single and for multi-channel. Stephan (1958) discussed the model of preemptive priority system with two priorities and Poisson arrival and Exponential holding time. In last few years, queuing system with catastrophes and balking is considerable with large importance. Haight (1957) studies an infinite M/M/1 queue model with balking in the case of the queue length encroaches k (k is a random variable). Takâcs (1961) assumes a queue with single server also with the condition that if a customer arrives at a time when the server is occupied then it may or may not join the queue. Ancker and Gafarian (1963) study M/M/1/N queuing with balking and reneging they derive its steady-state solution. Kumar and Pathasarthy (1993) used the generating function technique to study M/M/1 queue model with balking and obtained the transient solution for length of the queue.

Queuing systems with catastrophes correspond to manufacturing systems with sudden disasters. This type of queuing models become of great importance recently that it modeling many variety systems in our real life, such as virus attack or corruption of hard disk in computer systems and call centers with sudden power breakdowns. The happening of catastrophes leads to the extermination of all the customers in the system and temporarily pending the service till repairing the system. Chao (1995) discussed the network queuing model with catastrophes as an application to study the effect of Power outage or virus on queues in computer system networks. Di Crescenzo et al. (2003) studied a phenomenon of muscle contraction as a new application of the simple M/M/1 queuing model with catastrophes. Sudhesh (2010) studied a single server queuing system with catastrophes and customers impatience. He explicitly derived the solutions of the transient probabilites using the technique of generating function. Tarabia (2011) studied the transient and steady-state analysis of a single server Markovian queuing system with balking, catastrophes, server failures and repairs. Kumar and Sharma (2014) studied queuing model with reneging, balking and retention of reneged customers. They also studied in (2012) a Markovian feedback queue with balking and retention of retreated customers. Dharmaraja and Kumar (2015) obtained the transient solution of a queuing model with multiple heterogeneous servers in presence of catastrophes.Kumar and Arivudainambi (2015) insert the impact of catastrophes in a single server Markovian queuing system. They used generating function technique to deduce its transient solution explicitly. Sudesh et al. (2016) derived the solution of transient for the queuing system with a two-heterogeneous servers, catastrophes, server repair and customers' impatience. Yaseen and Tarabia (2017) analyzed the transient and steady-state attitude of Markovian queuing system with balking and reneging subject to catastrophes and server failures. Suranga and Liu (2018) studied the simple queue M/M/1 after adding reneging, server failures (catastrophes) and repairs. They obtained the transient and the steady-state solutions.

The systems of multi-server queuing with customer impatience have widely applications in many real life situations such as in hospitals, retail stores, computer communication etc. Choudhury and Medhi (2011) discussed a model designed as multi-server Markovian queuing system with balking and reneging, based on the history and the literature review they have adopting the procedures of Tarabia (2011) with balking, catastrophes, server failures and repairs.

Our motivation here is to study priority queuing system with two-class priority (high and low) with balking and catastrophes where balking is for high class priority and catastrophes for all the system. This paper organs as follows, the next section contains description of the model and the main assumptions will be discussed and considered, the transient solution of the model will be obtained in section 3. Finally, some special cases are listed in section 4.

Description and Analyzing of the Model:

A single server queuing system is considered which provides the service for two kinds of arrival customers, kind 1 defined as class-1 and the second kind as class-2, the both kinds independently possess its particular line. Class-1 is assigned to be a higher priority and the low priority assigned to class-2. We assume in each class the service act as FIFO with preemptive continued priority system, i.e. in the service time of a low priority customer, if any customer of high priority enters the system, consequently the customer's service of the low priority customer will be stopped and it can be resumed again if there is no customer in high priority class in the system. The newcomer customer either join the queue of the class-1 with probability one if the customers number in the class-1 is not exceeds the threshold value k. If there are k or more customers before him, then he enter the queue system with probability β or may he balk with complementary probability. The system has infinite capacity. When the system is idle or busy, catastrophes may happen with Poisson process by rate γ. The given system can be easily modeled by Markov process $\{(X_1(t), X_2(t)), t \geq 0\}$ where $X_i(t), i = 1,2$ denotes the number of customer's in the high and low class respectively .We denote $p_{i,j}(t)$= {the probabilities of the system where the first class is at the state i and the second class is at state j, and all of them are infinite}, The inter-arrival and service times are considered to be associated with negative exponentially distributed with mean $1/\lambda_1$ and $1/\mu_1$ respectively for the first class and the service and inter-arrival times are assumed to be negative exponentially distributed with mean $1/\lambda_2$ and $1/\mu_2$ respectively for the second class. As a result of all of the above supposition, we can deduce that the Chapman-Kolmogorov forward equations for the proposed system can be written as:

$$\frac{dp_{0,0}(t)}{dt} = -(\lambda_1 + \lambda_2 + \gamma)p_{0,0}(t) + \mu_1 p_{1,0}(t) + \mu_2\, p_{0,1}(t), \qquad (1)$$

$$\frac{dp_{0,j}(t)}{dt} = -(\lambda_1 + \lambda_2 + \mu_2 + \gamma)p_{0,j}(t) + \mu_1 p_{1,j}(t) + \mu_2 p_{0,j+1}(t) + \lambda_2\, p_{0,j-1}(t), \quad j \geq 1 \qquad (2)$$

$$\frac{dp_{i,0}(t)}{dt} = -(\lambda_1 + \lambda_2 + \mu_1 + \gamma)p_{i,0}(t) + \mu_1 p_{i+1,0}(t) + \lambda_1\, p_{i-1,0}(t), \quad 1 \leq i \leq k \qquad (3)$$

$$\frac{dp_{i,j}(t)}{dt} = -(\lambda_1 + \lambda_2 + \mu_1 + \gamma)p_{i,j}(t) + \mu_1 p_{i+1,j}(t) + \lambda_1 p_{i-1,j}(t) + \lambda_2\, p_{i,j-1}(t), \quad 1 \leq i < k, \quad j \geq 1, \qquad (4)$$

$$\frac{dp_{k,0}(t)}{dt} = -(\lambda_1\beta + \lambda_2 + \mu_1 + \gamma)p_{k,0}(t) + \mu_1 p_{k+1,0}(t) + \lambda_1\, p_{k-1,0}(t), \qquad (5)$$

$$\frac{dp_{k,j}(t)}{dt} = -(\lambda_1\beta + \lambda_2 + \mu_1 + \gamma)p_{k,j}(t) + \mu_1 p_{k+1,j}(t) + \lambda_1 p_{k-1,j}(t) + \lambda_2\, p_{k,j-1}(t), \quad j \geq 1 \qquad (6)$$

$$\frac{dp_{i,0}(t)}{dt} = -(\lambda_1\beta + \lambda_2 + \mu_1 + \gamma)p_{i,0}(t) + \mu_1 p_{i+1,0}(t) + \lambda_1\beta\, p_{I-1,0}(t), \quad i > k \qquad (7)$$

$$\frac{dp_{i,j}(t)}{dt} = -(\lambda_1\beta + \lambda_2 + \mu_1 + \gamma)p_{i,j}(t) + \mu_1 p_{i+1,j}(t) + \lambda_1\beta\, p_{i-1,j}(t) + \lambda_2\, p_{i,j-1}(t), \quad i > k, \quad j \geq 1, \qquad (8)$$

where $p_{i,j}(t) = p(X_1(t) = i, X_2(t) = j)$ and $p_{i,j}(0) = 1$ for $i = j = 0$ and else is zero.

