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*N*UMERICAL SCHEMES FOR BLACK-SCHOLES EQUATION WITH ERROR DYNAMICS

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Abstract. This paper focuses on the numerical solution of the Black-Scholes equation (BSE), which is used in finance to price options. The modified version of BSE to heat equation is subjected to two-time level finite difference method such as the Crank-Nicolson method and three-time level finite difference method such as the DuFort-Frankel method. The error dynamics is represented by the Global Spectral Analysis (GSA) method, which contradict the error dynamics of the von Neumann method, where the signal and error follow the same difference equation. For different maturities, volatilities and interest rates, both techniques are tested for accuracy. For the converted heat equation of BSE, the three-time level method is determined to be more accurate than the two-time level method. Finally, we conclude that risk can be reduced by short-term investment in a low interest, high-volatility market with a good approximation using the three-time level finite difference method for European call option for converted BSE to heat equation.

1. Introduction

Finance is one of the most rapidly growing and evolving sectors of the modern economy. Derivatives are financial products that allow the buyer or seller the right to acquire or sell an underlying asset at a later date. These contracts including as future, forward, swap and option are used in investing speculating and risk management. A financial contract that gives option holders the right to buy or sell an underlying asset from option writers by a certain date and price is known as an option. A call option gives the holder the right to buy a stock and a put option gives the holder the right to sell a stock. The Black-Scholes model, often known as the Black-Scholes-Merton (BSM) model, is one of the most well-known and generous models in current financial theory given by Black and Scholes [1] and Merton [2]. This model demonstrated the importance of mathematics in the financial world.

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Fischer Black and Myron Scholes first presented the Black-Scholes model in their key study "The Pricing of Options and Corporate Liabilities," published in the Journal of Political Economy in 1973. They also developed the Black-Scholes equation, a partial differential equation that forecasts the price of an option over time using current stock prices, expected dividends, the option's strike price, predicted interest rates, time to expiration and expected volatility in the same year. It is still regarded as one of the most effective methods for determining the price of an options contract. The term "Black-Scholes options pricing model" was coined by Robert C. Merton [2], who was the first to publish a paper describing the mathematical understanding and implementation of the option pricing model.
Under the Black–Scholes model, the renowned Black–Scholes equation is a partial differential equation that gives the price evolution of a European call or European put option.

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2(\frac{\partial^2 V}{\partial S^2})+rS\frac{\partial V}{\partial S}-rV=0 \tag{1}$$

According to Robert and Jarrow [5], the main idea behind the equation is that the risk can be eliminated by correctly hedging the option and removing risk (by making the dynamically hedged portfolio earn the risk-free interest rate). The option has exactly one price as a result of the hedge.
Following its early discovery, a large number of empirical studies, such as in 1979 Macbeth and Merville [4], in 2003 Hull [3], in 2006 Razali [7] and Rinalini [8], in 2019 Nwobe et al. [19] have examined the model's validity and discovered that it misprices options which will not beneficial to option traders in any way. That is to say the model's accuracy is still questionable and the main focus of the model is on predicting future volatility of the underlying asset (academic domain) and as a result, determining an appropriate option price.
The Black-Scholes equation is a second-order parabolic partial differential equation that simulates a time-dependent event. It is a variant of the well-known heat equation. The price variations of an option are linked to the flow of heat over a conducting medium. This problem's solution can be determined both analytically and numerically.
The existence of Black-Scholes model solutions has been extensively researched over the last few decades using a variety of methodologies. In general, some of these Black-Scholes PDEs do not have closed-form analytical solutions, hence one must use numerical techniques to resolve them. There is a large body of literature, including [18],[19],[20] and [21] that discusses the numerical solution of the original form of the Black–Scholes PDE in finance using diverse approaches. The best approximation comes from the original BSE solution, although another approach is presented here.
In this paper we shall be interested in numerical solution of converte
d Black-Scholes equation into heat equation using finite difference method. Finite difference methods are approaches for obtaining numerical solutions to partial differential equations. They are effective methods for generating approximate solutions to a wide range of partial differential equations that arise in a variety of scientific domains including finance. The basic concept of these methods is to use approximations created by Taylor expansions of functions around the point of interest to replace partial derivatives in equations as per Iserles and Arieh [16].
The solution of Black-Scholes equation is obtained analytically by converting it

