Bulletin of Faculty of Science, Zagazig University (BFSZU)
e-ISSN: 1110-1555
Volume-2023, Issue-3, pp-98-107
https://bfszu.journals.ekb.eg/journal
Research Paper DOI: **10.21608/bfszu.2023.182373.1231**
==

Separation for Schrodinger-type operators in weighted Hilbert spaces

H. A. Atia[1] , H. M. Abu-Donia[2] , N. A. Abdelsalam[3]

Mathematics Department, Faculty of Science, Zagazig University
Corresponding author: h_a_atia@hotmail.com

ABSTRACT **: The aim of this paper is to study the separation property of the Schrodinger operator L of the form $Lf(x) = -L_0 f(x) + V(x)f(x), x \in \mathbb{R}^n$, in the weighted Hilbert space $H^{\sim} = L_{2,k}(\mathbb{R}^n, H)$, the statement that achieve the separation, and the coercive estimate, with the operator potential $V(x) \in L(H)$ for every $x \in \mathbb{R}^n$, where $L(H)$ is the space of all bounded linear operators on the arbitrary Hilbert space H. The operator $L_0 = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i}$ is the differential operator with the real-valued continuous functions $a_{ij}(x)$ and $b_i(x)$. Furthermore, we study the existence and uniqueness of the solution of the second order differential equation $-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} f(x) - \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} f(x) + V(x)f(x) = W(x)$, where $W(x) \in H^{\sim}$, in the weighted Hilbert space $H^{\sim} = L_{2,k}(\mathbb{R}^n, H)$, such that $k \in C^1(\mathbb{R}^n)$ is positive weight function.**

KEYWORDS: Separation; Schrodinger-type operator; Operator potential; Hilbert space; Laplace operator; Coercive estimate; Existence and uniqueness.

Date of Submission: 23-12-2022 Date of acceptance: 20-07-2023

I. INTRODUCTION

It is noteworthy that Everitt and Giertz [6-9] were the first to obtain the basic results of defining the separation of differential expressions. Also Biomatov [3,4] and zettl [14] worked on the separation problem. For more fundamental results of separation see [1], [2], [5] and [12].

In [10] Mohamed and Atia have studied the separation of the Schrodinger operator of the form

$$Su(x) = -\Delta u(x) + V(x)u(x),$$

with the operator potential $V(x) \in C^1(\mathbb{R}^n, L(H_1))$, in the Hilbert space $L_2(\mathbb{R}^n, H_1)$, where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator in $\mathbb{R}^n$.

In [13] Zayed et al have studied the separation of the Laplace-Beltrami differential operator of the form

$$Au = -\frac{1}{\sqrt{\det g(x)}} \frac{\partial}{\partial x_i} \left[\sqrt{\det g(x)} g^{-1}(x) \frac{\partial u}{\partial x_i} \right] + V(x)u(x)$$

for every $x \in \Omega \subset \mathbb{R}^n$, in the Hilbert space $H = L_2(\Omega, H_1)$ with the operator potential $V(x) \in C^1(\Omega, L(H_1))$, where $L(H_1)$ is the space of all bounded linear operators on the Hilbert space $H_1, g(x) = g_{ij}(x)$ is the Riemannian matrix and $g^{-1}(x)$ is the inverse of the matrix $g(x)$.

In [11] Mohamed and Atia have studied the separation for the general secondorder elliptic differentialoperator

$$G = G_0 + V(x), \forall x \in \mathbb{R}^n$$

in the weighted Hilbert space $L_{2,k}(\mathbb{R}^n, H_1)$, with the operator potential $V(x)$, where

$$G_0 = \sum_{i,j=1}^{n} a_{ij}(x) D_i^j$$

is the differential operator with the real positive coefficients $a_{ij}(x) \in C^2(\mathbb{R}^n)$ and $D_i^j = \frac{\partial^2}{\partial x_i \partial x_j}$

To some positive weight function $k \in C^1(\mathbb{R}^n)$. Suppose that H be an arbitrary separable Hilbert space with norm $\|\cdot\|_H$ and scalar product $\langle , \rangle_H$, We introduce the weighted Hilbert space $H^\sim = L_{2,k}(\mathbb{R}^n, H)$ of all $u(x), x \in \mathbb{R}^n$, such that the norm of the vector function $u(x)$ is defined by

$$\| u \|_k = \left(\int_{R^n} k^2(x) \| u(x) \|_H^2 \, dx \right)^{1/2}.$$

The symbol $\langle u, v \rangle_k$, such that $u, v \in H^\sim$ denotes the scalar product in $H^\sim$ which is defined by $\langle u, v \rangle_k = \langle ku, kv \rangle$ where

$$\langle u, v \rangle = \int_{R^n} \langle u(x), v(x) \rangle_H dx, x \in R^n,$$

is the scalar product in the space $L_2(\mathbb{R}^n, H)$. The space of all $u(x), x \in \mathbb{R}^n$, has generalized derivatives $D^\gamma u(x), \gamma \leqslant 2$ such that $u(x)$ and its derivatives $D^\gamma u(x)$ belonges to $H^\sim$ denoted by $W_{2,k}^2(\mathbb{R}^n, H)$. The function $u(x) \in W_{2,k,loc}^2(\mathbb{R}^n, H)$ if $\forall \phi(x) \in C_0^\infty(\mathbb{R}^n)$ the vector function $\phi(x)u(x) \in W_{2,k}^2(\mathbb{R}^n, H)$.

II. THE SEPARATION PROBLEM

2.1. Definition: The differential operator

$$Lf(x) = -L_0 f(x) + V(x)f(x), x \in \mathbb{R}^n$$

is said to be separated in the weighted Hilbert space $H^\sim$ if the following statement holds:
If $f \in H^\sim \cap W_{2,k,loc}^2(\mathbb{R}^n, H)$ and $Lf \in H^\sim$ implies $L_0 f$ and $V(x)f \in H^\sim$. The basic result in this section has been formulated in the following theorem.