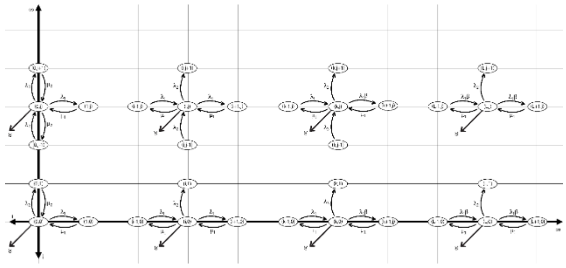

Fig . 1 Transient-State of Two –Class Priority Queueing System with Balking and Catastrophes

Solution of the Model:

The transient solution for the two-class priority queuing system with balking and catastrophes will be deduced in this section using the generating function technique.

Define $G_j(s,t)$ as the probability generating function:

$$G_j(s,t) = R_j(t) + \sum_{i=1}^{\infty} p_{k+i,j}(t)s^i, \quad j = 0,1,2,\ldots \qquad (9)$$

where $R_j(t) = \sum_{i=0}^{k} p_{i,j}(t)$, $j = 0,1,2,\ldots$.

Computing $\boldsymbol{G_0(s,t)}$ *:*

Using Eqs. (1), (3) and (5), we get:

$$\sum_{i=0}^{k}\frac{dp_{i,0}(t)}{dt} = -(\lambda_1 + \lambda_2 + \mu_1 + \gamma)\sum_{i=0}^{k} p_{i,0}(t) + \mu_1 \sum_{i=0}^{k} p_{i,0}(t) + \mu_1 p_{k+1,0}(t) + \lambda_1 \sum_{i=0}^{k} p_{i,0}(t) - \lambda_1\beta p_{k,0}(t) + \mu_2 p_{0,1}(t)$$
$$= -(\lambda_2 + \gamma)\sum_{i=0}^{k} p_{i,0}(t) + \mu_1 p_{k+1,0}(t) - \lambda_1\beta p_{k,0}(t) + \mu_2 p_{0,1}(t). \qquad (10)$$

Also, from Eq. (7), we have:

$$\sum_{i=1}^{\infty}\frac{dp_{k+i,0}(t)}{dt}s^i = -(\lambda_1\beta + \lambda_2 + \mu_1 + \gamma)\sum_{i=1}^{\infty} p_{k+i,0}(t)s^i + \mu_1 \sum_{i=1}^{\infty} p_{k+i+1,0}(t)s^i + \lambda_1\beta \sum_{i=1}^{\infty} p_{k+i-1,0}(t)\, s^i \qquad (11)$$

$$= \Big[-(\lambda_1\beta+\lambda_2+\mu_1+\gamma)+\frac{\mu_1}{s}+\lambda_1\beta s\Big]\sum_{i=1}^{\infty} p_{k+i,0}(t)s^i - \mu_1 p_{k+1,0}(t) + \lambda_1\beta s\, p_{k,0}(t).$$

Adding Eq. (10) to Eq. (11), we get

$$\frac{\partial G_0(s,t)}{\partial t} = -(\lambda_2+\gamma)R_0(t) + (\lambda_1\beta\, s - \lambda_1\beta)p_{k,0}(t) + \mu_2 p_{0,1}(t) + \Big[-(\lambda_1\beta+\lambda_2+\mu_1+\gamma)+\frac{\mu_1}{s}+\lambda_1\beta s\Big]\sum_{i=1}^{\infty} p_{k+i,0}(t)s^i \qquad (12)$$

$$= \Big[-(\lambda_1\beta+\lambda_2+\mu_1+\gamma)+\frac{\mu_1}{s}+\lambda_1\beta\, s\Big][G_0(s,t) - R_0(t)] - (\lambda_2+\gamma)R_0(t) + \lambda_1\beta(s-1)p_{k,0}(t) + \mu_2 p_{0,1}(t),$$

then,

$$\frac{\partial G_0(s,t)}{\partial t} - \Big[-(\lambda_1\beta+\lambda_2+\mu_1+\gamma)+\frac{\mu_1}{s}+\lambda_1\beta s\Big]G_0(s,t) = -\Big[-(\lambda_1\beta+\mu_1)+\frac{\mu_1}{s}+\lambda_1\beta s\Big]R_0(t) + \lambda_1\beta(s-1)p_{k,0}(t) + \mu_2 p_{0,1}(t). \qquad (13)$$

Solving for the generating function and using Bessel function for obtaining $G_0(s,t)$:

$$G_0(s,t) = G_0(s,0)\Big[\exp\Big[-(\lambda_1\beta+\lambda_2+\mu_1+\gamma)+\frac{\mu_1}{s}+\lambda_1\beta s\Big]t\Big] + \int_0^t \Big[\exp\Big[-(\lambda_1\beta+\lambda_2+\mu_1+\gamma)+\frac{\mu_1}{s}+\lambda_1\beta s\Big](t-u)\Big]\Big[(\lambda_1\beta+\mu_1) - \Big(\frac{\mu_1}{s}+\lambda_1\beta s\Big)\Big]R_0(u)du$$
$$+ \int_0^t \Big[\exp\Big[-(\lambda_1\beta+\lambda_2+\mu_1+\gamma)+\frac{\mu_1}{s}+\lambda_1\beta s\Big](t-u)\Big].\,[\lambda_1\beta(s-1)p_{k,0}(u)]du + \int_0^t [\exp\Big[-(\lambda_1\beta+\lambda_2+\mu_1+\gamma)+\frac{\mu_1}{s}+\lambda_1\beta s\Big](t-u)]\,[\mu_2 p_{0,1}(u)]du.$$

Suppose $\nu=\lambda_1\beta$, $\alpha = 2\sqrt{\nu\mu_1}$, $B=\sqrt{\frac{\nu}{\mu_1}}$ and $w = (\lambda_1\beta+\lambda_2+\mu_1+\gamma)$, then using the modified Bessel function of the first kind $I_n(.)$, we have $\exp\Big(\Big[\frac{\mu_1}{s}+\lambda_1 Bs\Big]t\Big) = \exp\Big(\Big[\frac{1}{Bs}+Bs\Big]\frac{\propto t}{2}\Big) = \sum_{n=-\infty}^{\infty}(Bs)^n I_n(\propto t)$.

$$G_0(s,t) = G_0(s,0)[\exp(-wt)]\sum_{n=-\infty}^{\infty}(Bs)^n I_n(\propto t) + \int_0^t [\exp[-w](t-u)]\sum_{n=-\infty}^{\infty}(Bs)^n I_n(\propto(t-u))\Big[(\nu+\mu_1) - \Big(\frac{\mu_1}{s}+\nu s\Big)\Big]R_0(u)du$$
$$+ \int_0^t [\exp[-w](t-u)]\sum_{n=-\infty}^{\infty}(Bs)^n\ I_n(\propto(t-u))\,[\nu(s-1)p_{k,0}(u)]du + \int_0^t [\exp[-w](t-u)]\sum_{n=-\infty}^{\infty}(Bs)^n I_n(\propto(t-u))\,[\mu_2 p_{0,1}(u)]du. \qquad (14)$$

Computing $\boldsymbol{p_{k+n,0}(t)}$:

We summaries the main results in the following theorems

Theorem1: The transient probabilities $\{p_{k+n,0}(t), n=1,2,\ldots\infty\}$ that the system under balking and catastrophes are obtained as:

$$p_{k+n,0}(t) = [\exp[-w]t]\sum_{m=0}^{\infty} p_{m,0}\, B^{1-T(m)}[I_{n-T(m)}(\propto t) - I_{n+T(m)}(\propto t)] + nB^n\int_0^t [\exp[-w](t-u)]\Big[\frac{I_n(\propto(t-u))}{(t-u)}\Big][p_{k,0}(u)]du,\quad n=1,2,\ldots\infty$$

Where $\nu=\lambda_1\beta$, $\alpha=2\sqrt{\nu\mu_1}$, $B=\sqrt{\frac{\nu}{\mu_1}}$, $w=(\lambda_1\beta+\lambda_2+\mu_1+\gamma)$,

$I_n(.)$ modified first kind Bessel function and $T(m) = (m-k)[1-\sum_{n=0}^{k}\delta_{in}]$.