into heat equation by Joyner and Charles D. [10]. Chenand and Chuan [11] applied Crank Nicolson method on converted heat equation. The heat equation, according to[11], sacrifices precision through the process of variable transformation. The work of Chenand and Chuan [11] is extended in this research for varied time periods of maturity, volatility and risk-free interest rate using two different time level finite difference methods for converted Heat equation. Error dynamics using GSA method is also discussed.
This paper is organised as follows. The Black-Scholes equation is described in detailed in Section 2. Section 3 contains the Black-Scholes equation transformed to a Heat equation with variable transformation. In sections 4 and 5, the transformed heat Equation is subjected to two - time level finite difference method (Crank-Nicolson Method) and three- time level finite difference method (DuFort- Frankel) respectively. Section 6 represents the error dynamics using GSA method. Section 7 deals with numerical results using Python. The results of numerical and analytical calculations are compared. The L_1 norm is used to compute relative error.

2. Black-Scholes Equation

The Black-Scholes model is a financial instrument pricing model that is used to assess the fair price of call and put stock options. The kind of option, volatility, underlying stock price, expiry time, strike price and risk-free rate are all factors in this model. The Black-Scholes model is based on the notion of hedging and removing risks associated with the volatility of underlying assets and stock options. It is based on the following fundamental assumptions:

- Asset prices follow the lognormal random walk $\frac{ds}{s} = \sigma dX + \mu dt$.
- The functions of risk-free interest rates and asset volatility over time are well-known.
- There are no transaction costs when it comes to trading and hedging a portfolio.
- During the option's lifetime, the underlying asset does not pay dividends.
- There are no possibilities for arbitrage. The return on all risk-free assets must be the same.
- Trading of the underlying asset can occur continuously.
- Short selling is permitted and the assets are divisible.

In this paper the attention is restricted to the European call option. According to Black and Fisher [1], the Black-Scholes equation with no dividend is:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 (\frac{\partial^2 C}{\partial S^2}) + rS\frac{\partial C}{\partial S} - rC = 0. \tag{2}$$

Where C(S,t) = value of an European call option, r = risk-free rate of interest, S = stock price of the underlying asset, E = exercise price, σ = volatility of the underlying asset, T = time of maturity, t = time in years.

According to [1], if the asset price S(t) falls to zero, the European call option will expire worthless at any time t. The initial condition is

$$C(0,t) = 0 \text{ for all } t. \tag{3}$$

The other condition is

$$C(S,t) \to S \text{ as } S \to \infty \text{ for all } t. \tag{4}$$

But for a large asset price S(t), the call option will be in the money at expiration and the payoff will be S(T) - K. The value at time t requires discounting back the term K and considering that the arbitrage-free price at time t for the underlying asset is simply S(t) . So the suitable boundary condition is

$$C(S_{max}, T) = S_{max} - Ke^{-r(T-t)} \text{ as } S \to \infty. \tag{5}$$

The terminal condition at expiration is

$$C(S, T) = \max\{S - K, 0\} \text{ for all } S. \tag{6}$$

3. Conversion of Black-Scholes Equation into Heat Equation

Black-Scholes equation is a linear backward parabolic partial differential equation. The linear diffusion equation has a number of interesting characteristics. Although we start with a discontinuity in the terminal data, due to a discontinuity in the pay out, these quickly gets smoothed out, because of the diffusive nature of the equation. Paul, Wilmott [12] derived that the equation has unique solution. The solution cannot grow too fast as S tends to infinity.
In this section, Black-Scholes equation is reduced to a simple heat equation by variable transformation. The converted heat equation is solved numerically and variables are back transformed into original financial variables.