2.2. Theorem: If the potential $V(x) \in C^1(\mathbb{R}^n, L(H))$ be a self-adjoint compact linear operator and for every $f \in H^\sim \cap W_{2,k,loc}^2(\mathbb{R}^n, H)$ such that $Lf \in H^\sim$, the following conditions are satisfied for all $x \in \mathbb{R}^n$,

$$\left| a_{ij}^{\frac{1}{2}}(x) V_0^{-\frac{1}{2}}(x) k^{-1}(x) \left(\frac{\partial}{\partial x_i} k(x) \right) \right|_1 \leqslant \frac{1}{2} \beta_1,$$

and

$$\left| a_{ij}^{\frac{1}{2}}(x) V_0^{-\frac{1}{2}}(x) V^{-1}(x) \left(\frac{\partial}{\partial x_i} V(x) \right) \right|_1 \leqslant \beta_2, i, j = 1, \dots, n,$$

where $0 < \beta_1 + \beta_2 < \frac{2}{n}, V_0(x) = \operatorname{Re} V(x)$, and $|\cdot|_1$ denotes the norm of the space $L(H)$. Then the following coercive estimate is true:

$$\| Vf \|_k + \sum_{i,j=1}^{n} \left\| a_{ij}^{\frac{1}{2}} V_0^{\frac{1}{2}} \frac{\partial}{\partial x_j} f \right\|_k + \sum_{i=1}^{n} \left\| b_i \frac{\partial}{\partial x_i} f \right\|_k + \|L_0 f\|_k \leqslant N \| Lf \|_k,$$

where N is a constant independent on f, that is the Schrodinger-type operator L is separated in the weighted Hilbert space $H^\sim$.

Proof. Obviously, by using integration by parts, we get

$$\left\langle \frac{\partial}{\partial x_i} f, w \right\rangle = -\left\langle f, \frac{\partial}{\partial x_i} w \right\rangle, i = 1, \dots, n, \text{ for all } f, w \in C_0^\infty(\mathbb{R}^n)$$

Hence

$$\begin{aligned}
\langle kLf, kVf \rangle &= \left\langle k \left[-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} f - \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} f + Vf \right], kVf \right\rangle \\
&= -\sum_{i,j=1}^{n} \left\langle k \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} f, kVf \right\rangle - \sum_{i=1}^{n} \left\langle kb_i(x) \frac{\partial}{\partial x_i} f, kVf \right\rangle + \langle kVf, kVf \rangle \\
&= \sum_{i,j=1}^{n} \left\langle a_{ij}(x) \frac{\partial}{\partial x_j} f, \frac{\partial}{\partial x_i} (k^2 Vf) \right\rangle - \sum_{i=1}^{n} \left\langle kb_i \frac{\partial}{\partial x_i} f, kVf \right\rangle + \langle kVf, kVf \rangle \\
&= \sum_{i,j=1}^{n} \left\langle a_{ij}(x) \frac{\partial}{\partial x_j} f, k^2 V \left(\frac{\partial}{\partial x_i} f \right) \right\rangle + \sum_{i,j=1}^{n} \left\langle a_{ij}(x) \frac{\partial}{\partial x_j} f, k^2 f \left(\frac{\partial}{\partial x_i} V \right) \right\rangle \\
&\quad + \sum_{i,j=1}^{n} \left\langle a_{ij}(x) \frac{\partial}{\partial x_j} f, 2kVf \left(\frac{\partial}{\partial x_i} k \right) \right\rangle - \sum_{i=1}^{n} \left\langle kb_i \frac{\partial}{\partial x_i} f, kVf \right\rangle + \langle kVf, kVf \rangle.
\end{aligned}$$

Taking the real parts of the two sides, we get

$$\begin{aligned}\mathrm{Re}\langle kLf, kVf\rangle = & \sum_{i,j=1}^{n}\left\langle ka_{ij}^{\frac{1}{2}}V_0^{\frac{1}{2}}\frac{\partial}{\partial x_j}f, ka_{ij}^{\frac{1}{2}}V_0^{\frac{1}{2}}\frac{\partial}{\partial x_j}f\right\rangle \\ & + \sum_{i,j=1}^{n}\mathrm{Re}\left\langle ka_{ij}^{\frac{1}{2}}V_0^{\frac{1}{2}}\frac{\partial}{\partial x_j}f, ka_{ij}^{\frac{1}{2}}V_0^{-\frac{1}{2}}V^{-1}\left(\frac{\partial}{\partial x_i}V\right)Vf\right\rangle \\ & + \sum_{i,j=1}^{n}\mathrm{Re}\left\langle ka_{ij}^{\frac{1}{2}}V_0^{\frac{1}{2}}\frac{\partial}{\partial x_j}f, 2ka_{ij}^{\frac{1}{2}}V_0^{-\frac{1}{2}}k^{-1}\left(\frac{\partial}{\partial x_i}k\right)Vf\right\rangle \\ & - \sum_{i=1}^{n}\mathrm{Re}\left\langle kb_i\frac{\partial}{\partial x_i}f, kVf\right\rangle + \langle kVf, kVf\rangle.\end{aligned}$$