Proof:

From Eq. (14), since $G_0(s,0)=\sum_{m=0}^{\infty} p_{m,0}s^{T(m)}$ and $T(m)=(m-k)[1-\sum_{n=0}^{k}\delta_{in}]$.

Comparing the coefficient of s^n on right and left –hand sides, we get

$$p_{k+n,0}(t) = [\exp[-w]t]\sum_{m=0}^{\infty} p_{m,0}s^{T(m)}\,B^n I_{n-T(m)}(\propto t) + \int_0^t [\exp[-w](t-u)]B^n I_n(\propto(t-u))\Big[(\nu+\mu_1) - \Big(\frac{\mu_1}{s}+\nu s\Big)\Big]R_0(u)du$$
$$+ \int_0^t [\exp[-w](t-u)]B^n I_n(\propto(t-u))\,[\nu(s-1)p_{k,0}(u)]du + \int_0^t [\exp[-w](t-u)]B^n I_n(\propto(t-u))\,[\mu_2 p_{0,1}(u)]du$$

$$B^{-n+1}p_{k+n,0}(t) = [\exp\,[-wt]]\sum_{m=0}^{\infty} p_{m,0}\,B^{1-T(m)} I_{n-T(m)}(\propto t) + (\nu+\mu_1)\int_0^t [\exp[-w](t-u)]BI_n(\propto(t-u))\,R_0(u)du - \int_0^t [\exp[-w](t-u)][B^2\mu_1 I_{n+1}(\propto(t-u)) + \nu I_{n-1}(\propto(t-u))]R_0(u)du$$

$$+\nu\int_0^t[\exp[-w](t-u)][I_{n-1}(\propto(t-u)-BI_n(\propto(t-u))][p_{k,0}(u)]du$$
$$+\int_0^t[\exp[-w](t-u)]BI_n(\propto(t-u))[\mu_2 p_{0,1}(u)]du\,.$$

For $n=0$, we obtain

$$B\ p_{k,0}(t)=[\exp[-wt]]\sum_{m=0}^{\infty}p_{m,0}\,B^{1-T(m)}I_{-T(m)}(\propto t)$$
$$+(\nu+\mu_1)\int_0^t[\exp[-w](t-u)]BI_0(\propto(t-u))\,R_0(u)du$$
$$-\int_0^t[\exp[-w](t-u)][B^2\mu_1 I_1(\propto(t-u))+\nu I_{-1}(\propto(t-u))]R_0(u)du$$
$$+\nu\int_0^t[\exp[-w](t-u)][I_{-1}(\propto(t-u)-BI_0(\propto(t-u))][p_{k,0}(u)]du$$
$$+\int_0^t[\exp[-w](t-u)]BI_0(\propto(t-u))[\mu_2 p_{0,1}(u)]du. \quad (15)$$

Since the probability function doesn't have any negative terms of powers then we replace (n) by (-n) in the equation must be zero so

$$0=[\exp[-w]t]\sum_{m=0}^{\infty}p_{m,0}\,B^{1-T(m)}I_{-n-T(m)}(\propto t)$$
$$+(\nu+\mu_1)\int_0^t[\exp[-w](t-u)]BI_{-n}(\propto(t-u))\,R_0(u)du$$
$$-\int_0^t[\exp[-w](t-u)][B^2\mu_1 I_{-n+1}(\propto(t-u))+\nu I_{-n-1}(\propto(t-u))]R_0(u)du$$
$$+\nu\int_0^t[\exp[-w](t-u)][I_{-n-1}(\propto(t-u)-BI_{-n}(\propto(t-u))][p_{k,0}(u)]du$$
$$+\int_0^t[\exp[-w](t-u)]BI_{-n}(\propto(t-u))[\mu_2 p_{0,1}(u)]du.$$

By using Bessel function property $I_{-n}(x)=I_n(x)$, then

$$0=[\exp[-w]t]\sum_{m=0}^{\infty}p_{m,0}\,B^{1-T(m)}I_{n+T(m)}(\propto t)$$
$$+(\nu+\mu_1)\int_0^t[\exp[-w](t-u)]BI_n(\propto(t-u))\,R_0(u)du$$
$$-\int_0^t[\exp[-w](t-u)][B^2\mu_1 I_{n-1}(\propto(t-u))+\nu I_{n+1}(\propto(t-u))]R_0(u)du$$
$$+\nu\int_0^t[\exp[-w](t-u)][I_{n+1}(\propto(t-u)-BI_n(\propto(t-u))][p_{k,0}(u)]du$$
$$+\int_0^t[\exp[-w](t-u)]BI_n(\propto(t-u))[\mu_2 p_{0,1}(u)]du.$$

By substituting in Eq. (15), we have

$$p_{k+n,0}(t)=[\exp[-w]t]\sum_{m=0}^{\infty}p_{m,0}\,B^{1-T(m)}[I_{n-T(m)}(\propto t)-I_{n+T(m)}(\propto t)]$$
$$+nB^n\int_0^t[\exp[-w](t-u)]\left[\frac{I_n(\propto(t-u))}{(t-u)}\right][p_{k,0}(u)]du,\quad n=1,2,\dots\infty \quad (16)$$

where $\frac{\propto}{2}=\frac{\nu}{B}$.

3.3 Computing $\boldsymbol{p_{m,0}(t)}$:

Theorem 2: The transient probabilities $\{p_{m,0}(t),\ m=0,1,2,\dots,k-1\}$ that the system under balking and catastrophes are obtained as:

$$p_{m,0}(t)=\mu_1\int_0^t b_{m,k-1}(t-u)\,p_{k,0}(u)du+\mu_2\int_0^t p_{0,1}(u)\ b_{m,0}(t-u)du+\sum_{i=0}^{k-1}p_{i,0}\ b_{m,i}(t)\ ,\ \ m=0,1,2,\dots,k-1,$$

where $p_{k,0}(t)=\mu_1\{\sum_{r=0}^{\infty}p_{r,0}(-1)^r\{\frac{(T(i)+r+1)}{B^{(T(i)+r+1)}}\int_0^t b_{k-1,0}^r(t-u)e^{-wu}\frac{I_{T(i)+r+1}(\alpha u)}{u}du\}\}$

$$+\mu_1\mu_2\sum_{r=0}^{\infty}p_{0,1}(u)(-1)^r\frac{(r+1)}{B^{(r+1)}}\int_0^t b_{k-1,0}^r(t-u)e^{-wu}\frac{I_{r+1}(\alpha u)}{u}du$$
$$-\mu_1\sum_{i=0}^{k-1}\sum_{r=0}^{\infty}p_{i,0}(-1)^r\{\frac{(r+1)}{B^{(r+1)}}.\int_0^t b_{i,o}(t-u)\int_0^u b_{k-1,0}^r(u-v)\,e^{-wu}\frac{I_{r+1}(\alpha v)}{v}dv\}du$$
$$+\mu_1\mu_2\sum_{r=0}^{\infty}p_{0,1}(u)(-1)^r\{\frac{(r+1)}{B^{(r+1)}}.\ \int_0^t b_{0,0}(t-u)\int_0^u b_{k-1,0}^r(u-v)\,e^{-wu}\frac{I_{r+1}(\alpha v)}{v}dv\}du$$

Proof:

Using Eqs. (1) and (3) of the given system, we have:

$$\frac{dp_{0,0}(t)}{dt}=-(\lambda_1+\lambda_2+\gamma)p_{0,0}(t)+\mu_1 p_{1,0}(t)+\mu_2 p_{0,1}(t)\,,$$
$$\frac{dp_{i,0}(t)}{dt}=-(\lambda_1+\lambda_2+\mu_1+\gamma)p_{i,0}(t)+\mu_1 p_{i+1,0}(t)+\lambda_1\,p_{i-1,0}(t)\,,\quad 1\le i<k\,,$$
$$p'(t)=Ap(t)+\mu_1 p_{k,0}(t)e_k+\mu_2 p_{0,1}(t)e_0,$$

where the matrix A = $(a_{m,n})_{kxk}$ is given as:

$$= \begin{pmatrix} -(\lambda_1+\lambda_2+\gamma) & \mu_1 & 0 & \cdots & & \vdots \\ \lambda_1 & -(\lambda_1+\lambda_2+\mu_1+\gamma) & \mu_1 & 0 & \cdots & \\ 0 & \lambda_1 & -(\lambda_1+\lambda_2+\mu_1+\gamma) & \mu_1 & 0 & \cdots \\ \vdots & & \ddots & \ddots & & 0 \\ \vdots & & & & & 0 \\ & & & \cdots & \lambda_1 & -(\lambda_1+\lambda_2+\mu_1+\gamma) \end{pmatrix}$$

$$P(t) = \left(p_{0,0}(t), p_{1,0}(t), p_{2,0}(t), \ldots \ldots p_{k-1,0}(t)\right), e_k = (0, \ldots ,0,1)^T,\ e_0 = (1,0,0, \ldots ,0).$$

Then by using Laplace transform we have:

$$zp^*(z) - p(0) = Ap^*(z) + \mu_1 p^*_{k,0}(z)e_k + \mu_2 p^*_{0,1}(z)e_0$$

$$p^*(z) = [zI - A]^{-1}\left[p(0) + \mu_1 p^*_{k,0}(z)e_k + \mu_2 p^*_{0,1}(z)e_0\right],$$

where $p^*(z)$ denote the Laplace transform of P(t) and p(0) =$p_{m,0}$, m=0, 1, 2…k-1. Writing $[zI - A]^{-1} = (b^*{}_{m,n}(z))_{kxk}$. , we get

$$p^*{}_{m,0}(z) = \mu_1\, b^*{}_{m,k-1}(z)p^*_{k,0}(z) + \mu_2 p^*_{0,1}(z)\, b^*{}_{m,0}(z) + \sum_{i=0}^{k-1} p_{i,0}\ b^*{}_{m,i}(z),$$

$m = 0,1,2, \ldots , k - 1.$

We will apply Raju and Bhat (1982) technique to calculate $b^*{}_{m,n}(z)$ from $[zI - A]^{-1} = (b^*{}_{m,n}(z))_{kxk}$.

For m=0, 1, 2…k-1,

$$b^*{}_{m,n}(z) = \begin{cases} \frac{1}{\mu_1}\left[\frac{u_{k,n+1}(z)u_{m,o}(z)-u_{m,n+1}(z)u_{k,o}(z)}{u_{k,o}(z)}\right] & , n = 0,1,2, \ldots k-2 \\ \frac{u_{n,o}(z)}{u_{k,o}(z)} & , n = k-1 \end{cases}.$$

With

$$u_{m,m}(z) = 1, m = 0,1,2, \ldots \ldots , k-1 ,$$

$$u_{m+1,m}(z) = \frac{1}{\mu_1}[z + (\lambda_1 + \lambda_2 + \mu_1(1 - \delta_{i,0}) + \gamma)], m = 0,1,2, \ldots \ldots , k-2,$$

$$u_{m+1,m-n}(z) = \frac{1}{\mu_1}\{[z + (\lambda_1 + \lambda_2 + \mu_1 + \gamma)]u_{m,m-n}(z) - (\lambda_1)u_{m-1,m-n}(z)\}$$

$$m = 0,1,2 \ldots k-2, n \le m ,$$

$$u_{k,n}(z) = \begin{cases} [z + (\lambda_1 + \lambda_2 + \mu_1 + \gamma)]u_{k-1,n}(z) - (\lambda_1)u_{k-2,n}(z) , & n = 0,1,2, \ldots \ldots , k-2 \\ [z + (\lambda_1 + \lambda_2 + \mu_1 + \gamma)] , & n = k-1 \end{cases}$$

and $u_{m,j}(z) = 0$, elsewhere.

Then

$p^*{}_{m,0}(z) = \mu_1\, b^*{}_{m,k-1}(z)p^*_{k,0}(z) + \mu_2 p^*_{0,1}(z)\, b^*{}_{m,0}(z) + \sum_{i=0}^{k-1} p_{i,0}\ b^*{}_{m,i}(z), m = 0,1,2 \ldots k-1.$

By apply the convolution theorem, we get

$$p_{m,0}(t) = \mu_1 \int_0^t b_{m,k-1}(t-u)\, p_{k,0}(u)du + \mu_2 \int_0^t p_{0,1}(u)\ b_{m,0}(t-u)du + \sum_{i=0}^{k-1} p_{i,0}\ b_{m,i}(t) , \quad m = 0,1,2 \ldots k-1.$$

Now we are going to obtain $p_{k,0}(u)$

From Eq. (16) since

$$Bp_{k,0}(t) = \sum_{m=0}^{\infty} p_{m,0}B^{1-T(m)}I_{-T(m)}(\alpha t) + (\nu + \mu_1)\int_0^t e^{-w(t-u)}\, BI_0(\alpha(t-u))R_0(u)du$$
$$- \int_0^t e^{-w(t-u)}\, [B^2\mu_1 I_1(\alpha(t-u)) + \nu I_{-1}(\alpha(t-u))]R_0(u)du$$
$$+ \nu\int_0^t e^{-w(t-u)}\, [I_{-1}(\alpha(t-u)) - BI_0(\alpha(t-u))]p_{k,0}(u)du$$
$$+ \int_0^t e^{-w(t-u)}\, [BI_0(\alpha(t-u))][\mu_2 p_{0,1}(u)]du.$$

Since $I_{-n}(x) = I_n(x)$ from Bessel function

$$p_{k,0}(t) = \exp[-w]t \sum_{m=0}^{\infty} p_{m,0}B^{-T(m)}I_{T(m)}(\alpha t) + (\nu + \mu_1)\int_0^t e^{-w(t-u)}\, I_0(\alpha(t-u))R_0(u)du$$
$$- \int_0^t e^{-w(t-u)}\, [(B\mu_1 + \frac{\nu}{B})I_1(\alpha(t-u))]R_0(u)du$$
$$+ \frac{\nu}{B}\int_0^t e^{-w(t-u)}\, [I_1(\alpha(t-u)) - BI_0(\alpha(t-u))]p_{k,0}(u)du$$
$$+ \int_0^t e^{-w(t-u)}\, [I_0(\alpha(t-u))][\mu_2 p_{0,1}(u)]du$$

By using Laplace transform, suppose $r = \sqrt{(z+w)^2 - \alpha^2}$ we have

$$p^*_{k,0}(z) = \sum_{m=0}^{\infty} p^*{}_{m,0}B^{-T(m)}\frac{[(z+w) - r]^{T(m)}}{\alpha^{T(m)}r} + (\nu + \mu_1)\frac{R^*_0(z)}{r} - \left(B\mu_1 + \frac{\nu}{B}\right)\frac{[(z+w) - r]}{\propto r}R^*_0(z) + \frac{\nu}{B}\left[\frac{[(z+w) - r]}{\propto r}\right]p^*_{k,0}(z) - \nu\frac{p^*_{k,0}(z)}{r} + \frac{\mu_2 p^*_{0,1}(z)}{r}$$

$$p^*_{k,0}(z) = \sum_{m=0}^{\infty} p^*{}_{m,0}(\alpha B)^{-T(m)}[(z+w) - r]^{T(m)} + [(\nu + \mu_1) - \frac{\left(B\mu_1 + \frac{\nu}{B}\right)}{\propto}[(z+w) - r]R^*_0(z) + \frac{\nu}{\alpha B}[(z+w) - r - \alpha B]p^*_{k,0}(z) + \mu_2 p^*_{0,1}(z).$$