Convert the Black-Scholes equation to heat equation by taking the variable transformation

$$S = Ee^x \quad , \quad t = T - \frac{2\tau}{\sigma^2} \quad , \quad C(S, t) = Ev(x, \tau). \tag{7}$$

$$\text{where } v \text{ is: } \quad v(x, \tau) = e^{-\frac{1}{2}(k-1)x - \frac{1}{4}(k+1)^2\tau}u(x, \tau) \quad , \quad k = \frac{2r}{\sigma^2}. \tag{8}$$

Converted form of heat equation from the Black-Scholes equation was given by Nwobi, F.N., Annorzie, M. N. and Amadi [19] is

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}. \tag{9}$$

According to D. V. Widder [9], the heat or diffusion equation is a partial differential equation that has been explored as a model of heat transport in a continuous medium for about two centuries. It is one of the most widely used and successful mathematical model, with a large amount of theory on its features and solution available .

According to Chen and Chuan [11], the converted initial and boundary conditions are as follows.

As the heat equation steps forward in time, final condition of Black-Scholes equation is converted to an initial condition

$$u(x, 0) = max(e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0). \tag{10}$$

First boundary condition at $x = 0$ is transformed into $u(0, \tau) = 0.$ (11)

and second boundary condition at $x \to \infty$ becomes

$$u(x,\tau) = e^{\frac{1}{4}(k+1)^2\tau}(e^{\frac{1}{2}(k+1)\tau} - e^{\frac{1}{2}(k-1)\tau}e^{-k\tau}). \quad (12)$$

4. Two Time Level Finite Difference Method(Crank-Nicolson Method)

Tuncer Cebeci [13] suggested that he Crank–Nicolson method is a finite difference method used for numerically solving the heat equation and similar partial differential equations. The method gives a relation between the function values at the two levels j+1 and j and is called a two time level formula. It is a second-order implicit method in time. G.D.Smith proved that [14] this method is useful generally when the initial data and its derivatives are continuous. Space index is denoted by i and time index is by j. The schematic representation of Crank–Nicolson method is given below.

$$(\frac{\partial u}{\partial \tau})_{i,j+\frac{1}{2}} = (\frac{\partial^2 u}{\partial x^2})_{i,j+\frac{1}{2}}, \quad (13)$$

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{1}{2}(\frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1} + u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}). \quad (14)$$

giving

$$-ru_{i-1,j+1} + (2+2r)u_{i,j+1} - ru_{i+1,j+1} = ru_{i-1,j} + (2-2r)u_{i,j} + ru_{i+1,j}, \quad (15)$$

where $r = \frac{k}{h^2}$, $k = \delta x =$ space step size and $h = \delta\tau =$ time step size

The matrix form with known boundry conditions is written as

$$\begin{bmatrix} 2+2r & -r & 0 & \dots & 0 & 0 \\ -r & 2+2r & -r & \dots & 0 & 0 \\ 0 & -r & 2+2r & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -r & 2+2r \end{bmatrix} \begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ u_{3,j+1} \\ \vdots \\ u_{N-1,j+1} \end{bmatrix} =$$
$$\begin{bmatrix} 2-2r & r & 0 & \dots & 0 & 0 \\ r & 2-2r & r & \dots & 0 & 0 \\ 0 & r & 2-2r & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & r & 2-2r \end{bmatrix} \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ u_{3,j} \\ \vdots \\ u_{N-1,j} \end{bmatrix} + \begin{bmatrix} ru_{0,j} + ru_{0,j+1} \\ 0 \\ \vdots \\ 0 \\ ru_{N,j} + ru_{N,j+1} \end{bmatrix} \quad (16)$$

According to [14], order of accuracy of this method is $0(k^2, h^2)$. This approach is unconditionally stable for all r > 0 [14].The unconditional stability remains the same after changing back into a financial variable.

5. Three Time Level Finite Difference Method(DuFort- Frankel Method)

Three time level schemes can be built to gain several advantages over two time level schemes, such as lower truncation error, more stability and is the better one to use when initial data is discontinuous or varies very rapidly with x , as per G.D.Smith [14].