Since we have

$$-|\langle kLf, kVf\rangle| \leqslant \mathrm{Re}\langle kLf, kVf\rangle \leqslant |\langle kLf, kVf\rangle|,$$

$$-\left|\left\langle ka_{ij}^{\frac{1}{2}}V_0^{\frac{1}{2}}\frac{\partial}{\partial x_j}f, 2ka_{ij}^{\frac{1}{2}}V_0^{-\frac{1}{2}}k^{-1}\left(\frac{\partial}{\partial x_i}k\right)Vf\right\rangle\right| \leqslant \mathrm{Re}\left\langle ka_{ij}^{\frac{1}{2}}V_0^{\frac{1}{2}}\frac{\partial}{\partial x_j}f, 2ka_{ij}^{\frac{1}{2}}V_0^{-\frac{1}{2}}k^{-1}\left(\frac{\partial}{\partial x_i}k\right)Vf\right\rangle \leqslant \left|\left\langle ka_{ij}^{\frac{1}{2}}V_0^{\frac{1}{2}}\frac{\partial}{\partial x_j}f, 2ka_{ij}^{\frac{1}{2}}V_0^{-\frac{1}{2}}k^{-1}\left(\frac{\partial}{\partial x_i}k\right)Vf\right\rangle\right|,$$

$$-\left|\left\langle ka_{ij}^{\frac{1}{2}}V_0^{\frac{1}{2}}\frac{\partial}{\partial x_j}f, ka_{ij}^{\frac{1}{2}}V_0^{-\frac{1}{2}}V^{-1}\left(\frac{\partial}{\partial x_i}V\right)Vf\right\rangle\right| \leqslant \mathrm{Re}\left\langle ka_{ij}^{\frac{1}{2}}V_0^{\frac{1}{2}}\frac{\partial}{\partial x_j}f, ka_{ij}^{\frac{1}{2}}V_0^{-\frac{1}{2}}V^{-1}\left(\frac{\partial}{\partial x_i}V\right)Vf\right\rangle \leqslant \left|\left\langle ka_{ij}^{\frac{1}{2}}V_0^{\frac{1}{2}}\frac{\partial}{\partial x_j}f, ka_{ij}^{\frac{1}{2}}V_0^{-\frac{1}{2}}V^{-1}\left(\frac{\partial}{\partial x_i}V\right)Vf\right\rangle\right|,$$

and

$$-\left|\left\langle kb_i\frac{\partial}{\partial x_i}f, kVf\right\rangle\right| \leqslant \mathrm{Re}\left\langle kb_i\frac{\partial}{\partial x_i}f, kVf\right\rangle \leqslant \left|\left\langle kb_i\frac{\partial}{\partial x_i}f, kVf\right\rangle\right|.$$

Hence by applying the cauchy-schwarz inequality, we get

$$\mathrm{Re}\langle kLf, kVf\rangle \leqslant |\langle kLf, kVf\rangle| \leqslant \| Lf \|_k \| Vf \|_k,$$

$$\mathrm{Re}\left\langle ka_{ij}^{\frac{1}{2}}V_0^{\frac{1}{2}}\frac{\partial}{\partial x_j}f, 2ka_{ij}^{\frac{1}{2}}V_0^{-\frac{1}{2}}k^{-1}\left(\frac{\partial}{\partial x_i}k\right)Vf\right\rangle \geqslant -\left\|a_{ij}^{\frac{1}{2}}V_0^{\frac{1}{2}}\frac{\partial}{\partial x_j}f\right\|_k \left\|2a_{ij}^{\frac{1}{2}}V_0^{-\frac{1}{2}}k^{-1}\left(\frac{\partial}{\partial x_i}k\right)Vf\right\|_k,$$

$$\mathrm{Re}\left\langle ka_{ij}^{\frac{1}{2}}V_0^{\frac{1}{2}}\frac{\partial}{\partial x_j}f, ka_{ij}^{\frac{1}{2}}V_0^{-\frac{1}{2}}V^{-1}\left(\frac{\partial}{\partial x_i}V\right)Vf\right\rangle \geqslant -\left\|a_{ij}^{\frac{1}{2}}V_0^{\frac{1}{2}}\frac{\partial}{\partial x_j}f\right\|_k \left\|a_{ij}^{\frac{1}{2}}V_0^{-\frac{1}{2}}V^{-1}\left(\frac{\partial}{\partial x_i}V\right)Vf\right\|_k,$$

and

$$\mathrm{Re}\left\langle kb_i\frac{\partial}{\partial x_i}f, kVf\right\rangle \geqslant -\left\|b_i\frac{\partial}{\partial x_i}f\right\|_k \| Vf \|_k.$$

Applying the conditions (2) and (3) on the inequalities (7) and (8) respectively, we get

$$\mathrm{Re}\left\langle ka_{ij}^{\frac{1}{2}}V_0^{\frac{1}{2}}\frac{\partial}{\partial x_j}f, 2ka_{ij}^{\frac{1}{2}}V_0^{-\frac{1}{2}}k^{-1}\left(\frac{\partial}{\partial x_i}k\right)Vf\right\rangle \geqslant -\beta_1\left\|a_{ij}^{\frac{1}{2}}V_0^{\frac{1}{2}}\frac{\partial}{\partial x_j}f\right\|_k \| Vf \|_k,$$

$$\mathrm{Re}\left\langle ka_{ij}^{\frac{1}{2}}V_0^{\frac{1}{2}}\frac{\partial}{\partial x_j}f, ka_{ij}^{\frac{1}{2}}V_0^{-\frac{1}{2}}V^{-1}\left(\frac{\partial}{\partial x_i}V\right)Vf\right\rangle \geqslant -\beta_2\left\|a_{ij}^{\frac{1}{2}}V_0^{\frac{1}{2}}\frac{\partial}{\partial x_j}f\right\|_k \| Vf \|_k,$$

From (10) and (11) where $\beta = \beta_1 + \beta_2$, it follows that

$$\operatorname{Re}\left\langle k a_{ij}^{\frac{1}{2}} V_0^{\frac{1}{2}} \frac{\partial}{\partial x_j} f, 2 k a_{ij}^{\frac{1}{2}} V_0^{-\frac{1}{2}} k^{-1}\left(\frac{\partial}{\partial x_i} k\right) V f\right\rangle$$
$$+\operatorname{Re}\left\langle k a_{ij}^{\frac{1}{2}} V_0^{\frac{1}{2}} \frac{\partial}{\partial x_j} f, k a_{ij}^{\frac{1}{2}} V_0^{-\frac{1}{2}} V^{-1}\left(\frac{\partial}{\partial x_i} V\right) V f\right\rangle$$
$$\geqslant -\beta\left\| a_{ij}^{\frac{1}{2}} V_0^{\frac{1}{2}} \frac{\partial}{\partial x_j} f\right\|_k \| Vf \|_k.$$