Since $\nu = \lambda_1\beta$, $\alpha = 2\sqrt{\nu\mu_1}$, $B = \sqrt{\frac{\nu}{\mu_1}}$, $\alpha B = 2\nu$, $B\mu_1 = \sqrt{\frac{\nu}{\mu_1}}\ \mu_1 = \sqrt{\nu\mu_1} = \frac{\alpha}{2}$, $\frac{\nu}{B} = \frac{\nu}{\sqrt{\frac{\nu}{\mu_1}}} = \sqrt{\nu\mu_1} = \frac{\alpha}{2}$, $\left(B\mu_1 + \frac{\nu}{B}\right) = \alpha$.

$$[[(z+w) - r]\ - (\nu + \mu_1)]R^*_0(z) = \sum_{m=0}^{\infty} p^*{}_{m,0}(\alpha B)^{-T(m)}[(z+w) - r]^{T(m)} + \frac{1}{2}[(z+w) - r\ -\ 2r - \alpha B]p^*_{k,0}(z) + \mu_2 p^*_{0,1}(z). [[(z+w) - r]\ - (\nu + \mu_1)]R^*_0(z)$$

$$= \sum_{m=0}^{\infty} p^*{}_{m,0}(\alpha B)^{-T(m)}[(z+w)-r]^{T(m)} + \frac{1}{2}[(z+w)-3r \ -2\nu]p^*_{k,0}(z) + \mu_2 p^*_{0,1}(z).$$

Suppose

$$N = \sum_{m=0}^{\infty} p^*{}_{m,0}(\alpha B)^{-T(m)}[(z+w)-r]^{T(m)} + \left[\frac{1}{2}\{(z+w)\ -3\,r\} - \nu\right] p^*_{k,0}(z)\ \ \mu_2 p^*_{0,1}(z).$$

$$R^*_0(z) = \frac{N}{[(z+w)-r-(\nu+\mu_1)]}. \quad (17)$$

But $R^*_0(z) = \sum_{i=0}^{k-1} p^*{}_{m,0}(z) + p^*_{k,0}(z)$,

$R^*_0(z) = e^T p^*(z) + p^*_{k,0}(z)$,

where $e = (1 \dots 1)^T, p^*(z) = [zI-A]^{-1}[p(0) + \mu_1 p^*_{k,0}(z)e_k + \mu_2 p^*_{0,1}(z)e_0]$.

Then

$$R^*_0(z) = e^T\{[zI-A]^{-1}[p(0)+\mu_1 p^*_{k,0}(z)e_k + \mu_2 p^*_{0,1}(z)e_0]\} + p^*_{k,0}(z) = e^T[zI-A]^{-1}p(0) + \mu_1 e^T[zI-A]^{-1}e_k p^*_{k,0}(z) + \mu_2 e^T[zI-A]^{-1}e_0 p^*_{0,1}(z) + p^*_{k,0}(z). \quad (18)$$

Substituting from Eq. (18) into Eq. (17) we get

$$e^T[zI-A]^{-1}p(0) + \mu_1 e^T[zI-A]^{-1}e_k p^*_{k,0}(z)\ + \mu_2 e^T[zI-A]^{-1}e_0 p^*_{0,1}(z) + p^*_{k,0}(z) = \frac{N}{[(z+w)-r-(\nu+\mu_1)]},$$

$$[1+\mu_1 e^T[zI-A]^{-1}e_k]p^*_{k,0}(z) = \frac{N}{[(z+w)-r-(\nu+\mu_1)]} - \{e^T[zI-A]^{-1}p(0) + \mu_2 e^T[zI-A]^{-1}e_0 p^*_{0,1}(z)\}.$$

$$\{[1 + \mu_1 e^T[zI-A]^{-1}e_k] + \frac{\left[\nu - \frac{1}{2}\{(z+w)\ -3\,r\}\right]}{[(z+w)-r-(\nu+\mu_1)]}\}p^*_{k,0}(z) = \frac{\sum_{m=0}^{\infty} p^*{}_{m,0}(\alpha B)^{-T(m)}[(z+w)-r]^{T(m)} + \mu_2 p^*_{0,1}(z)}{[(z+w)-r-(\nu+\mu_1)]} - \{e^T[zI-A]^{-1}p(0) + \mu_2 e^T[zI-A]^{-1}e_0 p^*_{0,1}(z)\},$$

Then

$$p^*_{k,0}(z) = 1\Big/\{\ [(z+w)-r-(\nu+\mu_1)][1+\mu_1 e^T[zI-A]^{-1}e_k] + \left[\nu - \frac{1}{2}\{(z+w)\ -3\,r\}\right]\}$$
$$\times\{\sum_{m=0}^{\infty} p^*{}_{m,0}(\alpha B)^{-T(m)}[(z+w)-r]^{T(m)} + \mu_2 p^*_{0,1}(z) - [(z+w)-r-(\nu+\mu_1)]\{e^T[zI-A]^{-1}p(0) + \mu_2 e^T[zI-A]^{-1}e_0 p^*_{0,1}(z)\}\}.$$

Using $[z\,I-A]^{-1} = (\,b^*{}_{m,n}(z))_{kxk}$, then

$$p^*_{k,0}(z)\ = 1\Big/\Big\{(z+w)-r-(\nu+\mu_1)\Big[\mu_1 \sum_{m=0}^{k-1} b^*{}_{m,k-1}(z)\Big] + \left[\tfrac{1}{2}\{(z+w)+\ r\} - \mu_1\right]\Big\} \quad (19)$$
$$\times\{\sum_{m=0}^{\infty} p^*{}_{m,0}(\alpha B)^{-T(m)}[(z+w)-r]^{T(m)} + \mu_2 p^*_{0,1}(z) - [(z+w)-r-(\nu+\mu_1)]\Big\{\sum_{i=0}^{k-1}\sum_{m=0}^{k-1}\ p_{I,0}\, b^*{}_{m,i}(z) + \mu_2 \sum_{m=0}^{k-1} b^*{}_{m,0}(z)\, p^*_{0,1}(z)\Big\}$$

Obviously that $\det[z\,I-A] = (\theta-\chi)^{-1}\{\theta^{k+1} - \chi^{k+1} - \mu_1(\theta^k - \chi^k)\}$ see Tarabia (2001) where

$$\theta(z) = \frac{1}{2}\Big[(z+(\lambda_1+\lambda_2+\mu_1+\gamma)) + \sqrt{(z+(\lambda_1+\lambda_2+\mu_1+\gamma))^2 - 4\lambda_1\mu_1}\ \Big]$$

$$\chi(z) = \frac{1}{2}\Big[(z+(\lambda_1+\lambda_2+\mu_1+\gamma)) - \sqrt{(z+(\lambda_1+\lambda_2+\mu_1+\gamma))^2 - 4\lambda_1\mu_1}\ \Big].$$

