ESTIMATION AND OPTIMAL DESIGN IN STEP ACCELERATED LIFE TESTS FOR THE GENERALIZED BURR DISTRIBUTION USING TYPE-II CENSORING *

Gamila M. Nasr **

This paper considers simple failure step-stress accelerated life testing (ALT) under mixture distribution where the experiment is subject to type-II censoring. A failure step test runs until specified proportion of units fail at each stress. The life test model consists of generalized Burr lifetime distribution with scale parameter is affected by the stress through the inverse power law model, and a cumulative exposure model for the effect of changing stress. Maximum Likelihood estimators (MLE) of the model are obtained. Also, Confidence intervals estimation of the parameters is presented. Moreover, optimum plans for simple failure step-stress ALT are developed. Such plans determine the best choice of the proportion of test units allocated to each stress, depending on minimizing the generalized asymptotic variance (GAV) of the model parameters. An example is included for numerical illustration.

1 Introduction:

Testing the life time of some products or materials under normal condition often requires long periods of time. So, in order to short the testing time, all or some of test units may be subjected to conditions more sever than normal case. In this case, quick information on the reliability of a product components or materials can be collected.

* Part of Ph. D. Thesis Submitted to Faculty of Economics & Political Science, Cairo University. Supervised by: Prof. Dr. Abdalla A. Abdel-Ghaly, and Eman H. El-Khodary, Associate Professor of Statistics, Department of Statistics, Faculty of Economics & Political Science, Cairo University.

** Researcher of Statistics, The National Center for Social and Criminological Research, Cairo.

In ALT, all test items run only at accelerated conditions. According to Nelson (1990)[1], the stress can be applied in various ways. In some branches of reliability testing, stress on the same units is changed during the test. In case of step-stress, the stress on the surviving units is turned up in order to force all or most of the units to fail more quickly than the case of constant stress.

Stress on each unit is increased at pre-specified times (time-step stress) or upon the occurrence of a fixed number failures (failure-step stress). The step-stress pattern is chosen to assure failures quickly. Usually all units go through the same specified pattern of stress levels and test times. As with constant stress test, the parameters of a model for life under step-stress are estimated. The test data and the relation between the cumulative distribution function of product life under constant stress and the cumulative distribution function under step stress, are used to estimate product reliability. It is assumed that changing the stress from one level to another affects the value of the parameters only and not the functional form of the lifetime distribution, this is a major assumption of ALT.

Several models are available in the literature concerning the relationship between certain parameters of the life time distribution and the stress levels at which the experiment is conducted. The power rule model is the most widely used model as an acceleration function.

A functional relationship , $\theta = g(\underline{s},\underline{\alpha})$, where $(\underline{\alpha})$ is a vector of unknown parameters and $(\underline{s})$ denotes the vector of stresses. It is assumed that changing $(\underline{s})$ affects the value of θ only and not the functional form of $f(t,\theta)$.

There are different models showing how the stress $\underline{s}$ is affecting the failure distribution. Among these models, the most famous models are the inverse power law, the Arrhenius, the Erying relationships and the log linear relationship.

The Inverse Power Law:

This model is mostly used for flash lamps and simple fatigue due to mechanical loading. This relation is given by :

$$\theta = v / s^{p},$$

where θ is a parameter of life distribution, s is the applied stress, v is the constant of proportionality and p is the power of the applied stress, where v and p are the parameters to be estimated.

One method of constructing a new distribution is to use the known parametric form of a distribution and allow one (or more) of the parameters to vary according to a special probability law. The new distribution is called a Mixture of distribution. This theory has useful applications in industrial reliability and medical survivorship analysis.

If $f(t|\theta)$ is a probability density function depending on a m dimensional parameter vector $\underline{\theta}$ and if $G(\underline{\theta})$ is called a m-dimensional cumulative distribution function, then :

$f(t) = \int_{\underline{\theta}} f(t|\underline{\theta})g(\underline{\theta})$ is called a mixture density, and $g(\underline{\theta})$ is called the mixing distribution[2].

Dubey (1968)[3] obtained a (generalized burr) distribution by mixing the Weibull distribution in the form

$$f(t|\phi,\theta) = \phi\theta t^{\phi-1} e^{-\theta t^{\phi}}, \qquad t > 0,\ \phi,\theta > 0,$$

over the Gamma distribution in the form :

$$g(\theta|\alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}, \qquad \theta > 0, \alpha, \beta > 0$$

The resulting probability density function (pdf) has the following form:

$$f(t|\alpha,\beta,\phi) = \frac{\alpha\,\beta^{\alpha}\,\phi\,t^{\phi-1}}{(\beta+t^{\phi})^{\alpha+1}}, \qquad t>0\,,\ \phi,\alpha,\beta>0,$$

which is a generalized Burr distribution with three parameters (α,β,ϕ).

The distribution function is:

$$F(t|\alpha,\beta,\phi) = 1-\left(1+\frac{t^{\phi}}{\beta}\right)^{-\alpha} \qquad t>0\ .$$

The reliability function has the following form:

$$R(t|\alpha,\beta,\phi) = \left(1+\frac{t^{\phi}}{\beta}\right)^{-\alpha}, \qquad t>0\ .$$

and the hazard rate function is

$$h(t) = \frac{\alpha\phi\,t^{\phi-1}}{\beta+t^{\phi}}, \qquad t>0\ .$$

The step stress ALT is widely used in major research area, metal fatigue. Researchers in this area developed many cumulative exposure models. It also play an important role in electronic applications.

Statisticians have an important role in supporting expert in engineering to develop cumulative damage models. Nelson (1980)[4] was the first used maximum likelihood (ML) methods for estimating a model for life as a function of constant stress from step-stress test data. He obtained the MLE of the parameters of Weibull distribution under the inverse power law model depending on data of cable insulation.

Miller and Nelson (1983)[5] presented the optimum plans for simple (two stresses) step-stress for accelerated life testing for the case where all units are observed until fail. Such plans minimize the asymptotic variance of the maximum likelihood estimator of the mean at a design stress and the test units have exponential distribution.

Bai et. al (1989)[6] discussed the optimum simple time-step and failure-step stress accelerated life tests for the case where a pre-specified censoring time is involved as an extension of the results of Miller and Nelson (1983)[7]. They obtained the stress change time and the number of items failed at low stress which minimize the asymptotic variance of MLE of the log mean life at normal condition.

The optimum simple step-stress accelerated life tests for products with competing causes of failure was presented by Bai and Chun (1991)[8].They assumed that the life distribution of each cause is exponential.

LuValle and Hines (1992)[9] used a case study to show a procedure for designing and graphically analysing step stress experiments in order to gain information about Kinetics of the processes governing failure.

Bai et. al (1993)[10] presented an optimum simple step-stress accelerated life test for the weibull distribution under type-I censoring. It is assumed that a log-linear relationship exists between the weibuull scale parameter and the stress and that a certain cumulative exposure model for the effects of changing stress holds.

Lu Valle (1993)[11] studied the behavior of a large class of physical processes that specify how multiple steps interact in producing failure.

This paper considers simple failure-step-stress ALT which uses two levels of stresses higher than the level of normal stress. The aim of such experiment is to have more failure data in a limited time without using a high stress to all test units.

2-The cumulative Exposure (CE) Model:

In a failure-step stress, units are run at a specified low stress, which is still larger than normal stress. If they do not fail at occurrence of a pre-specified number of failures, stress is repeatedly increased and held, until the units fail. As with the constant-stress test, one estimates parameters under step stress. The parameter estimates are used to estimate life at a constant design stress.