For any $\alpha > 0$ and $y_1, y_2 \in \mathbb{R}$, we have

$$|y_1||y_2| \leqslant \frac{\alpha}{2}|y_1|^2 + \frac{1}{2\alpha}|y_2|^2.$$

Consequently, we get

$$\operatorname{Re}\left\langle k a_{ij}^{\frac{1}{2}} V_0^{\frac{1}{2}} \frac{\partial}{\partial x_j} f, 2 k a_{ij}^{\frac{1}{2}} V_0^{-\frac{1}{2}} k^{-1}\left(\frac{\partial}{\partial x_i} k\right) V f\right\rangle$$
$$+\operatorname{Re}\left\langle k a_{ij}^{\frac{1}{2}} V_0^{\frac{1}{2}} \frac{\partial}{\partial x_j} f, k a_{ij}^{\frac{1}{2}} V_0^{-\frac{1}{2}} V^{-1}\left(\frac{\partial}{\partial x_i} V\right) V f\right\rangle$$
$$\geqslant -\frac{\beta\alpha}{2}\left\| a_{ij}^{\frac{1}{2}} V_0^{\frac{1}{2}} \frac{\partial}{\partial x_j} f\right\|_k^2 - \frac{\beta}{2\alpha} \| Vf \|_k^2.$$

and (9) become

$$\operatorname{Re}\left\langle k b_i \frac{\partial}{\partial x_i} f, kVf\right\rangle \geqslant -\frac{\alpha}{2}\left\| b_i \frac{\partial}{\partial x_i} f\right\|_k^2 - \frac{1}{2\alpha}\| Vf \|_k^2.$$

From (6), (12) and (13) the equation (5) takes the form

$$\| Lf \|_k \| Vf \|_k \geqslant \sum_{i,j=1}^{n}\left\| a_{ij}^{\frac{1}{2}} V_0^{\frac{1}{2}} \frac{\partial}{\partial x_j} f\right\|_k^2 - \frac{\beta\alpha}{2}\sum_{i,j=1}^{n}\left\| a_{ij}^{\frac{1}{2}} V_0^{\frac{1}{2}} \frac{\partial}{\partial x_j} f\right\|_k^2$$
$$-\frac{n^2\beta}{2\alpha}\| Vf \|_k^2 + \frac{\alpha}{2}\sum_{i=1}^{n}\left\| b_i \frac{\partial}{\partial x_i} f\right\|_k^2 + \frac{n^2}{2\alpha}\| Vf \|_k^2 + \| Vf \|_k^2.$$

Therefore

$$\left[1 - \frac{n^2(\beta-1)}{2\alpha}\right]\| Vf \|_k^2 + \left[1 - \frac{\beta\alpha}{2}\right]\sum_{i,j=1}^{n}\left\| a_{ij}^{\frac{1}{2}} V_0^{\frac{1}{2}} \frac{\partial}{\partial x_j} f\right\|_k^2 + \frac{\alpha}{2}\sum_{i=1}^{n}\left\| b_i \frac{\partial}{\partial x_i} f\right\|_k^2 \leqslant \| Lf \|_k \| Vf \|_k .$$

By choosing $\frac{n^2(\beta-1)}{2} < \alpha < \frac{2}{\beta}$, we find from (14) that

$$\| Vf \|_k \leqslant \left[1 - \frac{n^2(\beta-1)}{2}\right]^{-1} \| Lf \|_k,$$
$$\sum_{i,j=1}^{n}\left\| a_{ij}^{\frac{1}{2}} V_0^{\frac{1}{2}} \frac{\partial}{\partial x_j} f\right\|_k^2 \leqslant \left[1 - \frac{\beta\alpha}{2}\right]^{-1}\left[1 - \frac{n^2(\beta-1)}{2}\right]^{-1} \| Lf \|_k^2,$$

and

$$\sum_{i=1}^{n}\left\| b_i \frac{\partial}{\partial x_i} f\right\|_k^2 \leqslant \left[\frac{\alpha}{2}\right]^{-1}\left[1 - \frac{n^2(\beta-1)}{2}\right]^{-1} \| Lf \|_k^2.$$

From the inequalities (16) and (17) we can respectively obtain

$$\sum_{i,j=1}^{n}\left\| a_{ij}^{\frac{1}{2}} V_0^{\frac{1}{2}} \frac{\partial}{\partial x_j} f\right\|_k \leqslant n^2\left[1 - \frac{\beta\alpha}{2}\right]^{-\frac{1}{2}}\left[1 - \frac{n^2(\beta-1)}{2}\right]^{-\frac{1}{2}} \| Lf \|_k,$$

and

$$\sum_{i=1}^{n}\left\|b_i\frac{\partial}{\partial x_i}f\right\|_k \leqslant n\left[\frac{\alpha}{2}\right]^{-\frac{1}{2}}\left[1-\frac{n^2(\beta-1)}{2}\right]^{-\frac{1}{2}}\parallel Lf\parallel_k.$$

On the other hand, we have

$$Lf = -L_0 f + V(x)f.$$

So it is clear that

$$\begin{aligned}\|L_0 f\|_k &\leqslant \parallel Lf\parallel_k + \parallel Vf\parallel_k\\ &\leqslant \left[1+\left[1-\frac{n^2(\beta-1)}{2}\right]^{-1}\right]\parallel Lf\parallel_k.\end{aligned}$$

From (15), (18), (19) and (20), we get

$$\parallel Vf\parallel_k + \sum_{i,j=1}^{n}\left\|a_{ij}^{\frac{1}{2}}V_0^{\frac{1}{2}}\frac{\partial}{\partial x_j}f\right\|_k + \sum_{i=1}^{n}\left\|b_i\frac{\partial}{\partial x_i}f\right\|_k + \|L_0 f\|_k \leqslant N\parallel Lf\parallel_k,$$

where

$N = 1+2\left[1-\frac{n^2(\beta-1)}{2}\right]^{-1} + n^2\left[1-\frac{\beta\alpha}{2}\right]^{-\frac{1}{2}}\left[1-\frac{n^2(\beta-1)}{2}\right]^{-\frac{1}{2}} + n\left[\frac{\alpha}{2}\right]^{-\frac{1}{2}}\left[1-\frac{n^2(\beta-1)}{2}\right]^{-\frac{1}{2}}$, is a constant independent on f.