Using partial fraction technique, then the cofactor $b^*{}_{m,n}(z)$ for the arranged element (*m, n*) of the matrix $[zI-A]$ can be rewritten as

$$b^*{}_{m,j}(z) = \sum_{n=0}^{k-1} \frac{c^n_{m,j}}{z - z_n}, \quad (20)$$

where z_n are the eigenvalues of the matrix A for $n = 0,1,2,\dots,k-1$ and $c^*_{m,j} = \lim_{z\to z_n}(z - z_n)\, b^*{}_{m,j}(z)$. After inverting Eq. (19) we obtain

$$b_{m,j}(t) = \sum_{n=0}^{k-1} c^n_{m,j}\ e^{z_n t},$$

where $w = (\lambda_1\beta + \lambda_2 + \mu_1 + \gamma)$, $\nu = \lambda_1\beta$ then

$$[(z+w)-r-(\nu+\mu_1)] = z + (\lambda_1\beta+\lambda_2+\mu_1+\gamma) - r - (\lambda_1\beta+\mu_1) = z + \lambda_2 + \gamma - r$$

Then we have

$$\sum_{m=0}^{k-1}(z+\lambda_2+\gamma-r\,)\, b^*{}_{m,j}(z) = 1 + \sum_{m=0}^{k-1}\frac{B^{(m)}_j}{z - z_m} = 1 +\ b^*{}_j(z),$$

where

$$b_j(t) = 1 + \sum_{m=0}^{k-1} B^{(m)}_j e^{z_m t}\ \ , j = 0,1,2,\dots,k-1.$$

Then Eq. (19) can be rewritten as

$$p^*_{k,0}(z) = \frac{1}{\Big\{[\mu_1\, b^*{}_{k-1\,,0}(z)] + \left[\frac{1}{2}\{(z+w)+\ r\} - \mu_1\right]\Big\}} \times \left\{\sum_{m=0}^{\infty} p^*{}_{m,0}(\alpha B)^{-T(m)}[(z+w)-r]^{T(m)} + \mu_2 p^*_{0,1}(z) - \sum_{i=0}^{k-1}\ p_{i,0}\, b^*{}_{i,0}(z) + \mu_2\, b^*{}_{0,0}(z) p^*_{0,1}(z)\right\}$$

$$p^*_{k,0}(z) = \{(z+w)-\ r\}\Big/\Big[\frac{1}{2}\{(z+w)^2 -\ r^2\} + \mu_1\{(z+w)-\ r\}\, b^*{}_{k-1\,,0}(z) - \mu_1\{(z+w)-\ r\}\Big]$$

$$\times\left\{\sum_{m=0}^{\infty} p^*_{m,0}(\alpha B)^{-T(m)}[(z+w)-r]^{T(m)} + \mu_2 p^*_{0,1}(z) - \sum_{i=0}^{k-1} p_{i,0}\, b^*_{i,0}(z) + \mu_2\, b^*_{0,0}(z) p^*_{0,1}(z)\right\}$$

$$p^*_{k,0}(z) = \frac{2}{\alpha}\frac{\{(z+w)-r\}}{\alpha}\{1 + \frac{2\mu_1\{(z+w)-r\}(b^*_{k-1,0}(z)-1)}{\alpha^2}\}^{-1} \times\{\sum_{m=0}^{\infty} p^*_{m,0}(\alpha B)^{-T(m)}[(z+w)-r]^{T(m)} + \mu_2 p^*_{0,1}(z) - \sum_{i=0}^{k-1} p_{i,0}\, b^*_{i,0}(z) + \mu_2\, b^*_{0,0}(z) p^*_{0,1}(z)\}$$

$$= \frac{2\sum_{m=0}^{\infty} p^*_{m,0}(\alpha B)^{-T(m)}[(z+w)-r]^{T(m)+1}}{\frac{\alpha^2}{2\mu_1} + \{(z+w)-r\}\, b^*_{k-1,0}(z)} + \frac{2\mu_2 p^*_{0,1}(z)\{(z+w)-r\}}{\frac{\alpha^2}{2\mu_1} + \{(z+w)-r\}\, b^*_{k-1,0}(z)} - \frac{2\{(z+w)-r\}\sum_{i=0}^{k-1} p_{i,0}\, b^*_{i,0}(z)}{\frac{\alpha^2}{2\mu_1} + \{(z+w)-r\}\, b^*_{k-1,0}(z)} + \frac{2\mu_2\, b^*_{0,0}(z) p^*_{0,1}(z)\{(z+w)-r\}}{\frac{\alpha^2}{2\mu_1} + \{(z+w)-r\}\, b^*_{k-1,0}(z)}.$$

Since $\alpha = 2\sqrt{\nu\mu_1}$ then $\frac{\alpha^2}{2\mu_1} = \frac{4\nu\mu_1}{2\mu_1} = 2\nu$

$$p^*_{k,0}(z) = \frac{2\sum_{m=0}^{\infty} p^*_{m,0}(\alpha B)^{-T(m)}[(z+w)-r]^{T(m)+1}}{2\nu + \{(z+w)-r\}\, b^*_{k-1,0}(z)} + \frac{2\mu_2 p^*_{0,1}(z)\{(z+w)-r\}}{2\nu + \{(z+w)-r\}\, b^*_{k-1,0}(z)} - \frac{2\{(z+w)-r\}\sum_{i=0}^{k-1} p_{i,0}\, b^*_{i,0}(z)}{2\nu + \{(z+w)-r\}\, b^*_{k-1,0}(z)} + \frac{2\mu_2\, b^*_{0,0}(z) p^*_{0,1}(z)\{(z+w)-r\}}{2\nu + \{(z+w)-r\}\, b^*_{k-1,0}(z)}.$$

Since $B = \sqrt{\frac{\nu}{\mu_1}}$ then $\alpha B = 2\nu$,

$$p^*_{k,0}(z) = \frac{\mu_1\sum_{m=0}^{\infty} p^*_{m,0}(\alpha B)^{-T(m)}[(z+w)-r]^{T(m)+1}}{\alpha B + \{(z+w)-r\}\, b^*_{k-1,0}(z)} + \frac{\mu_1\mu_2 p^*_{0,1}(z)\{(z+w)-r\}}{\alpha B + \{(z+w)-r\}\, b^*_{k-1,0}(z)} - \frac{\mu_1\{(z+w)-r\}\sum_{i=0}^{k-1} p_{i,0}\, b^*_{i,0}(z)}{\alpha B + \{(z+w)-r\}\, b^*_{k-1,0}(z)} + \frac{\mu_1\mu_2\, b^*_{0,0}(z) p^*_{0,1}(z)\{(z+w)-r\}}{\alpha B + \{(z+w)-r\}\, b^*_{k-1,0}(z)}.$$

By using the inverse Laplace transform

$$p_{k,0}(u) = \mu_1\{\sum_{r=0}^{\infty} p_{r,0}(-1)^r\{\frac{(T(i)+r+1)}{B^{(T(i)+r+1)}}\int_0^t b^{(r)}_{k-1,0}(t-u)e^{-wu}\frac{I_{T(i)+r+1}(\alpha u)}{u}du\}\} \qquad (21)$$

$$+\mu_1\mu_2\sum_{r=0}^{\infty} p_{0,1}(u)(-1)^r\frac{(r+1)}{B^{(r+1)}}\int_0^t b^{(r)}_{k-1,0}(t-u)e^{-wu}\frac{I_{r+1}(\alpha u)}{u}du - \mu_1\sum_{i=0}^{k-1}\sum_{r=0}^{\infty} p_{i,0}(-1)^r\{\frac{(r+1)}{B^{(r+1)}}\int_0^t b_{i,0}(t-u)\int_0^u b^{(r)}_{k-1,0}(u-v)\, e^{-wu}\frac{I_{r+1}(\alpha v)}{v}dv\}du + \mu_1\mu_2\sum_{r=0}^{\infty} p_{0,1}(u)(-1)^r\{\frac{(r+1)}{B^{(r+1)}}$$

Using Eq. (21) in the following

$$p_{m,0}(t) = \mu_1\int_0^t b_{m,k-1}(t-u)\, p_{k,0}(u)du + \mu_2\int_0^t p_{0,1}(u)\; b_{m,0}(t-u)du + \sum_{i=0}^{k-1} p_{i,0}\; b_{m,i}(t)\ ,\quad m = 0,1,2,\dots,k-1$$

Therefore all probabilities from 0 to k-1 can be obtained.