So to analyze data from step-stress, one needs a model that relates the distribution (or cumulative exposure) under step-stress to the distribution (or exposure) under constant stress. This model assumes that the remaining life of a unit depends only on the current cumulative fraction failed and current stress. The unit does not remember how the exposure was accumulated. Moreover, if held at the current stress, survivors will continue failing according to the cumulative distribution function of that stress but starting at the age corresponding to the previous fraction failed. This model is called the cumulative exposure (CE) model [12].

In the experiment, the number of steps equal *2*. The model of constant stress is considered in the first step. Such a model affects the lifetime of the unit by a certain level of stress c_1, where c_1 is larger than the usual stress c_u. For the second step, other stress is considered as c_2 where $c_u < c_1 < c_2$. Then the cumulative exposure model reflects the effect of moving from the first stress to the second one on the cumulative exposure distribution of the failure time.

In the experiment of step stress testing, the following assumptions are taken:

1. For any stress c_1, c_2 the life time distribution is Generalized Burr (α, β, ϕ) in the form:

$$f\left(t_{ij} \middle| \alpha, \beta, \phi_j\right) = \frac{\alpha\, \beta^{\alpha}\, \phi_j\, t_{ij}^{\phi-1}}{\left(\beta + t_{ij}^{\phi_j}\right)^{\alpha+1}}$$

$$t_{ij} > 0, \alpha, \beta, \phi_j > 0, i = 1, \ldots\ldots, n_j\ , j = 1,.2. \tag{2.1}$$

2. β,α are constant with respect to the stress c, and the scale parameter ϕ_j is affected by the stress $c_j, j=1,2.$ through the inverse power law model in the form

$$\phi_j = v s_j^p . \tag{2.2}$$

Where v is the constant of proportionality, p is the power of applied stress, are the parameters of this model, and

$$s_j = \frac{c^*}{c_j} \quad , c^* = \prod_{j=1}^{k} c_j^{b_j} \quad , b_j = \frac{n_j}{\sum_{j=1}^{k} n_j} \quad ,v>0 \ ,p>0 \ .$$

Suppose that, for a particular pattern of stress, units run at stress c_j starting at time $\tau_{n_{j-1}}$, and reaching to time $\tau_{n_j}, j=1,2 \ (\tau_0 = 0)$. The behavior of such units is as follows:

In Step 1:

The population fraction $F_1(t)$ of units failing by time t under constant stress c_1 is

$$F_1(t) = 1 - \left(1 + \frac{t^{v s_1^p}}{\beta}\right)^{-\alpha} \qquad 0 < t < \tau_{n_1}, \alpha, \beta, v, p > 0 \ . \tag{2.3}$$

If we let $H(t)$ be the population cumulative fraction of units failing under step stress. Then in the first step:

$$H(t) = F_1(t) \ . \qquad 0 < t < \tau_{n_1} \ . \tag{2.4}$$

Where τ_{n_1} is the time when the stress is raised from c_1 to c_2 where number of failures is n_1. Therefore

$$H(t) = 1 - \left(1 + \frac{t^{v s_1^p}}{\beta}\right)^{-\alpha} \tag{2.5}$$

<u>In step 2:</u>

For the second step, the cumulative exposure model is as follows: When step 2 starts, units have equivalent age u_1, which have produced the same fraction failed seen at the end of step 1. In other meaning the survivors at time τ_{n_1} will be switched to the stress c_2 beginning at the point u_1, which can be determined as the solution of

$$F_2(u_1) = F_1(\tau_{n_1}) \tag{2.6}$$

i.e. $1-(1+\psi_1)^{-\alpha} = 1-(1+\omega_1)^{-\alpha}$

where

$$\psi_1 = \frac{u_1^{vs_2^p}}{\beta}. \qquad \text{and} \qquad \omega_1 = \frac{\tau_{n_1}^{vs_1^p}}{\beta}.$$

$$\therefore 1+\psi_1 = 1+\omega_1 \qquad \therefore \psi_1 = \omega_1$$

$$\therefore u_1^{vs_2^p} = \tau_{n_1}^{vs_1^p}$$

$$\therefore u_1 = \tau_{n_1}^{(s_1/s_2)^p} \tag{2.7}$$

where $s_2 = \dfrac{c^*}{c_2}$ and $c^* = \prod_{j=1}^{k} c_j^{b_j}$, $b_j = n_j \Big/ \sum_{j=1}^{k} n_j$ $,v,p>0.$

The population cumulative fraction of units failing in step 2 by time t is expressed as follows:

$$H(t) = F_2\left[(t-\tau_{n_1})+u_1\right] \tag{2.8}$$

$$= 1-\left[1+\frac{\left[(t-\tau_{n_1})+u_1\right]^{vs_2^p}}{\beta}\right]^{-\alpha} \tag{2.9}$$

by substituting from (2.7) we have

$$H(t)=1-\left[1+\frac{\left[\left(t-\tau_{n_1}\right)+\tau_{n_1}^{(s_1/s_2)^p}\right]^{vs_2^p}}{\beta}\right]^{-\alpha} ,\tau_{n_1}\leq t\leq\tau_{n_2} \qquad (2.10)$$

It is seen that H(t) for a step-stress pattern consists of segments of the cumulative distributions F_1, F_2 . Then H(t) can be written in the form:

$$H(t)=\begin{cases}0 & t\leq\tau_0 \\ F_1(t); & \tau_0\leq t\leq\tau_{n_1} \\ F_2\left(\left(t-\tau_{n_1}\right)+u_1\right); & \tau_{n_1}\leq t\leq\tau_{n_2}\end{cases} \qquad (2.11)$$

and the associated density function h(t) is shown as the following form:

$$h(t)=\begin{cases}0 & t\leq\tau_0 \\ f_1(t); & \tau_0\leq t\leq\tau_{n_1} \\ f_2\left(\left(t-\tau_{n_1}\right)+u_1\right); & \tau_{n_1}\leq t\leq\tau_{n_2}\end{cases} \qquad (2.12)$$

3- Maximum Likelihood Estimation:

Grimshaw (1993) [13] indicated that the ML method is commonly used for most theoretical models and censored data. Although the exact sampling distribution of maximum likelihood estimators (MLE) is sometimes unknown, MLE have the desirable properties of being consistent and asymptotically normal for large samples. Also, it is shown by Bugaighis (1988)(14) that the ML procedure generally yields efficient estimators. However, these estimators do not always exist in closed form, so, numerical methods are used to compute them.

The experiment under failure-step stress has the following assumptions:

[1] There are 2 levels of stress c_1, c_2, where $c_1 < c_2$ are applied, such that each unit is initially put under stress c_1.

[2] We assumed that we begin the experiment with N units. It is considered that at the first step, when stress c_1 is applied, n_1 failure times $t_{i1}, i = 1,2,......n_1$ of test units are observed. At the second step, stress c_2 is applied and n_2 failure times $t_{i2}, i = 1,2,......n_2$ are noticed.