Now the coercive estimate (4) is obtained, and so the differential operator (1) is separated in the weighted Hilbert space $H^{\sim}$. Hence the proof is completed.

III. THE EXISTENCE AND UNIQUENESS PROBLEM

In the following theorem, we show that the differential equation

$$Lf(x) = -\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}a_{ij}(x)\frac{\partial}{\partial x_j}f(x) - \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i}f(x) + V(x)f(x) = W(x),$$

with the operator potential $V(x) \in C^1(\mathbb{R}^n, L(H))$, has one and only one solution in the space $H^{\sim}$, where $W(x) \in H^{\sim}$.

3.1. Theorem:

Suppose that the elliptic differential operator (1) is separated in the weighted Hilbert space $H^{\sim}$. If there exist a positive function $t(x) \in C^1(\mathbb{R}^n)$ such that for all $x \in \mathbb{R}^n$ the following conditions are valid:

$$\left|a_{ij}^{\frac{1}{2}}(x)V_0^{-\frac{1}{2}}(x)t^{-1}(x)\left(\frac{\partial}{\partial x_i}t(x)\right)\right|_1 \leqslant \frac{2}{\sqrt{2}}r_1^{\frac{1}{2}},$$

and

$$\left|a_{ij}^{\frac{1}{2}}(x)V_0^{-\frac{1}{2}}(x)k^{-1}(x)\left(\frac{\partial}{\partial x_i}k(x)\right)\right|_1 \leqslant \frac{1}{\sqrt{2}}r_2^{\frac{1}{2}}, i,j = 1,\dots,n,$$

where $0 < r_1 + r_2 - \frac{1}{2} < \frac{1}{n^2}$ and $V_0(x) = \operatorname{Re} V(x) \geqslant CE$. then the second order differential equation

$$Lf(x) = -L_0 f(x) + V(x)f(x) = W(x),$$

with the operator potential $V(x)$, has a unique solution in the space $H^{\sim}$, where $W(x) \in H^{\sim}$.

Proof. Firstly, we prove that the homogeneous equation of the form

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} f(x) + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} f(x) = V(x) f(x),$$

has only the zero solution $f(x) \equiv 0$, for all $x \in \mathbb{R}^n$. Let $\Phi(x) \in C^2(\mathbb{R}^n)$ be a non-negative function, then

$$\begin{aligned}
\langle kVf, kt\Phi f\rangle &= \left\langle k\left[\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} f + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} f\right], kt\Phi f\right\rangle \\
&= \left\langle k \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} f, kt\Phi f\right\rangle + \left\langle k \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} f, kt\Phi f\right\rangle \\
&= -\sum_{i,j=1}^{n} \left\langle a_{ij}(x) \frac{\partial}{\partial x_j} f, \frac{\partial}{\partial x_i}(k^2 t\Phi f)\right\rangle + \sum_{i=1}^{n} \left\langle kb_i \frac{\partial}{\partial x_i} f, kt\Phi f\right\rangle \\
&= -\sum_{i,j=1}^{n} \left\langle a_{ij}(x) \frac{\partial}{\partial x_j} f, k^2 t\Phi \frac{\partial}{\partial x_i} f\right\rangle - \sum_{i,j=1}^{n} \left\langle a_{ij}(x) \frac{\partial}{\partial x_j} f, k^2 tf \frac{\partial}{\partial x_i} \Phi\right\rangle \\
&\quad -\sum_{i,j=1}^{n} \left\langle a_{ij}(x) \frac{\partial}{\partial x_j} f, k^2 \Phi f \frac{\partial}{\partial x_i} t\right\rangle - \sum_{i,j=1}^{n} \left\langle a_{ij}(x) \frac{\partial}{\partial x_j} f, 2kt\Phi f \frac{\partial}{\partial x_i} k\right\rangle + \sum_{i=1}^{n} \left\langle kb_i \frac{\partial}{\partial x_i} f, kt\Phi f\right\rangle.
\end{aligned}$$

Equating the real parts of both sides, we get

$$\begin{aligned}
\langle kV_0 f, kt\Phi f\rangle = &-\sum_{i,j=1}^{n} \left\langle ka_{ij}(x) \frac{\partial}{\partial x_j} f, kt\Phi \frac{\partial}{\partial x_j} f\right\rangle - \sum_{i,j=1}^{n} \operatorname{Re}\left\langle ka_{ij}(x) \frac{\partial}{\partial x_j} f, ktf \frac{\partial}{\partial x_i} \Phi\right\rangle \\
&-\sum_{i,j=1}^{n} \operatorname{Re}\left\langle ka_{ij}(x) \frac{\partial}{\partial x_j} f, k\Phi f \frac{\partial}{\partial x_i} t\right\rangle - \sum_{i,j=1}^{n} \operatorname{Re}\left\langle ka_{ij}(x) \frac{\partial}{\partial x_j} f, 2kk^{-1} t\Phi f \frac{\partial}{\partial x_i} k\right\rangle \\
&+\sum_{i=1}^{n} \operatorname{Re}\left\langle kb_i \frac{\partial}{\partial x_i} f, kt\Phi f\right\rangle.
\end{aligned}$$