Computing $\boldsymbol{G_j(s,t), j \geq 1}$ *:*

Otherwise for $j \geq 1$:

Using Eq. (2), Eq. (4), Eq. (6) and Eq. (8), we get:

$$\sum_{i=0}^{k}\frac{dp_{i,j}(t)}{dt} = -(\lambda_1+\lambda_2+\mu_2+\gamma)p_{0,j}(t) + \mu_1 p_{1,j}(t) + \mu_2 p_{0,j+1}(t) + \lambda_2\, p_{0,j-1}(t) - (\lambda_1+\lambda_2+\mu_1+\gamma)\sum_{i=1}^{k-1} p_{i,j}(t) - (\lambda_1\beta+\lambda_2+\mu_1+\gamma)p_{k,j}(t) + \mu_1\sum_{i=1}^{k-1} p_{i+1,j}(t) + \lambda_1\sum_{i=1}^{k-1} p_{i-1,j}(t) + \mu_1 p_{k+1,j}(t) + \lambda_1 p_{k-1,j}(t) + \lambda_2\, p_{k,j-1}(t) + \lambda_2\sum_{i=1}^{k-1} p_{i,j-1}(t)$$

$$= -(\lambda_2+\gamma)\sum_{i=0}^{k} p_{i,j}(t) + \mu_1 p_{k+1,j}(t) + \mu_2\left(p_{0,j+1}(t) - p_{0,j}(t)\right) + \sum_{i=0}^{k} p_{i,j-1}(t) - \lambda_1\beta p_{k,j}(t). \qquad (22)$$

$$\sum_{i=1}^{\infty}\frac{dp_{k+i,j}(t)}{dt}s^i \qquad (23)$$

$$= \left[-(\lambda_1\beta + \lambda_2 + \mu_1 + \gamma) + \frac{\mu_1}{s} + \lambda_1\beta\, s\right]\sum_{i=1}^{\infty} p_{k+i,j}(t)\, s^i - \mu_1 p_{k+1,j}(t) + \lambda_1\beta\, s p_{k,j}(t) + \lambda_2 \sum_{i=1}^{\infty} p_{k+i,j-1}(t)\; s^i$$

Adding the last equation to Eq. (22) we get

$$\frac{\partial G_j(s,t)}{\partial t} = \left[-(\lambda_1\beta + \lambda_2 + \mu_1 + \gamma) + \frac{\mu_1}{s} + \lambda_1\beta\, s\right]\left[G_j(s,t) - R_j(t)\right] - (\lambda_2+\gamma)R_j(t) + \lambda_2\left[\sum_{i=0}^{k} p_{I,j-1}(t) + \sum_{i=1}^{\infty} p_{k+I,j-1}(t)\; s^i\right] + \lambda_1\beta\,(s-1)p_{k,j}(t) + \mu_2\left(p_{0,j+1}(t) - p_{0,j}(t)\right) \quad ,j \geq 1$$

$$\frac{\partial G_j(s,t)}{\partial t} = \left[-(\lambda_1\beta + \lambda_2 + \mu_1 + \gamma) + \frac{\mu_1}{s} + \lambda_1\beta\, s\right]\left[G_j(s,t) - R_j(t)\right] - (\lambda_2+\gamma)R_j(t) + \lambda_2 G_{j-1}(s,t) + \lambda_1\beta(s-1)p_{k,j}(t) + \mu_2\left(p_{0,j+1}(t) - p_{0,j}(t)\right), j \geq 1. \qquad (24)$$

From Eq. (24) we have

$$\frac{\partial G_j(s,t)}{\partial t} = \left[-(\lambda_1 B + \lambda_2 + \mu_1 + \gamma) + \frac{\mu_1}{s} + \lambda_1 Bs\right]\left[G_j(s,t) - R_j(t)\right] - (\lambda_2+\gamma)R_j(t) + \lambda_2 G_{j-1}(s,t) + \lambda_1 B(s-1)p_{k,j}(t) + \mu_2\left(p_{0,j+1}(t) - p_{0,j}(t)\right), j \geq 1,$$

where $G_j(s,0) = \sum_{m=0}^{\infty} p_{m,j} s^{T(m)}$, $T(m) = (m-k)[1 - \sum_{n=0}^{k} \delta_{in}]$ and $R_j(t) = \sum_{i=0}^{k} p_{i,j}(t)$, with the assume that $=h = \left[-(\lambda_1 B + \lambda_2 + \mu_1 + \gamma) + \frac{\mu_1}{s} + \lambda_1 Bs\right]$, then

$$\frac{\partial G_j(s,t)}{\partial t} - hG_j(s,t) - \lambda_2 G_{j-1}(s,t) = \left[(\lambda_1 B + \lambda_2 + \mu_1 + \gamma) - \frac{\mu_1}{s} - \lambda_1 Bs - (\lambda_2+\gamma)\right]R_j(t) + \lambda_1 B(s-1)p_{k,j}(t) + \mu_2\left(p_{0,j+1}(t) - p_{0,j}(t)\right)$$

$$= \left[(\lambda_1 B - \lambda_1 Bs) + \left(\mu_1 - \frac{\mu_1}{s}\right)\right]R_j(t) + \lambda_1 B(s-1)p_{k,j}(t) + \mu_2\left(p_{0,j+1}(t) - p_{0,j}(t)\right), j \geq 1.$$

By using Laplace transform:

$$zG_j^*(s,z) - G_j(s,0) - hG_j^*(s,z) - \lambda_2 G_{j-1}^*(s,z) = \left[(\lambda_1 B - \lambda_1 Bs) + \left(\mu_1 - \frac{\mu_1}{s}\right)\right]R_j^*(z) + \lambda_1 B(s-1)p^*_{k,j}(z) + \mu_2\left(p^*_{0,j+1}(z) - p^*_{0,j}(z)\right), \qquad j \geq 1$$

$$[z-h]G_j^*(s,z) - \lambda_2 G_{j-1}^*(s,z) = \sum_{m=0}^{\infty} p_{m,j} s^{T(m)} + \left[(\lambda_1 B - \lambda_1 Bs) + \left(\mu_1 - \frac{\mu_1}{s}\right)\right]\sum_{i=0}^{k} p_{i,j}^*(z) + \lambda_1 B(s-1)p^*_{k,j}(z) + \mu_2\left(p^*_{0,j+1}(z) - p^*_{0,j}(z)\right), j \geq 1.$$

By using Laplace inverse transform:

$$[t-h]G_j(s,t) - \lambda_2 G_{j-1}(s,t) = \sum_{m=0}^{\infty} p_{m,j} s^{T(m)} + \left[(\lambda_1 B - \lambda_1 Bs) + \left(\mu_1 - \frac{\mu_1}{s}\right)\right]\sum_{i=0}^{k} p_{i,j}(t) + \lambda_1 B(s-1)p_{k,j}(t) + \mu_2\left(p_{0,j+1}(t) - p_{0,j}(t)\right), j \geq 1 \qquad (25)$$