[3] The experiment begins at stress level c_1 . If the unit doesn't fail till the occurrence of predetermined n_1 failures, the stress is raised to c_2 and held until the occurrence of n_2 failures. In general , if the unit doesn't fail during the interval $[\tau_{n\,j-2}, \tau_{n\,j-1}]$ (until the occurrence of n_{j-1} failures) at stress c_{j-1} , then the stress is raised to c_j at $\tau_{n\,j-1}$, $j = 2$, and held until τ_{n_j} (n_j failures)

[4] the test is continued till all units, N fail or till a pre specified number of failures $= \sum_{j=1}^{k} n_j$. At this time, there are n_c units still survived, where $n_c = N - \sum_{j=1}^{k} n_j$, it is known that N is the total number of units run on the experiment. At the second step, the data would be the failure times of $(N - n_c)$ failed units arranged in order, and units which survived beyond $\tau_{n_k}(\eta)$.

[5] The failure time distribution is assumed to be generalized Burr distribution in the form (2.1) and the scale parameter is shown as

a function of the stress through the inverse power law model. We pay attention to the case of censored samples. The likelihood function of the experiment is assumed to have the following form:

$$L(t_{ij}, i=1,.....n_j, j=1,......k) \propto [\prod_{i=1}^{n_1} f_1(t_{i1})]$$

$$[\prod_{j=2}^{k}\prod_{i=1}^{n_j} f_j(t_{ij}-\tau_{n-1}+u_{j-1})]\ [1-F_k(\eta-\tau_{k-1}+u_{k-1})]^{n_c} \qquad (3.1)$$

It is the general form of the likelihood function in time step-stress accelerated life testing with censoring. It is shown from equation (3.1) that this likelihood function consists of three parts. The first one represents likelihood of the first step which is the same as the case of constant stress. The second part shows the likelihood function of the $(k-1)$ other stresses. The third part shows the likelihood function of the survived units by time τ_{n_k} (η).

Depending on the previous assumptions and considering the cumulative exposure model to relate cdf under step stress to the cdf under constant stress it is evident that:

$$f_j(t_{ij}-\tau_{j-1}+u_{j-1})=\frac{v s_j^p \alpha}{\beta}\left((t_{ij}-\tau_{j-1})+u_{j-1}\right)^{\left(v s_j^p-1\right)}$$

$$\left(1+\frac{\left[(t_{ij}-\tau_{j-1})+u_{j-1}\right]^{vs_j^p}}{\beta}\right)^{-(\alpha+1)} \quad , j=2,3,......,k. \qquad (3.2)$$

Where $u_{j-1}=\left(\Delta_{j-2}+u_{j-2}\right)\left(s_{j-1}/s_j\right)^p$

Then, the likelihood function can be expressed in the following form:

$$L \propto \prod_{i=1}^{n_1} \frac{v s_1^p \alpha}{\beta} t_{i1}^{\left(vs_1^p - 1\right)} \left(1 + \frac{t_{i1}^{vs_1^p}}{\beta}\right)^{-(\alpha+1)} *$$

$$\prod_{j=2}^{k} \prod_{i=1}^{n_j} \frac{v s_j^p \alpha}{\beta} \left(\left(t_{ij} - \tau_{j-1}\right) + u_{j-1}\right)^{\left(v s_j^p - 1\right)} \left(1 + \frac{\left[\left(t_{ij} - \tau_{j-1}\right) + u_{j-1}\right]^{vs_j^p}}{\beta}\right)^{-(\alpha+1)}$$

$$* \left(1 + \frac{\left[\left(\eta - \tau_{k-1}\right) + u_{k-1}\right]^{vs_k^p}}{\beta}\right)^{-\alpha(n_c)} . \qquad (3.3)$$

The likelihood function of the experiment in the case of failure-step stress is considered to have the same form as equation (3.3) but the stress change point τ_{j-1} is replaced by $\tau_{n_{j-1}}, j = 2,3,.........,k$ and censoring point η is replaced by τ_{n_k}. It is clear that in failure-step stress, n_{j-1}, $j = 2,3,.....,k+1$ are pre specified but τ_{j-1} , $j = 2,3,.....,k$ and η are random variables.

It is known that the ML estimators of v, p, α and β are obtained by maximizing the logarithm of the likelihood function expressed in the form:

$$ln\, L = ln\, A + (N - n_c) ln\, v + p \sum_{j=1}^{k} n_j\, ln\, s_j + (N - n_c) ln\, \alpha - (N - n_c) ln\, \beta$$

$$+ \left(vs_1^p - 1\right) \sum_{i=1}^{n_1} ln\, t_{i1} - (\alpha + 1) \sum_{i=1}^{n_1} ln \left(1 + \frac{t_{i1}^{vs_j^p}}{\beta}\right) + \sum_{j=2}^{k} \left(vs_j^p - 1\right) \sum_{i=1}^{n_j} ln\left(\left(t_{ij} - \tau_{j-1}\right) + u_{j-1}\right)$$

$$-(\alpha+1)\sum_{j=2}^{k}\sum_{i=1}^{n_j} ln\left(1+\frac{\left[\left(t_{ij}-\tau_{j-1}\right)+u_{j-1}\right]^{vs_j^p}}{\beta}\right)$$

$$-\alpha(n_c)ln\left(1+\frac{\left[\left(\eta-\tau_{k-1}\right)+u_{k-1}\right]^{vs_k^p}}{\beta}\right). \qquad (3.4)$$

Where A is constant.

4-The Maximum Likelihood Estimation In The Case of Type-II Censoring When K=2 As a special case:

As a special case, let $k=2$, $\tau_{n_1}=\tau_1$ and $\tau_{n_2}=\eta$ it is shown that:

$$F_2(u_1)=F_1(\tau_1), \qquad (4.1)$$

Then $u_1=\tau_1^{(s_1/s_2)^p}$ (4.2)

So the population cumulative fraction of specimens failing in step 2 by time t is given by:

$$H(t)=F_2[(t-\tau_1)+u_1]$$

$$=1-\left[1+\frac{\left[(t-\tau_1)+u_1\right]^{vs_2^p}}{\beta}\right]^{-\alpha}. \qquad (4.3)$$

When $k=2$ there are two steps only with two levels of stress c_1 and c_2. In this case the likelihood function has the following form:

$$L=B*\prod_{i=1}^{n_1}\frac{v\,s_1^p\,\alpha}{\beta}t_{i1}^{\left(vs_1^p-1\right)}\left(1+\frac{t_{i1}^{vs_1^p}}{\beta}\right)^{-(\alpha+1)}$$

$$\prod_{i=1}^{n2} \frac{v s_2^p \alpha}{\beta} \left((t_{i2} - \tau_1) + u_1\right)^{\left(v s_2^p - 1\right)} \left(1 + \frac{\left[(t_{i2} - \tau_1) + u_1\right]^{v s_j^p}}{\beta}\right)^{-(\alpha+1)}$$

$$\left(1 + \frac{\left[(\eta - \tau_1) + u_1\right]^{v s_2^p}}{\beta}\right)^{-\alpha(n_c)} . \qquad (4.4)$$

Where B is a constant.