Dufort-Frankel Method is an explicit three time level method. The method gives a relation between the function values at the three time levels j+1, j and j-1 and is called a three time level formula. The difference scheme for Dufort-Frankel method is

$$u_i^{j+1} = u_i^{j-1} + 2r(u_{i-1}^j - u_i^{j+1} - u_i^{j-1} + u_{i+1}^j), \tag{17}$$

which may be written as

$$(1+2r)u_i^{j+1} = (1-2r)u_i^{j-1} + 2r(u_{i-1}^j + u_{i+1}^j). \tag{18}$$

The matrix form is

$$(1+2r)\begin{bmatrix} u_1^{j+1} \\ u_2^{j+1} \\ u_3^{j+1} \\ \vdots \\ u_{N-1}^{j+1} \end{bmatrix} = (1-2r)\begin{bmatrix} u_1^{j-1} \\ u_2^{j-1} \\ u_3^{j-1} \\ \vdots \\ u_{N-1}^{j-1} \end{bmatrix} + 2r\begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 1 & 0 & 1 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}\begin{bmatrix} u_1^j \\ u_2^j \\ u_3^j \\ \vdots \\ u_{N-1}^j \end{bmatrix} + 2r\begin{bmatrix} u_0^j \\ 0 \\ 0 \\ \vdots \\ u_N^j \end{bmatrix} \tag{19}$$

It is necessary to calculate a solution along the first time level by some other method in order to solve the first set of equations for u_i^2, with the assumption that the initial data at t=0 are known. The accuracy of the first time level solution must match that of the three time level equations. In this paper Crank-Nicolson method is applied to calculate first set of equations. According to Salih, A [15] and Abraham Bouwer [17], order of accuracy of the scheme is $O(\Delta t^2, \Delta x^2, (\Delta t/\Delta x)^2)$.
Theorem 1 [Lax equivalence theorem:] It states that for a well-posed linear initial-boundary value problem, a consistent finite difference approach is convergent if and only if it is stable.

6. Global Spectral Analysis(GSA) Method for Error Dynamics

According to von Neumann method of stability analysis the Crank-Nicolson method is unconditionally stable for all choices of r [14]. In von Neumann method it is considered that the signal and the error follow the same difference equation. But according to T.K.Sengupta et.([22]),([23]) and ([26]) the error do not follow the same difference equation.

According to [27], the GSA defined in Fourier spectral plane for space, is different from von Neumann analysis to study error dynamics. Error is defined as, $e(x,t) = u(x,t) - u_N(x,t)$, with u_N the numerically computed solution expressed by the hybrid representation at the m^{th} node (x_m) of a uniform grid with spacing h by,

$$u_{mn}(x_m, t^n) = \int \hat{u}(k, t^n) e^{ikx_m} dk \tag{20}$$

where $x_m = mh$ and current time is indicated by t^n. In the integrand, $\hat{u}$ represents the bi-lateral Laplace transform and the integration range is determined by the Nyquist limit. We have defined the numerical amplification factor at the j^{th} node by,

$$G(kh, N_c) = \frac{\hat{u}(k, t + \Delta t)}{\hat{u}(k, t)} \tag{21}$$

to characterize numerical stability or instability for the combined space-time discretization methods, where N_c is the CFL number. The complex quentity is furthermore denoted as, $G_m = G_{rm} + iG_{im}$. The values of G_{rm} and G_{im} determine the numerical phase shift caused per time step β_m and is responsible for the numerical solution of the equation propogating from left to right with a numerical phase speed, which can deviate from the physical phase speed . The numerical phase shift is given by

$$tan\beta_m = \frac{-G_{im}}{G_{rm}} \tag{22}$$

,which need not be equal to the physical phase shift.

7. Spectral Analysis for Numerical Method:

Two Time Level Method: To analyse Eqn.(15), substitute Eqn.(20), and equating individual k components, one gets

$$-r\hat{u}(k,t{+}\Delta t)e^{-ikh}{+}2(1{+}r)\hat{u}(k,t{+}\Delta t){-}r\hat{u}(k,t{+}t)e^{ikh} = r\hat{u}(k,t)e^{-ikh}{+}2(1{-}r)\hat{u}(k,t){-}r\hat{u}(k,t)e^{ikh} \tag{23}$$

Divide above Eqn.(23) by $\hat{u}(k, t)$, we get

$$G(kh, Pe) = \frac{1 - 4Pe(1 - \lambda)sin^2(kh/2)}{1 + 4Pe\lambda sin^2(kh/2)} \tag{24}$$

Here $\lambda = 0.5$ and $N_c = Pe = r$.
$|G|$ - contours in (kh, λ)-plane are shown for various values of Pe in FIGURE - 1. For Pe = 0.01, one observe that for a very small range of kh, the values of $|G|$ with near-neutral behaviour. For Pe = 0.1, this method still indicate numerical stability. The Crank-Nicolson approach is stable for Pe = 1, but the higher kh values will be attenuated up to a reasonable range, above which the solution will remain stable ($|G(kh| < 1$) but be oscillatory. Such highly monotonic decaying components along with high wavenumber oscillatory attenuating components will give rise to large error.