At j=1 then

$$[t-h]G_1(s,t) - \lambda_2 G_0(s,t) = \sum_{m=0}^{\infty} p_{m,1} s^{T(m)} + \left[(\lambda_1 B - \lambda_1 Bs) + \left(\mu_1 - \frac{\mu_1}{s}\right)\right]\sum_{i=0}^{k} p_{i,1}(t) + \lambda_1 B(s-1)p_{k,1}(t) + \mu_2\left(p_{0,2}(t) - p_{0,1}(t)\right).$$

Since

$$G_0(s,t) = G_0(s,0)\left[\exp\left[-w + \frac{\mu_1}{s} + \lambda_1\beta s\right]t\right] + \int_0^t \left[\exp\left[-w + \frac{\mu_1}{s} + \lambda_1\beta s\right](t-u)\right]\left[(\lambda_1\beta + \mu_1) - \left(\frac{\mu_1}{s} + \lambda_1\beta s\right)\right]R_0(u)du + \int_0^t \left[\exp\left[-w + \frac{\mu_1}{s} + \lambda_1\beta s\right](t-u)\right]\left[\lambda_1\beta(s-1)p_{k,0}(u)\right]du + \int_0^t [\exp\left[-w + \frac{\mu_1}{s} + \lambda_1\beta s\right](t-u)]\left[\mu_2 p_{0,1}(u)\right]du.$$

Then

$$G_1(s,t) = \left[\lambda_2\left[\sum_{m=0}^{\infty} p_{m,0} s^{T(m)}\left[\exp\left[-w + \frac{\mu_1}{s} + \lambda_1\beta s\right]t\right] + \int_0^t \left[\exp\left[-w + \frac{\mu_1}{s} + \lambda_1\beta s\right](t-u)\right]\left[(\lambda_1\beta + \mu_1) - \left(\frac{\mu_1}{s} + \lambda_1\beta s\right)\right]R_0(u)du + \int_0^t \left[\exp\left[-w + \frac{\mu_1}{s} + \lambda_1\beta s\right](t-u)\right]\left[\lambda_1\beta(s-1)p_{k,0}(u)\right]du + \int_0^t \left[\exp\left[-w + \frac{\mu_1}{s} + \lambda_1\beta s\right](t-u)\right]\left[\mu_2 p_{0,1}(u)\right]du\right]\right.$$

$$+\sum_{m=0}^{\infty} p_{m,1}s^{T(m)} + \Big[(\lambda_1 B - \lambda_1 Bs) + \left(\mu_1 - \frac{\mu_1}{s}\right)\Big]\sum_{i=0}^{k} p_{i,1}(t) + \lambda_1 B(s-1)p_{k,1}(t) + \mu_2\left(p_{0,2}(t) - p_{0,1}(t)\right)\Big]/[t-h]$$

Since $G_1(s,t) = \sum_{i=0}^{k} p_{i,1}(t) + \sum_{i=1}^{\infty} p_{k+i,1}(t)\ s^i$, then

$$G_1(0,t) = \sum_{i=0}^{k} p_{i,1}(t) = \Bigg[\lambda_2\Big[\int_0^t \exp[-w(t-u)]\,[(\lambda_1\beta+\mu_1)]R_0(u)du + \int_0^t \exp[-w(t-u)]\,[-\lambda_1\beta p_{k,0}(u)]du + \int_0^t \exp[-w(t-u)]\,[\mu_2 p_{0,1}(u)]du\Big] + [\lambda_1 B + \mu_1]\sum_{i=0}^{k} p_{i,1}(t) - \lambda_1 B p_{k,1}(t) + \mu_2\left(p_{0,2}(t) - p_{0,1}(t)\right)\Bigg]/[t+w]$$

$$\sum_{i=0}^{k} p_{i,1}(t) = \Bigg[\lambda_2\Big[\int_0^t \exp[-w(t-u)]\,[(\lambda_1\beta + \mu_1)R_0(u)du + (-\lambda_1\beta)p_{k,0}(u)du + \mu_2 p_{0,1}(u)du\Big]\Big] - \lambda_1 B p_{k,1}(t) + \mu_2\left(p_{0,2}(t) - p_{0,1}(t)\right)\Bigg] / (t+w)\left(\left(1-(\lambda_1 B + \mu_1)\right)\right).$$

Similarly, we can obtain all $G_j(s,t)\ , j \geq 1$.

Special Cases

Here we state various special cases as follow:

(a) If $\lambda_2 = \mu_2 = 0\ , \gamma = 0\ , \ \lambda_1 = \lambda,\ \ \mu_1 = \mu\ \ , \beta = \rho$ then

$$\frac{\partial G_0(s,t)}{\partial t} - \left[-(\lambda\rho+\mu) + \frac{\mu}{s} + \lambda\rho s\right]G_0(s,t) = -\left[-(\lambda\rho+\mu) + \frac{\mu}{s} + \lambda\rho s\right]R_0(t) + \lambda\rho(s-1)p_{k,0}(t)$$

Then $G_0(s,t) = p(s,t)\ , R_0(t) = q_k(t)\ ,\ \ p_{k,0}(t) = p_k(t)$, these results agree with Kumar (1993).

(b) If $\lambda_2 = \mu_2 = 0\ ,\ \lambda_1 = \lambda,\ \mu_1 = \mu$ we get that

$$\frac{\partial G_0(s,t)}{\partial t} = \left[-(\lambda\beta+\mu+\gamma) + \frac{\mu}{s} + \lambda\beta s\right]G_0(s,t) + \left[(\lambda\beta+\mu) - \left(\frac{\mu}{s} + \lambda\beta s\right)\right]R_0(t) + \lambda\beta(s-1)p_{k,0}(t)$$

Then $G_0(s,t) = H(s,t)\ , Q(t) = 0\ , R_0(t) = r_{i,k}(t),\ \ p_{k,0}(t) = p_{i,k}(t)$, which agree with Tarabia (2011).

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الملخص العربي

عنوان البحث: حل عابر لنظام انتظار ذي أولوية من طبقتين مع التصدع والكوارث

أحمد محمد كامل طرابيه[١], رحاب القاضي*[١], غدير الشريف[١]
[١] قسم الرياضيات – كلية العلوم – جامعة دمياط

يُنظر إلى نظام انتظار الخادم الفردي الذي يوفر الخدمة لنوعين من عملاء الوصول، النوع ١ يُعرف بالفئة ١ والنوع الثاني بالفئة ٢، ويمتلك كلا النوعين بشكل مستقل خطه الخاص .تم تعيين الفئة ١ لتكون ذات أولوية أعلى ويتم تعيين أولوية منخفضة للفئة ٢ . نحن نفترض في كل فئة أن الخدمة تعمل كـ FIFO مع نظام أولوية مستمر، أي في وقت الخدمة لعميل ذي أولوية منخفضة، إذا دخل أي عميل ذي أولوية عالية إلى النظام، سيتم إيقاف خدمة العميل للعميل ذي الأولوية المنخفضة و يمكن استئنافه مرة أخرى إذا لم يكن هناك عميل في فئة الأولوية العالية في النظام .العميل الوافد الجديد إما أن ينضم إلى قائمة انتظار الفئة ١ مع احتمال واحد إذا كان رقم العملاء في الفئة ١ لا يتجاوز قيمة k إذا كان هناك k أو أكثر من العملاء قبله، فإنه يدخل نظام قائمة الانتظار باحتمالية β أو قد يرفض الاحتمال التكميلي .النظام لديه قدرة لا نهائية .عندما يكون النظام خاملاً أو مشغولاً، قد تحدث الكوارث مع عملية بواسون حسب المعدل γ يمكن تصميم النظام المعطى بسهولة بواسطة عملية ماركوف حيث تشير عدد العملاء في الدرجة العالية والمنخفضة على التوالي بالرموز I,j .