By substituting from (4.2) we get

$$L = B * \prod_{i=1}^{n1} \frac{v s_1^p \alpha}{\beta} t_{i1}^{\left(v s_1^p - 1\right)} \left(1 + \frac{t_{i1}^{v s_1^p}}{\beta}\right)^{-(\alpha+1)}$$

$$* \prod_{i=1}^{n2} \frac{v s_2^p \alpha}{\beta} \left((t_{i2} - \tau_1) + \tau_1^{(s_1/s_2)^p}\right)^{\left(v s_2^p - 1\right)}$$

$$* \left(1 + \frac{\left[(t_{i2} - \tau_1) + \tau_1^{(s_1/s_2)^p}\right]^{v s_2^p}}{\beta}\right)^{-(\alpha+1)}$$

$$* \left(1 + \frac{\left[(\eta - \tau_1) + \tau_1^{(s_1/s_2)^p}\right]^{v s_2^p}}{\beta}\right)^{-\alpha(n_c)} . \qquad (4.5)$$

$$\ln L = a1 + a2 + a3 + a4 + a5 . \qquad (4.6)$$

Where

$$a1 = \ln B + n_1 \ln v + n_1 p \ln s_1 + n_1 \ln \alpha - n_1 \ln \beta + \left(v s_1^p - 1\right)\sum_{i=1}^{n_1} \ln t_{i1} .$$

$$a2 = -(\alpha+1)\sum_{i=1}^{n_1} \ln\left(1 + \frac{t_{i1}^{v s_j^p}}{\beta}\right) + n_2 \ln v + n_2 p \ln s_2 + n_2 \ln \alpha - n_2 \ln \beta .$$

$$a3 = +\left(v s_2^p - 1\right)\sum_{i=1}^{n_2} \ln\left(\left(t_{i2} - \tau_1\right) + \tau_1^{(s_1/s_2)^p}\right).$$

$$a4 = -(\alpha+1)\sum_{i=1}^{n_2} \ln\left(1 + \frac{\left[\left(t_{i2} - \tau_1\right) + \tau_1^{(s_1/s_2)^p}\right]^{v s_2^p}}{\beta}\right).$$

$$a5 = -\alpha\left(n_c\right)\ln\left(1 + \frac{\left[\left(\eta - \tau_1\right) + \tau_1^{(s_1/s_2)^p}\right]^{v s_2^p}}{\beta}\right).$$

$$\text{So, } \ln L = b1 + b2 + b3 + b4 + b5 . \qquad (4.7)$$

Where

$$b1 = \ln B + \left(N - n_c\right)\ln v + p\sum_{j=1}^{k} n_j \ln s_j + \left(N - n_c\right)\ln \alpha - \left(N - n_c\right)\ln \beta .$$

$$b2 = +\left(v s_1^p - 1\right)\sum_{i=1}^{n_1} \ln t_{i1} - (\alpha+1)\sum_{i=1}^{n_1} \ln\left(1 + \frac{t_{i1}^{v s_j^p}}{\beta}\right).$$

$$b3 = +\left(v s_2^p - 1\right)\sum_{i=1}^{n_2} \ln\left(\left(t_{i2} - \tau_1\right) + \tau_1^{(s_1/s_2)^p}\right).$$

$$b4=-(\alpha+1)\sum_{i=1}^{n2} ln\left(1+\frac{\left[\left(t_{i2}-\tau_1\right)+\tau_1^{(s_1/s_2)^p}\right]^{vs_2^p}}{\beta}\right).$$

$$b5=-\alpha(n_c)ln\left(1+\frac{\left[\left(\eta-\tau_1\right)+\tau_1^{(s_1/s_2)^p}\right]^{vs_2^p}}{\beta}\right).$$

Differentiating the logarithm likelihood function in (4.7) with respect to v, p, α and β, we can obtain the MLE's depending on the following equations:

$$\frac{\partial \ln L}{\partial v}=\frac{N-n_c}{v}+s_1^p\sum_{i=1}^{n1} ln\, t_{i1}-\frac{(\alpha+1)}{\beta}\sum_{i=1}^{n1}\frac{s_1^p t_{i1}^{vs_1^p}\ ln\, t_{i1}}{\left(1+\frac{t_{i1}^{vs_1^p}}{\beta}\right)}$$

$$+s_2^p\sum_{i=2}^{n2} ln\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)$$

$$-\frac{(\alpha+1)}{\beta}s_2^p\sum_{i=1}^{n2}\frac{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p} ln\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)}{\left(1+\frac{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)}$$

$$-\frac{\alpha(n_c)s_2^p}{\beta}\frac{\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p} ln\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)}{\left(1+\frac{\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)}. \tag{4.8}$$

$$\frac{\partial ln L}{\partial p}=vs_1^p\, ln s_1\sum_{i=1}^{n_1} ln t_{i1}-\frac{v(\alpha+1)s_1^p\, ln s_1}{\beta}\sum_{i=1}^{n_1}\frac{t_{i1}^{vs_1^p}\, ln t_{i1}}{\left(1+\frac{t_{i1}^{vs_1^p}}{\beta}\right)}$$

$$+vs_2^p\, ln s_2\sum_{i=1}^{n_2} ln\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)$$

$$+\left(\frac{s_1}{s_2}\right)^p\left(vs_2^p-1\right)\tau_1^{(s_1/s_2)^p}\, ln\left(\frac{s_1}{s_2}\right) ln\tau_1\sum_{i=1}^{n_2}\frac{1}{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)}$$

$$-\frac{(\alpha+1)}{\beta}\sum_{i=1}^{n_2}\left[\frac{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\left(1+\frac{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)}\left(\frac{\left(\frac{s_1}{s_2}\right)^p vs_2^p\tau_1^{(s_1/s_2)^p}\, ln\left(\frac{s_1}{s_2}\right) ln\tau_1}{t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}}\right.\right.$$

$$+ vs_2^p \ln s_2 \sum_{i=1}^{n2} \ln\left(t_{i2} - \tau_1 + \tau_1^{(s_1/s_2)^p} \right)\Bigg)\Bigg]$$

$$-\frac{n_c \alpha}{\beta}\left[\frac{\left(\eta - \tau_1 + \tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\left(1 + \frac{\left(\eta - \tau_1 + \tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)} \left(\frac{\left(\frac{s_1}{s_2}\right)^p vs_2^p \tau_1^{(s_1/s_2)^p} \ln\left(\frac{s_1}{s_2}\right) \ln \tau_1}{\eta - \tau_1 + \tau_1^{(s_1/s_2)^p}} \right.\right.$$

$$\left.\left. + vs_2^p \ln s_2 \sum_{i=1}^{n2} \ln\left(\eta - \tau_1 + \tau_1^{(s_1/s_2)^p} \right)\right)\right]. \tag{4.9}$$

$$\frac{\partial \ln L}{\partial \alpha} = \frac{(N - n_c)}{\alpha} - \sum_{i=1}^{n1} \ln\left(1 + \frac{t_{i1}^{vs_1^p}}{\beta}\right) - \sum_{i=1}^{n2} \ln\left(1 + \frac{\left(t_{i2} - \tau_1 + \tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)$$

$$n_c \ln\left(1 + \frac{\left(\eta - \tau_1 + \tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right). \tag{4.10}$$

$$\frac{\partial \ln L}{\partial \beta}=\frac{(\alpha+1)}{\beta^2}\sum_{i=1}^{n_1}\frac{t_{i1}^{vs_1^p}}{\left(1+\frac{t_{i1}^{vs_1^p}}{\beta}\right)}+\frac{(\alpha+1)}{\beta^2}\sum_{i=1}^{n_2}\frac{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\left(1+\frac{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)}$$

$$-\frac{N-n_c}{\beta}+\frac{n_c\alpha}{\beta^2}\frac{\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\left(1+\frac{\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)}. \qquad (4.11)$$

Therefore the MLE may be found by setting (4.8), (4.9), (4.10) and (4.11) equal to zero. As shown they are nonlinear equations, their solutions are numerically obtained by using Newton-Raphson method as will be seen later. They are solved numerically to obtain v, p, β, α.