7.1. Spectral Analysis for Numerical Method: Three Time Level Method:

Substituting Eqn.(20) in Eqn.(18) and equating individual k components, one gets

$$\hat{u}(k,t{+}\Delta t){-}\hat{u}(k,t) = Pe[\hat{u}(k,t)e^{ikh}{-}\hat{u}(k,t{+}\Delta t){-}\hat{u}(k,t{-}\Delta t){+}\hat{u}(k,t)e^{-ikh}] \tag{25}$$

$$G - \frac{1}{G} = 2Pe[e^{ikh} - G - \frac{1}{G} + e^{-ikh}] \tag{26}$$

$$G - \frac{1}{G} = 2Pe[2coskh - G - \frac{1}{G}] \tag{27}$$

Roots of this quadratic equation are given by

$$G_{1,2} = \frac{2Pe}{1 + 2Pe}coskh \pm \frac{\sqrt{1 - 4Pe^2sin^2kh}}{1 + 2Pe} \tag{28}$$

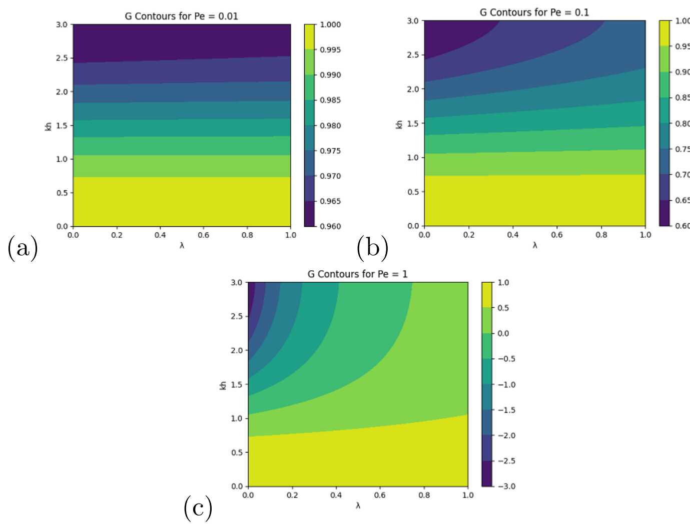


Figure 1. (a) The contour plots for $|G|$ in (λ, kh)-plane for the implicit method with Pe=0.01. (b) The contour plots for $|G|$ in (λ, kh)-plane for the implicit method with Pe=0.1. (c) The contour plots for $|G|$ in $(, kh)$-plane for the implicit method with Pe=1.