But on the other hand, we have

$$\begin{aligned}
2\mathrm{Re}\left\langle k a_{ij}(x)\frac{\partial}{\partial x_j}f, ktf\frac{\partial}{\partial x_i}\Phi\right\rangle = & \left\langle k a_{ij}(x)\frac{\partial}{\partial x_j}f, ktf\frac{\partial}{\partial x_i}\Phi\right\rangle + \left\langle k a_{ij}(x)\frac{\partial}{\partial x_j}f, ktf\frac{\partial}{\partial x_i}\Phi\right\rangle \\
= & \left\langle k a_{ij}(x)\frac{\partial}{\partial x_j}f, ktf\frac{\partial}{\partial x_i}\Phi\right\rangle + \left\langle f, k^2 a_{ij} t\frac{\partial}{\partial x_i}\Phi\frac{\partial}{\partial x_j}f\right\rangle \\
= & -\left\langle f, \frac{\partial}{\partial x_j}\left(k^2 a_{ij} tf\frac{\partial}{\partial x_i}\Phi\right)\right\rangle + \left\langle f, k^2 a_{ij} t\frac{\partial}{\partial x_i}\Phi\frac{\partial}{\partial x_j}f\right\rangle \\
= & -\left\langle f, k^2 a_{ij} tf\frac{\partial^2}{\partial x_j\,\partial x_i}\Phi\right\rangle - \left\langle f, k^2 a_{ij} t\frac{\partial}{\partial x_i}\Phi\frac{\partial}{\partial x_j}f\right\rangle \\
& -\left\langle f, k^2 a_{ij} f\frac{\partial}{\partial x_i}\Phi\frac{\partial}{\partial x_j}t\right\rangle - \left\langle f, k^2 tf\frac{\partial}{\partial x_i}\Phi\frac{\partial}{\partial x_j}a_{ij}\right\rangle \\
& -\left\langle f, 2k a_{ij} tf\frac{\partial}{\partial x_i}\Phi\frac{\partial}{\partial x_j}k\right\rangle + \left\langle f, k^2 a_{ij} t\frac{\partial}{\partial x_i}\Phi\frac{\partial}{\partial x_j}f\right\rangle
\end{aligned}$$

$$\begin{aligned}
&= -\left\langle kf, k\left(a_{ij} t\frac{\partial^2}{\partial x_j\,\partial x_i}\Phi + a_{ij}\frac{\partial}{\partial x_i}\Phi\frac{\partial}{\partial x_j}t\right.\right. \\
&\left.\left. + t\frac{\partial}{\partial x_i}\Phi\frac{\partial}{\partial x_j}a_{ij} + 2k^{-1}a_{ij} t\frac{\partial}{\partial x_i}\Phi\frac{\partial}{\partial x_j}k\right)f\right\rangle \\
&= -\left\| \left(a_{ij} t\frac{\partial^2}{\partial x_j\,\partial x_i}\Phi + a_{ij}\frac{\partial}{\partial x_i}\Phi\frac{\partial}{\partial x_j}t + t\frac{\partial}{\partial x_i}\Phi\frac{\partial}{\partial x_j}a_{ij} + 2k^{-1}a_{ij} t\frac{\partial}{\partial x_i}\Phi\frac{\partial}{\partial x_j}k\right)^{\frac{1}{2}} f\right\|_k^2, \\
&\mathrm{Re}\left\langle k a_{ij}\frac{\partial}{\partial x_j}f, k\Phi f\frac{\partial}{\partial x_i}t\right\rangle = \mathrm{Re}\left\langle k a_{ij}^{\frac{1}{2}} t^{\frac{1}{2}}\Phi^{\frac{1}{2}}\frac{\partial}{\partial x_j}f, kt^{\frac{1}{2}}\Phi^{\frac{1}{2}}\left(a_{ij}^{\frac{1}{2}} V_0^{-\frac{1}{2}} t^{-1}\frac{\partial}{\partial x_i}t\right)V_0^{\frac{1}{2}}f\right\rangle \\
&\leqslant \left\| a_{ij}^{\frac{1}{2}} t^{\frac{1}{2}}\Phi^{\frac{1}{2}}\frac{\partial}{\partial x_j}f\right\|_k \left\| t^{\frac{1}{2}}\Phi^{\frac{1}{2}}\left(a_{ij}^{\frac{1}{2}} V_0^{-\frac{1}{2}} t^{-1}\frac{\partial}{\partial x_i}t\right)V_0^{\frac{1}{2}}f\right\|_k, \\
&\mathrm{Re}\left\langle k a_{ij}\frac{\partial}{\partial x_j}f, 2kk^{-1}t\Phi f\frac{\partial}{\partial x_i}k\right\rangle = \mathrm{Re}\left\langle k a_{ij}^{\frac{1}{2}} t^{\frac{1}{2}}\Phi^{\frac{1}{2}}\frac{\partial}{\partial x_j}f, 2kt^{\frac{1}{2}}\Phi^{\frac{1}{2}}\left(a_{ij}^{\frac{1}{2}} V_0^{-\frac{1}{2}} k^{-1}\frac{\partial}{\partial x_i}k\right)V_0^{\frac{1}{2}}f\right\rangle \\
&\leqslant \left\| a_{ij}^{\frac{1}{2}} t^{\frac{1}{2}}\Phi^{\frac{1}{2}}\frac{\partial}{\partial x_j}f\right\|_k \left\| 2t^{\frac{1}{2}}\Phi^{\frac{1}{2}}\left(a_{ij}^{\frac{1}{2}} V_0^{-\frac{1}{2}} k^{-1}\frac{\partial}{\partial x_i}k\right)V_0^{\frac{1}{2}}f\right\|_k,
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Re}\left\langle k b_i\frac{\partial}{\partial x_i}f, kt\Phi f\right\rangle &= \mathrm{Re}\left\langle kt^{\frac{1}{2}}\Phi^{\frac{1}{2}}V_0^{-\frac{1}{2}} b_i\frac{\partial}{\partial x_i}f, kt^{\frac{1}{2}}\Phi^{\frac{1}{2}}V_0^{\frac{1}{2}}f\right\rangle \\
&\leqslant \left\| t^{\frac{1}{2}}\Phi^{\frac{1}{2}}V_0^{-\frac{1}{2}} b_i\frac{\partial}{\partial x_i}f\right\|_k \left\| t^{\frac{1}{2}}\Phi^{\frac{1}{2}}V_0^{\frac{1}{2}}f\right\|_k.
\end{aligned}$$