The asymptotic variance-covariance matrix of the estimators of v, p, β, α is obtained depending on the inverse of Fisher information matrix, where its elements are the negative of the second derivatives of the natural logarithm of likelihood function defined in equation (4.7).

The elements of the matrix are given as follows:

$$\frac{\partial^2 \ln L}{\partial v^2} = -\frac{N-n_c}{v^2} + \frac{s_1^{2p}(\alpha+1)}{\beta}\sum_{i=1}^{n_1}\frac{t_{i1}^{vs_1^p}(\ln t_{i1})^2}{\left(1+\frac{t_{i1}^{vs_1^p}}{\beta}\right)}\left(\frac{t_{i1}^{vs_1^p}}{\left(1+\frac{t_{i1}^{vs_1^p}}{\beta}\right)\beta}-1\right)$$

$$+\frac{s_2^{2p}(\alpha+1)}{\beta}\sum_{i=1}^{n_2}\frac{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}\left(\ln\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)\right)^2}{\left(1+\frac{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)}$$

$$\left(\frac{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta\left(1+\frac{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)}-1\right)$$

$$+\frac{s_2^{2p} n_c \alpha}{\beta} \frac{\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}\left(ln\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)\right)^2}{\left(1+\frac{\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)}$$

$$\left(\frac{\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta\left(1+\frac{\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)}-1\right). \qquad (4.12)$$

$$\frac{\partial^2 \ln L}{\partial \alpha^2}=-\frac{N-n_c}{\alpha^2}. \qquad (4.13)$$

$$\frac{\partial^2 \ln L}{\partial p^2}=vs_1^p\left(\ln s_1\right)^2\sum_{i=1}^{n_1}\ln t_{i1}-\frac{(\alpha+1)vs_1^p\left(\ln s_1\right)^2}{\beta}\sum_{i=1}^{n_2}\frac{t_{i1}^{vs_1^p}\ \ln t_{i1}}{\left(1+\frac{t_{i1}^{vs_1^p}}{\beta}\right)}$$

$$\left[1-\frac{vs_1^p t_{i1}^{vs_1^p}\ ln t_{i1}}{\beta\left(1+\frac{t_{i1}^{vs_1^p}}{\beta}\right)}+vs_1^p\ ln t_{i1}\right]+\sum_{i=1}^{n2}\frac{2\left(\frac{s_1}{s_2}\right)^p vs_2^p \tau_1^{\left(\frac{s_1}{s_2}\right)^p}\ ln\frac{s_1}{s_2} ln\tau_1\ ln s_2}{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)}$$

$$+\sum_{i=1}^{n2}\frac{\left(\frac{s_1}{s_2}\right)^p\left(vs_2^p-1\right)\tau_1^{\left(\frac{s_1}{s_2}\right)^p}\left(ln\frac{s_1}{s_2}\right)^2 ln\tau_1}{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)}\left(1-\frac{\left(\frac{s_1}{s_2}\right)^p\tau_1^{\left(\frac{s_1}{s_2}\right)^p}\ ln\tau_1}{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)}-\left(\frac{s_1}{s_2}\right)^p ln\tau_1\right)$$

$$\sum_{i=1}^{n2}s_2^p\ v(ln s_2)^2\ ln\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)+\sum_{i=1}^{n2}\frac{(1+\alpha)\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta\left(1+\frac{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)}$$

$$\left[\frac{\left(\frac{s_1}{s_2}\right)^p vs_2^p\tau_1^{\left(\frac{s_1}{s_2}\right)^p}\ ln\frac{s_1}{s_2} ln\tau_1}{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)}+vs_2^p\ ln s_2\ ln\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)\right]^2$$

$$\left[\frac{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta\left(1+\frac{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)}-1\right]-\sum_{i=1}^{n_2}\frac{(1+\alpha)\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta\left(1+\frac{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)}$$

$$\left[\frac{\left(\frac{s_1}{s_2}\right)^p vs_2^p \tau_1^{\left(\frac{s_1}{s_2}\right)^p} ln\frac{s_1}{s_2} ln\tau_1}{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)}\right]\left[ln\frac{s_1}{s_2}+2lns_2+\left(\frac{s_1}{s_2}\right)^p ln\frac{s_1}{s_2} ln\tau_1+\right.$$

$$\left.(lns_2)^2\, ln\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)-\frac{\left(\frac{s_1}{s_2}\right)^p \tau_1^{\left(\frac{s_1}{s_2}\right)^p} ln\frac{s_1}{s_2} ln\tau_1}{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)}\right]$$

$$+\frac{n_c\alpha\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta\left(1+\frac{\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)}$$

$$\left[\frac{\left(\frac{s_1}{s_2}\right)^p vs_2^p\tau_1^{\left(\frac{s_1}{s_2}\right)^p} ln\frac{s_1}{s_2} ln\tau_1}{\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)}+vs_2^p\, ln s_2\, ln\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)\right]^2.$$

$$\left[\frac{\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta\left(1+\frac{\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)}-1\right]-\frac{n_c\alpha\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta\left(1+\frac{\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)}.$$

$$\left[\frac{\left(\frac{s_1}{s_2}\right)^p v s_2^p \tau_1^{\left(\frac{s_1}{s_2}\right)^p} ln\frac{s_1}{s_2} ln\tau_1}{\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)}\right]\left[ln\frac{s_1}{s_2}+2\,ln s_2+\left(\frac{s_1}{s_2}\right)^p ln\frac{s_1}{s_2} ln\tau_1+\right.$$

$$\left.(ln s_2)^2\, ln\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)-\frac{\left(\frac{s_1}{s_2}\right)^p \tau_1^{\left(\frac{s_1}{s_2}\right)^p} ln\frac{s_1}{s_2} ln\tau_1}{\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)}\right]. \qquad (4.14)$$

$$\frac{\partial^2 \ln L}{\partial\beta^2}=\frac{N-n_c}{\beta^2}+\frac{(\alpha+1)}{\beta^3}\sum_{i=1}^{n_1}\frac{t_{i1}^{vs_1^p}}{\left(1+\frac{t_{i1}^{vs_1^p}}{\beta}\right)}\left(\frac{t_{i1}^{vs_1^p}}{\beta\left(1+\frac{t_{i1}^{vs_1^p}}{\beta}\right)}-2\right)$$

$$+\frac{(\alpha+1)}{\beta^3}\sum_{i=1}^{n_2}\frac{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\left(1+\frac{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)}\left(\frac{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta\left(1+\frac{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)}-2\right)$$

$$+\frac{n_c}{\beta^3}\frac{\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\left(1+\frac{\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)}\left(\frac{\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta\left(1+\frac{\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)}-2\right)$$