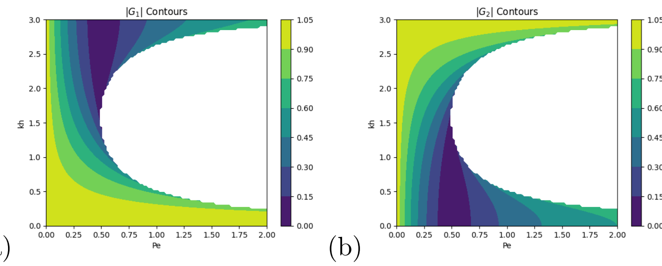
$G_{1,2}$ will be complex conjugate if

$$1 - 4Pe^2 sin^2 kh < 0 \tag{29}$$

Satisfaction of this condition implies that after each time step, phase is shifted for those k's which satisfies Eqn.(29). This is best viewed in the plots of contour- lines of $|G_1|$ and $|G_2|$. Thus, these k's components will numerically propogate like a wave as shown in FIGURE - 2. When condition of Eqn.(29) is satisified, one identifies a region in (Pe, kh)-plane where G_i becomes complex. As the governing equation has a first derivative, so one should get a single amplification factor. However, for Du Fort-Frankel method the presence of two amplification factors indicate one of these, G_1 will approach to physical mode, while the second one is the spurious numerical mode, G_2. These are shown in FIGURE - 2. The presence of this numerical mode will contribute to error. Hence, all three time level methods suffers from the presence of spurious numerical modes.(according to [24],[25]) There is also the second source of problem one notices from FIGURE(2) is the existence of regions in the parameter space for values of Pe greater than a critical value for which the parabolic partial differential equation becomes hyperbolic in nature. This then violates the Lax's Equivalence Theorem. This is the problem of inconsistency of numerical method.

The same change of behaviour of equation can be now shown by the Taylor series expansion of the terms about the $(m, n)^{th}$ node, whose amplification yields

$$\frac{\partial u}{\partial t} + \frac{(\Delta t)^2}{6}\frac{\partial^3 u}{\partial t^3} + ... = \frac{\partial^2 u}{\partial x^2} + (\frac{\Delta t}{h})^2 \frac{\partial^2 u}{\partial t^2} + \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4} - \frac{(\Delta t)^4}{12h^2}\frac{\partial^4 u}{\partial t^4} \tag{30}$$



(a) (b)

FIGURE 2. (a)The contour plots for $|G_1|$ in (Pe, kh)-plane for the Du-Fort-Frankel method.(b)The contour plots for $|G_2|$ in (Pe, kh)-plane for the Du-Fort-Frankel method.

Thus,the leading truncation error terms of the difference equation is

$$-\frac{(\Delta t)^2}{6}\frac{\partial^3 u}{\partial t^3}+\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}+(\frac{\Delta t}{h})^2\frac{\partial^2 u}{\partial t^2}-\frac{(\Delta t)^4}{12h^2}\frac{\partial^4 u}{\partial t^4} \tag{31}$$

a highly peculiar quality of Eqn.(31) emerges from parabolic to hyperbolic when the grid spacing and time step are progressively refined such that $\frac{\Delta t}{h}(=\beta)$ and $\frac{\Delta t^2}{h}(=\gamma)$, attain finite values as indicated in the paranthesis. In the limit of $\Delta t, h \to 0$, the equivalent differential Eqn.(31) simplifies to

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}-\beta\frac{\partial^2 u}{\partial t^2}+\frac{\gamma^2}{12}\frac{\partial^4 u}{\partial t^4}=0 \tag{32}$$

8. Numerical results

For the European call option, the converted form of the BSE to Heat equation is calculated using the Crank-Nicolson method and the DuFort-Frankel method. Three cases are observed for European call option with no dividend. Tabular data and graphs are produced using PYTHON.

(1) **Case:1** Consider a strike price of E=$1000, S=4E with a risk-free interest rate r = 0.05 and a volatility sigma = 0.5. As the number of time steps and space steps increases, the relative inaccuracy for both systems decreases. So both the methods are convergent as $h \to 0$ and $k \to 0$. The Crank-Nicolson method is found less accurate than the DuFort- Frankel method. Table:1 shows the results.

No.of time steps	No.of space steps	Crank-Nicolson method	DuFort-Frankel method
500	50	0.081026674	0.079738851
750	75	0.080915189	0.078964707
1000	100	0.080860865	0.078250081
1200	125	0.080828724	0.077425155
1500	150	0.080807484	0.076883418
1600	160	0.080800878	0.076615361
1700	180	0.080789900	0.075809832
1800	300	0.080755048	0.067985163
1900	500	0.080734341	0.050757233
2000	600	0.080729188	0.044597646

Table 1. For E=1000, S=4E, r=0.05, sigma=0.5, and T=1 year, compare the Crank-Nicolson and DuFort Frankel methods as the number of time and space steps increases.