By using the conditions (21) and (22),

$$\mathrm{Re}\left\langle k a_{ij}\frac{\partial}{\partial x_j}f, k\Phi f\frac{\partial}{\partial x_i}t\right\rangle \leqslant \left\| a_{ij}^{\frac{1}{2}} t^{\frac{1}{2}}\Phi^{\frac{1}{2}}\frac{\partial}{\partial x_j}f\right\|_k \left(\frac{2}{\sqrt{2}} r_1^{\frac{1}{2}}\right) \left\| t^{t^{\frac{1}{2}}}\Phi^{\frac{1}{2}}V_0^{\frac{1}{2}}f\right\|_k,$$

and

$$\mathrm{Re}\left\langle ka_{ij}\frac{\partial}{\partial x_j}f, 2kk^{-1}a_{ij}t\Phi f\frac{\partial}{\partial x_i}k\right\rangle \leqslant \left\|a_{ij}^{\frac{1}{2}}t^{\frac{1}{2}}\Phi^{\frac{1}{2}}\frac{\partial}{\partial x_j}f\right\|_k \left(\frac{1}{\sqrt{2}}r_2^{\frac{1}{2}}\right)\left\|2t^{\frac{1}{2}}\Phi^{\frac{1}{2}}V_0^{\frac{1}{2}}f\right\|_k.$$

For any $\alpha > 0$ and $y_1, y_2 \in \mathbb{R}$, we have

$$|y_1||y_2| \leqslant \frac{\alpha}{2}|y_1|^2 + \frac{1}{2\alpha}|y_2|^2.$$

Consequently (28)-(30) becomes:

$$\mathrm{Re}\left\langle kb_i\frac{\partial}{\partial x_i}f, kt\Phi f\right\rangle \leqslant \frac{\alpha}{2}\left\|t^{\frac{1}{2}}\Phi^{\frac{1}{2}}V_0^{-\frac{1}{2}}b_i\frac{\partial}{\partial x_i}f\right\|_k^2 + \frac{1}{2\alpha}\left\|t^{\frac{1}{2}}\Phi^{\frac{1}{2}}V_0^{\frac{1}{2}}f\right\|_k^2,$$

$$\mathrm{Re}\left\langle ka_{ij}\frac{\partial}{\partial x_j}f, k\Phi f\frac{\partial}{\partial x_i}t\right\rangle \leqslant \frac{\alpha}{2}\left\|a_{ij}^{\frac{1}{2}}t^{\frac{1}{2}}\Phi^{\frac{1}{2}}\frac{\partial}{\partial x_j}f\right\|_k^2 + \frac{1}{2\alpha}\left(\frac{4}{2}r_1\right)\left\|t^{\frac{1}{2}}\Phi^{\frac{1}{2}}V_0^{\frac{1}{2}}f\right\|_k^2,$$

and

$$\mathrm{Re}\left\langle ka_{ij}\frac{\partial}{\partial x_j}f, 2kk^{-1}t\Phi f\frac{\partial}{\partial x_i}k\right\rangle \leqslant \frac{\alpha}{2}\left\|a_{ij}^{\frac{1}{2}}t^{\frac{1}{2}}\Phi^{\frac{1}{2}}\frac{\partial}{\partial x_j}f\right\|_k^2 + \frac{4}{2\alpha}\left(\frac{1}{2}r_2\right)\left\|t^{\frac{1}{2}}\Phi^{\frac{1}{2}}V_0^{\frac{1}{2}}f\right\|_k^2.$$

For $\alpha = 1$, we have

$$\mathrm{Re}\left\langle kb_i\frac{\partial}{\partial x_i}f, kt\Phi f\right\rangle \leqslant \frac{1}{2}\left\|t^{\frac{1}{2}}\Phi^{\frac{1}{2}}V_0^{-\frac{1}{2}}b_i\frac{\partial}{\partial x_i}f\right\|_k^2 + \frac{1}{2}\left\|t^{\frac{1}{2}}\Phi^{\frac{1}{2}}V_0^{\frac{1}{2}}f\right\|_k^2,$$

$$\mathrm{Re}\left\langle ka_{ij}\frac{\partial}{\partial x_j}f, k\Phi f\frac{\partial}{\partial x_i}t\right\rangle + \mathrm{Re}\left\langle ka_{ij}\frac{\partial}{\partial x_j}f, 2kk^{-1}t\Phi f\frac{\partial}{\partial x_i}k\right\rangle$$
$$\leqslant \left\|a_{ij}^{\frac{1}{2}}t^{\frac{1}{2}}\Phi^{\frac{1}{2}}\frac{\partial}{\partial x_j}f\right\|_k^2 + (r_1 + r_2)\left\|t^{\frac{1}{2}}\Phi^{\frac{1}{2}}V_0^{\frac{1}{2}}f\right\|_k^2.$$