(4.15)

$$\frac{\partial^2 \ln L}{\partial v \partial p} =$$

$$\sum_{i=1}^{n_1} s_1^p \ln s_1 \ln t_{i1} - \sum_{i=1}^{n_1} \frac{s_1^p t_{i1}^{vs_1^p} \ln t_{i1} \ln s_1 (\alpha+1)}{\left(1+\frac{t_{i1}^{vs_1^p}}{\beta}\right)\beta} \left[1 - \frac{vs_1^p t_{i1}^{vs_1^p} \ln t_{i1}}{\left(1+\frac{t_{i1}^{vs_1^p}}{\beta}\right)\beta} + vs_1^p \ln t_{i1}\right]$$

$$+ \sum_{i=1}^{n_2} \frac{\left(\frac{s_1}{s_2}\right)^p s_2^p \tau_1^{\left(\frac{s_1}{s_2}\right)^p} \ln \frac{s_1}{s_2} \ln \tau_1}{\left(\eta - \tau_1 + \tau_1^{(s_1/s_2)^p}\right)} - \frac{\left(\frac{s_1}{s_2}\right)^p s_2^p \tau_1^{\left(\frac{s_1}{s_2}\right)^p} \ln \frac{s_1}{s_2} \ln \tau_1}{\beta}$$

$$\left[(\alpha+1)\sum_{i=1}^{n_2} \frac{\left(t_{i2} - \tau_1 + \tau_1^{(s_1/s_2)^p}\right)^{vs_2^p - 1}}{\left(1 + \frac{\left(t_{i2} - \tau_1 + \tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)} + n_c \alpha \frac{\left(\eta - \tau_1 + \tau_1^{(s_1/s_2)^p}\right)^{vs_2^p - 1}}{\left(1 + \frac{\left(\eta - \tau_1 + \tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)}\right]$$

$$+\sum_{i=1}^{n_2} s_2^p \ln s_2 \ln\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)\left[1-\frac{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}(\alpha+1)}{\left(1+\frac{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)\beta}\right]$$

$$+\sum_{i=1}^{n_2}\frac{(1+\alpha)s_2^p\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}\ln\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)}{\beta\left(1+\frac{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)}$$

$$\left[\frac{\left(\frac{s_1}{s_2}\right)^p vs_2^p \tau_1^{\left(\frac{s_1}{s_2}\right)^p}\ln\frac{s_1}{s_2}\ln\tau_1}{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)}+vs_2^p\ln s_2\ln\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)\right]$$

$$\left[\frac{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta\left(1+\frac{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)}-1\right]+$$

$$\frac{n_c\alpha\, s_2^p\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}\ ln\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)}{\beta\left(1+\frac{\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)}$$

$$\left[\frac{\left(\frac{s_1}{s_2}\right)^p vs_2^p\tau_1^{\left(\frac{s_1}{s_2}\right)^p}\ ln\frac{s_1}{s_2}ln\tau_1}{\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)}+vs_2^p\ ln s_2\ ln\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)\right].$$

$$\left[\frac{\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta\left(1+\frac{\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)}-1\right]$$

$$-\frac{n_c\alpha\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p} s_2^p \ln s_2 \ln\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)}{\left(1+\frac{\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)\beta}. \tag{4.16}$$

$$\frac{\partial^2 \ln L}{\partial v \partial \alpha}=-s_2^p\sum_{i=1}^{n_2}\frac{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p} \ln\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)}{\left(1+\frac{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)\beta}$$

$$-\sum_{i=1}^{n_1}\frac{s_1^p t_{i1}^{vs_1^p}\ ln\,t_{i1}}{\left(1+\frac{t_{i1}^{vs_1^p}}{\beta}\right)\beta}-\frac{(n_c)s_2^p}{\beta}\frac{\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}\ ln\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)}{\left(1+\frac{\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)}$$

(4.17)

$$\frac{\partial^2 \ln L}{\partial v\partial\beta}=-\frac{s_1^p(\alpha+1)}{\beta^2}\sum_{i=1}^{n_1}\frac{t_{i1}^{vs_1^p}\ ln\,t_{i1}}{\left(1+\frac{t_{i1}^{vs_1^p}}{\beta}\right)}\left(\frac{t_{i1}^{vs_1^p}}{\left(1+\frac{t_{i1}^{vs_1^p}}{\beta}\right)\beta}-1\right)$$

$$-\frac{s_2^p(\alpha+1)}{\beta^2}\sum_{i=1}^{n_2}\frac{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}\left(ln\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)\right)^2}{\left(1+\frac{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)}.$$

$$\left(\frac{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta\left(1+\frac{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)}-1\right)-\frac{s_2^p n_c\alpha}{\beta^2}\left[\left(\frac{\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta\left(1+\frac{\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)}-1\right)\right.$$

$$\left.\left(\frac{\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}\left(ln\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)\right)^2}{\left(1+\frac{\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)}\right)\right]. \qquad (4.18)$$

$$\frac{\partial^2 \ln L}{\partial\alpha\partial p}=-\frac{vs_1^p \ln s_1}{\beta}\sum_{i=1}^{n_1}\frac{t_{i1}^{vs_1^p}\ \ln t_{i1}}{\left(1+\frac{t_{i1}^{vs_1^p}}{\beta}\right)}-\frac{1}{\beta}\sum_{i=1}^{n_2}\frac{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\left(1+\frac{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)}$$

$$\left[\frac{\left(\frac{s_1}{s_2}\right)^p \tau_1^{\left(\frac{s_1}{s_2}\right)^p}\ vs_2^p \ln\tau_1 \ln\frac{s_1}{s_2}}{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)}+vs_2^p \ln s_2 \ln\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}\right]$$

$$-\frac{n_c}{\beta}\frac{\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\left(1+\frac{\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)}\left[\frac{\left(\frac{s_1}{s_2}\right)^p \tau_1^{\left(\frac{s_1}{s_2}\right)^p}\ vs_2^p \ln\tau_1 \ln\frac{s_1}{s_2}}{\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)}\right.$$

$$\left.+vs_2^p \ln s_2 \ln\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}\right]. \qquad (4.19)$$

$$\frac{\partial^2 \ln L}{\partial\beta\partial p} = -\frac{vs_1^p(\alpha+1)\ln s_1}{\beta^2}\sum_{i=1}^{n_1}\frac{t_{i1}^{vs_1^p}\ln t_{i1}}{\left(1+\frac{t_{i1}^{vs_1^p}}{\beta}\right)}\left(\frac{t_{i1}^{vs_1^p}}{\beta\left(1+\frac{t_{i1}^{vs_1^p}}{\beta}\right)}-1\right)$$

$$-\frac{(\alpha+1)}{\beta^2}\sum_{i=1}^{n_2}\frac{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\left(1+\frac{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)}\left[\frac{\left(\frac{s_1}{s_2}\right)^p\tau_1^{\left(\frac{s_1}{s_2}\right)^p}vs_2^p\ln\tau_1\ln\frac{s_1}{s_2}}{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)}\right.$$

$$\left.+vs_2^p\ln s_2\ln\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}\right]\left(\frac{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta\left(1+\frac{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)}-1\right)$$

$$-\frac{n_c\alpha}{\beta^2}\frac{\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\left(1+\frac{\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)}\left[\frac{\left(\frac{s_1}{s_2}\right)^p\tau_1^{\left(\frac{s_1}{s_2}\right)^p} vs_2^p\ ln\tau_1\ ln\frac{s_1}{s_2}}{\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)}\right.$$

$$\left.+vs_2^p\ ln s_2\ ln\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}\right]\left(\frac{\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta\left(1+\frac{\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)}-1\right) \tag{4.20}$$

$$\frac{\partial^2 lnL}{\partial\alpha\partial\beta}=\sum_{i=1}^{n_1}\frac{t_{i1}^{vs_1^p}}{\beta^2\left(1+\frac{t_{i1}^{vs_1^p}}{\beta}\right)}+\frac{n_c\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta^2\left(1+\frac{\left(\eta-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)}$$

$$+\sum_{i=1}^{n_2}\frac{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta^2\left(1+\frac{\left(t_{i2}-\tau_1+\tau_1^{(s_1/s_2)^p}\right)^{vs_2^p}}{\beta}\right)} \quad . \tag{4.21}$$