(2) **Case:2** For the strike price of E=$1000, S=4E with a risk-free interest rate r = 0.05 and a volatility sigma = 0.5, these two strategies are also found to be the most effective for exercising the option for short periods of time.As the time interval of expiration decreases, the relative error decreases. It is shown in Table : 2

Time of Exercise in year	Crank-Nicolson method	DuFort-Frankel method
12/12	0.080729188	0.044597646
9/12	0.061200170	0.034464710
6/12	0.041441092	0.024045510
3/12	0.021403385	0.013153445
1/12	0.007688528	0.005241659
2/52	0.003827781	0.002814674
1/52	0.002089616	0.001638972

TABLE 2. Comparison of the Crank-Nicolson and DuFort Frankel methods for r=0.05, sigma=0.5, E=1000, S=4E, number of time steps=2000, number of space steps=600 as the exercise period is reduced.

(3) **Case:3** For E=$1000, S=4E, sigma=0.5, T=1/52, it is also noticed that as the risk-free interest rate r grew, the error increased. The DuFort-Frankel method is more accurate than the Crank-Nicolson method. It is observed in Table : 3

Rate of Interest r	Crank-Nicolson method	DuFort-Frankel method
0.05	0.002089616	0.001638972
0.07	0.002461203	0.001818904
0.09	0.002933668	0.002201061
0.12	0.003850461	0.003101067
0.14	0.004619725	0.003898241
0.16	0.005536690	0.004856385
0.18	0.006624770	0.005991288
0.2	0.007912376	0.007327268

TABLE 3. Comparison of the Crank-Nicolson and DuFort- Frankel methods for the transformed heat equation for sigma=0.5, E=1000, S=4E, T=1/52(for 1 week) when the rate of interest r is changed.

(4) **Case:4** The graphical representation of relative error with respect to volatility is presented as the risk-free interest rate r increased for both the methods. From Figure 3 and Figure 4 it is observed that the DuFort-Frankel method is found to be more accurate than the Crank-Nicolson method.

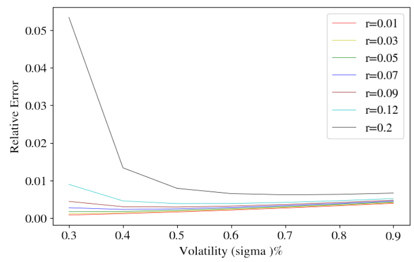


FIGURE 3. For transformed heat equation utilising Crank-Nicolson method, relative error increases as volatility increases for various risk-free rates of interest.

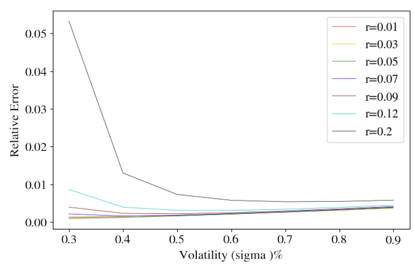


FIGURE 4. For transformed heat equation utilising DuFort-Frankel method, relative error increases as volatility increases for various risk-free rates of interest.

(5) **Case:5** Comparision is shown with analytical solution for E = 1000, S = 4000, sigma = 0.5, r = 0.05

Comparosion of solutions	T=1 year	T=0.5 year	T=1/12 year
BSE Analytic	880.73	833.80	805.50
Converted form of BSE to heat with CR method	760.20	780.77	797.91
Converted form of BSE to heat with Dufort-Frankel method	798.88	801.17	801.46

9. Conclusion

The numerical solution of the transformed Black-Scholes equation to heat equation was the subject of this paper. The Crank-Nicolson method (Two Time Level Method) and the DuFort- Frankel method (Three Time Level Method) were used to solve the heat equation. In L_1 norms, the relative error was calculated by comparing the numerical and analytical solutions. These two approaches are compared for clarity. This work contradict the error dynamics using von Neumann method where the signal and error follows the same difference equation. According to von Neumann error analysis, these two methods are inextricably stable and as the number of time and space steps grows, so does the accuracy. But GSA contradict this statement. GSA method can capable of handling the entire analysis of error generated in these two mentioned methods. It's also been established that as the time interval of exercises and the rate of interest are reduced, these two strategies provide improved accuracy. For high-volatility situations the accuracy of the three time level method is better than the two time level method. For the converted heat equation of BSE, the three time level method is determined to be more accurate than the two time level method. Finally we conclude that by investing for a short period of time at a low rate of interest in a high volatility market with good approximation, one can hedge the risk using three time level method.

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