Therefore, the MLE $\hat{v},\hat{p},\hat{\alpha}$ and $\hat{\beta}$ have an asymptotic variance-covariance matrix defined by inverting the information matrix I.

$$I=-\begin{bmatrix}\frac{\partial^2 \ln L}{\partial v^2} & \frac{\partial^2 \ln L}{\partial v\partial p} & \frac{\partial^2 \ln L}{\partial v\partial\alpha} & \frac{\partial^2 \ln L}{\partial v\partial\beta}\\ \frac{\partial^2 \ln L}{\partial p\partial v} & \frac{\partial^2 \ln L}{\partial p^2} & \frac{\partial^2 \ln L}{\partial p\partial\alpha} & \frac{\partial^2 \ln L}{\partial p\partial\beta}\\ \frac{\partial^2 \ln L}{\partial\alpha\partial v} & \frac{\partial^2 \ln L}{\partial\alpha\partial p} & \frac{\partial^2 \ln L}{\partial\alpha^2} & \frac{\partial^2 \ln L}{\partial\alpha\partial\beta}\\ \frac{\partial^2 \ln L}{\partial\beta\partial v} & \frac{\partial^2 \ln L}{\partial\beta\partial p} & \frac{\partial^2 \ln L}{\partial\beta\partial\alpha} & \frac{\partial^2 \ln L}{\partial\beta^2}\end{bmatrix}\downarrow\left(\hat{v},\hat{p},\hat{\alpha},\hat{\beta}\right) \tag{4.22}$$

Since the maximum likelihood estimates are consistent and asymptotically normally distributed, then, the confidence intervals of the estimators are as the following.

To define a confidence interval for a population value ω ; suppose

$\omega_*=\ \omega_*(y_1,\ldots\ldots\ldots\ldots,y_n)$ and $\omega_{**}=\ \omega_{**}(y_1,\ldots\ldots\ldots\ldots,y_n)$ are functions of the sample data $y_1,\ldots\ldots\ldots\ldots,y_n$ such that:

$$p_\omega(\omega_*\le\omega\le\omega_{**})=\gamma\,,$$

where the interval $[\omega_{*},\omega_{**}]$ is called a two sided $100\gamma\%$ confidence interval for ω , where ω_{*} and ω_{**} are the random lower and upper confidence limits that enclose ω with probability γ .

For large sample size, the maximum likelihood estimates under appropriate regularity conditions, are consistent and asymptotically normally distributed. Therefore, the two-sided approximate $100\gamma\%$ confidence limits for the maximum likelihood estimate $\hat{\omega}$ of a population value ω can be obtained by :

$$p\left[-z\le\frac{\hat{\omega}-\omega}{\sigma(\hat{\omega})}\le z\right]\cong\gamma, \tag{4.23}$$

where z is the $\left[\frac{100(1-\gamma)}{2}\right]^{th}$ standard normal percentile. Therefore, the two-sided approximate $100\gamma\%$ confidence limits for v,p,α,β will be respectively, as follows:

$$\left.\begin{array}{lll} L_v=\hat{v}-z\sigma(\hat{v}) & , & U_v=\hat{v}+z\sigma(\hat{v}) \\ L_p=\hat{p}-z\sigma(\hat{p}) & , & U_p=\hat{p}+z\sigma(\hat{p}) \\ L_\alpha=\hat{\alpha}-z\sigma(\hat{\alpha}) & , & U_\alpha=\hat{\alpha}+z\sigma(\hat{\alpha}) \\ L_\beta=\hat{\beta}-z\sigma(\hat{\beta}) & , & U_\beta=\hat{\beta}+z\sigma(\hat{\beta}) \end{array}\right\} \tag{4.24}$$

5- Simulation Studies :

To obtain the maximum likelihood estimates, for v,p,α and β put equations (4.8), (4.9), (4.10) and (4.11) equal to zero, it is shown they are nonlinear equations, their solutions are numerically obtained by using Newton-Raphson method depending on Mathematica 5.0. Different sized samples are generated from the generalized Burr lifetime distribution. The population is with parameters $v=0.7, p=0.8, \alpha=0.8$ and $\beta=15$; given $n_1=0.4N$, $n_2=0.5N$ and

$n_c = 0.1N$, $c_1 = 1.5$ and $c_2 = 2$. For each set of data, 500 samples are obtained randomly. The sets of data are of sample sizes $N = 100(100)500$.

Evaluating the performance of the parameters, v, p, α and β, has been considered through the measures of accuracy such as the mean relative absolute bias (MRA bias), the relative absolute bias ((RA bias), mean square error (MSE), the relative error (RE) and the variances - covariances matrix of the estimators.

Table (5.1) demonstrates the average times ($\bar{\tau}_{n1}$ and $\bar{\eta}_{n2}$) at which n_1 and n_2 units failed. Also, table (5.1) summarizes the performance of the parameters, v, p, α and β. It demonstrates the maximum likelihood estimates of these two populations respectively. Their MRA bias, RA bias, MSE and RE are obtained. As it is seen in the tables, the estimators are near to the true value of the parameters when N is increasing. Also, the MRA bias, RA bias, MSE and RE are decreasing when the sample size is increasing.

Table (5.2) shows the corresponding asymptotic variance-covariance matrices for these two populations respectively with their different sized samples. It is clear that the asymptotic variances of the estimators are decreasing when N is increasing.

Table (5.3) presents the estimated values of the scale parameter and the reliability function. In general it is obvious that the reliability decreases when the mission time (t_0) increases. Also the same table shows that the relative absolute bias RA Bias (the absolute difference between the predicted reliability function and its true value divided by its true value).

Depending on the equation (4.24) the confidence intervals estimation for the parameters are obtained. Table (5.4) presents the two-sided confidence limits at significant level 5% for each population, respectively with different sized samples of

$N = 100(100)500$. As shown from the results, the intervals of the parameters appear to be narrow as the sample size increases.

by setting the following equations equal to zero, π_j, $j = 1,2$ can be optimally determined by solving them simultaneously and applying the Newton-Raphson method:

$$\frac{\partial |I|}{\partial \pi_j}, j = 1,2. \tag{5.1}$$

Where I is as shown in (4.22),

Then, optimum test plans are developed numerically. Table (5.5) includes the expected number of items that must be allocated to each step of stress represented by n_1^* and n_2^* which minimize the generalized asymptotic variance (GAV) which defined as the reciprocal of the determinant of the Fisher information matrix I [15]. That is:

$$GAV(\hat{v}, \hat{p}, \hat{\alpha}, \hat{\beta}) = |I|^{-1}$$

Where minimization of the GAV is equivalent to maximization of the determinant of I.

It is clear that the optimum test plans do not allocate the same number of the test units to each step. Also, the same tables include the average of expected times ; $\bar{\tau}^*$ at which the stress changes from c_1 to c_2 and $\bar{\eta}^*$; at which each the experiment terminates. As indicated from the results, the optimal GAV of the MLE of the model parameters is decreasing as the sample size N is increasing.

Table(5.1): The Estimates, MRA Bias, RA Bias, MSE, RE of the parameters $\nu = 0.7, p = 0.8, \alpha = 0.8, \beta = 15$ given $n_1 = 0.4N$ and $n_2 = 0.5N$ for different sample size

N	$\overline{\tau}_{n1}$	$\overline{\eta}_{n2}$	Estimates	MRA Bias	RAB	MSE	RE
100	25.67	6642.6	$\hat{\nu}$=0.7355	0.157095	0.0507145	0.0246165	0.224138
			$\hat{p}$=0.860	0.320998	0.0752562	0.106891	0.408677
			$\hat{\alpha}$=1.192	0.750412	0.489564	1.74164	1.64964
			$\hat{\beta}$=23.30	0.752401	0.553742	480.435	1.46126
200	26.34	6845.4	$\hat{\nu}$=0.7187	0.107622	0.0266837	0.0097511	0.141068
			$\hat{p}$=0.811	0.205697	0.0136197	0.0441923	0.262775
			$\hat{\alpha}$=0.898	0.33865	0.123032	0.25103	0.626286
			$\hat{\beta}$=17.47	0.342159	0.164845	68.6803	0.55249
300	26.19	6423.7	$\hat{\nu}$=0.7033	0.0797966	0.0100401	0.0048527	0.099516
			$\hat{p}$=0.817	0.165408	0.020705	0.0277368	0. 20818
			$\hat{\alpha}$=0.888	0.254164	0.109448	0.095633	0.386557
			$\hat{\beta}$=16.94	0.261103	0.129577	33.9203	0.388274
400	26.19	6617	$\hat{\nu}$=0.7086	0.0746699	0.0122089	0.0043372	0.0940818
			$\hat{p}$=0.811	0.153395	0.0133033	0.024119	0.194129
			$\hat{\alpha}$=0.838	0.209695	0.0458555	0.0525382	0.286515
			$\hat{\beta}$=16.02	0.208297	0.0678079	17.0588	0.275348
500	26.06	6409	$\hat{\nu}$=0.7054	0.0642988	0.0076941	0.0030981	0.0795148
			$\hat{p}$=0.816	0.131288	0.0196851	0.017507	0.165392
			$\hat{\alpha}$=0.829	0.17695	0.411469	0.0335491	0.228955
			$\hat{\beta}$=15.89	0.18654	0.0590975	13.4327	0.244337

Table(5.2): Asymptotic Variances and Covariances of estimates for different samples size of the parameters $\nu = 0.7, p = 0.8, \alpha = 0.8, \beta = 15$ given $n_1 = 0.4N$ and $n_2 = 0.5N$ using type-II censoring

N	Variance-Covariance Matrix			
	ν	p	α	β
100	0.00250891 0.00448581 -0.00140873 0.0943901	0.00448581 0.0632464 0.0057448 0.402278	-0.00140873 0.0057448 0.0129116 0.26522	0.0943901 0.402278 0.26522 20.5028
200	0.00219023 0.00232707 -0.0024237 0.0545601	0.00232707 0.0343436 0.0054295 0.287591	-0.0024237 0.0054295 0.0126348 0.225934	0.0545601 0.287591 0.225934 13.4504
300	0.00184494 0.00146541 -0.0022956 0.0327988	0.00146541 0.0238454 0.00532425 0.23673	-0.0022956 0.00532425 0.0124887 0.232029	0.0327988 0.23673 0.232029 11.2415
400	0.00158013 0.00105116 -0.00239865 0.023915	0.00105116 0.0183235 0.00425654 0.185147	-0.00239865 0.00425654 0.0108499 0.172894	0.023915 0.185147 0.172894 8.39498
500	0.00147617 0.00068079 -0.00262597 0.0127523	0.00068079 0.0148991 0.0041652 0.159172	-0.00262597 0.0041652 0.0105098 0.170596	0.0127523 0.159172 0.170596 7.22553

Table(5.3): The Estimated Scale Parameter and Reliability under use condition at different samples size when $v = 0.7, p = 0.8, \alpha = 0.8, \beta = 15$ given $n_1 = 0.4N$ and $n_2 = 0.5N$

N	$\hat{\phi}_u$	t_0	$\hat{R}_u(t_0)$	Relative Bias
100	2.17126	3.6	0.53417	0.154439
		3.8	0.503447	0.172824
		4	0.474162	0.191323
200	1.99395	3.6	0.60923	0.035623
		3.8	0.58397	0.040523
		4	0.55964	0.045545
300	1.96514	3.6	0.614313	0.0275765
		3.8	0.589596	0.0312792
		4	0.56577	0.0350863
400	1.96521	3.6	0.619057	0.0200681
		3.8	0.594813	0.022708
		4	0.57145	0.025399
500	1.96906	3.6	0.617434	0.0226367
		3.8	0.593142	0.0254531
		4	0.569742	0.0283111

Table(5.4):Confidence Bounds of the estimates at Confidence Level 95% when $v = 0.7,\ p = 0.8, \alpha = 0.8, \beta = 15$ given $n_1 = 0.4N$ and $n_2 = 0.5N$

N	Parameter	Estimates	Standard Devision	Lower Bound	Upper Bound
100	v	0.7355	0.050089	0.637325	0.833675
	p	0.860	0.251488	0.367288	1.35312
	α	1.192	0.113629	0.968938	1.41436
	β	23.30	4.528	14.4312	32.181
200	v	0.7187	0.0467999	0.626951	0.810406
	p	0.811	0.18532	0.447668	1.17412
	α	0.898	0.112404	0.678112	1.11874
	β	17.47	3.66748	10.2844	24.6609
300	v	0.7033	0.0429528	0.619017	0.787446
	p	0.817	0.15442	0.513901	1.11923
	α	0.888	0.111753	0.669523	1.10659
	β	16.94	3.35284	10.3721	23.5152
400	v	0.7086	0.0397509	0.630635	0.786458
	p	0.811	0.135364	0.545328	1.07596
	α	0.838	0.104163	0.632526	1.04084
	β	16.02	2.89741	10.3382	21.696
500	v	0.7054	0.0384209	0.630081	0.780691
	p	0.816	0.122062	0.576507	1.05499
	α	0.829	0.102517	0.631984	1.03385
	β	15.89	2.68803	10.6179	21.155

Table(5.5):The results of optimal design of the life test for different sized samples under type-II censoring in step-stress FALT given $n_1 = 0.4N$ and $n_2 = 0.5N$

N	$\bar{\tau}_{n1}$	$\bar{\eta}_{n2}$	π_1^*	π_2^*	n_1^*	n_2^*	$\bar{\tau}^*$	$\bar{\eta}^*$	GAV
100	25.67	6642.6	0.320921	0.520921	32	52	12.483	747.02	0.000008
200	26.34	6845.4	0.344726	0.544726	69	109	17.163	3046.3	0.000002
300	26.19	6423	0.349918	0.549918	105	165	18.738	4289.15	0.0000008
400	26.19	6617	0.350909	0.550909	140	220	18.506	4822.35	0.0000003
500	26.06	6409	0.352933	0.552933	176	276	18.666	5360.93	0.0000002

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تقدير المعالم والتصميم الأمثل فى اختبارات الحياة المعجلة المتدرجة لتوزيع بيير العام باستخدام العينات المراقبة من النوع الثانى

جميلـــة نصـــر

تم فى هذه الدراسة استخدام طريقة الإمكان الأعظم فى تقدير معالم التوزيع للاختبارات المعجلة المتدرجة بتعريض الوحدات للضغط على مرحلتين ؛ وذلك لسرعة إنهاء الاختبار والتجربة .

كما تم دراسة خصائص المقدرات وذلك بالحصول على مصفوفة فيشر للتباين والتغاير (مصفوفة المعلومات) ، وكذلك التصميم الأمثل للاختبار .