By substituting (25), (34) and (35) into (24), we get

$$\langle kV_0 f, kt\Phi f\rangle = \left\langle kt^{\frac{1}{2}}\Phi^{\frac{1}{2}}V_0^{\frac{1}{2}}f, kt^{\frac{1}{2}}\Phi^{\frac{1}{2}}V_0^{\frac{1}{2}}f\right\rangle = \left\|t^{\frac{1}{2}}\Phi^{\frac{1}{2}}V_0^{\frac{1}{2}}f\right\|_k^2$$

$$\leqslant -\sum_{i,j=1}^{n}\left\|a_{ij}^{\frac{1}{2}}t^{\frac{1}{2}}\Phi^{\frac{1}{2}}\frac{\partial}{\partial x_j}f\right\|_k^2$$

$$+\frac{1}{2}\sum_{i,j=1}^{n}\|\left(a_{ij}t\frac{\partial^2}{\partial x_j\,\partial x_i}\Phi + a_{ij}\frac{\partial}{\partial x_i}\Phi\frac{\partial}{\partial x_j}t + t\frac{\partial}{\partial x_i}\Phi\frac{\partial}{\partial x_j}a_{ij}\|^2\|_{ij}\frac{\partial}{\partial x_i}\Phi\frac{\partial}{\partial x_j}k\right)^{\frac{1}{2}}f$$

$$+\sum_{i,j=1}^{n}\left\|a_{ij}^{\frac{1}{2}}t^{\frac{1}{2}}\Phi^{\frac{1}{2}}\frac{\partial}{\partial x_j}f\right\|_k^2 + n^2(r_1+r_2)\left\|t^{\frac{1}{2}}\Phi^{\frac{1}{2}}V_0^{\frac{1}{2}}f\right\|_k^2$$

$$-\frac{1}{2}\sum_{i=1}^{n}\left\|t^{\frac{1}{2}}\Phi^{\frac{1}{2}}V_0^{-\frac{1}{2}}b_i\frac{\partial}{\partial x_i}f\right\|_k^2 - \frac{1}{2}n^2\left\|t^{\frac{1}{2}}\Phi^{\frac{1}{2}}V_0^{\frac{1}{2}}f\right\|_k^2$$

$$\left[1-n^2(r_1+r_2)+\frac{1}{2}n^2\right]\left\|t^{\frac{1}{2}}\Phi^{\frac{1}{2}}V_0^{\frac{1}{2}}f\right\|_k^2 + \frac{1}{2}\sum_{i=1}^{n}\left\|t^{\frac{1}{2}}\Phi^{\frac{1}{2}}V_0^{-\frac{1}{2}}b_i\frac{\partial}{\partial x_i}f\right\|_k^2$$

$$\leqslant \frac{1}{2}\sum_{i,j=1}^{n}\|\left(a_{ij}t\frac{\partial^2}{\partial x_j\,\partial x_i}\Phi + a_{ij}\frac{\partial}{\partial x_i}\Phi\frac{\partial}{\partial x_j}t + t\frac{\partial}{\partial x_i}\Phi\frac{\partial}{\partial x_j}a_{ij}\|^2\right.$$

$$\left.+2k^{-1}a_{ij}t\frac{\partial}{\partial x_i}\Phi\frac{\partial}{\partial x_j}k\right)^{\frac{1}{2}}f$$

By choosing $\Phi(x) \equiv 1$ for all $x \in \mathbb{R}^n$, it follows that

$$\left[1-n^2\left(r_1+r_2-\frac{1}{2}\right)\right]\int_{\mathbb{R}^n}k^2(x)\left\|t^{\frac{1}{2}}(x)V_0^{\frac{1}{2}}(x)f(x)\right\|_H^2 dx$$

$$+\frac{1}{2}\int_{\mathbb{R}^n}k^2(x)\sum_{i=1}^{n}\left\|t^{\frac{1}{2}}(x)V_0^{-\frac{1}{2}}(x)b_i(x)\frac{\partial}{\partial x_i}f(x)\right\|_H^2 dx \leqslant 0,$$

which holds only for $f(x) \equiv 0$.

Now it is easy to check that the linear manifold $M = \{W: W = Lf; f \in C_0^\infty(\mathbb{R}^n)\}$ is dense everywhere in $H^\sim$. So for $W \in H^\sim$ there exist a sequence $\{Ly_m\}$ in M such that $\|Ly_m - W\|_k \longrightarrow 0$ as $m \longrightarrow \infty$.

Applying the coercive estimate (4), we find that

$$\|V(y_p - y_m)\|_k + \sum_{i,j=1}^{n}\left\|a_{ij}^{\frac{1}{2}}V_0^{\frac{1}{2}}\frac{\partial}{\partial x_j}(y_p - y_m)\right\|_k + \sum_{i=1}^{n}\left\|b_i\frac{\partial}{\partial x_i}(y_p - y_m)\right\|_k$$
$$+\|L_0(y_p - y_m)\|_k \leqslant N\|L(y_p - y_m)\|_k,$$

where $f = (y_p - y_m), p, m = 1,2,3, \ldots$

As $p \longrightarrow \infty$ and $m \longrightarrow \infty$, we see that the sequences $\{Vy_m\}$, $\left\{a_{ij}^{\frac{1}{2}}V_0^{\frac{1}{2}}\frac{\partial}{\partial x_j}y_m\right\}$, $\left\{b_i\frac{\partial}{\partial x_i}y_m\right\}$ and $\{L_0 y_m\}$ are convergent and consequently, they are cauchy sequences in $H^\sim$. Then there exist vector functions Ψ_0, Ψ_1, Ψ_2 and Ψ_3 in $H^\sim$ such that $\|Vy_m - \Psi_0\|_k \longrightarrow 0$, $\left\|a_{ij}^{\frac{1}{2}}V_0^{\frac{1}{2}}\frac{\partial}{\partial x_j}y_m - \Psi_1\right\|_k \longrightarrow 0$, $\left\|b_i\frac{\partial}{\partial x_i}y_m - \Psi_2\right\|_k \longrightarrow 0$ and $\|L_0 y_m -$

$\Psi_3\|_k \longrightarrow 0$ as $m \longrightarrow \infty$, and so we have $\{Vy_m\}, \left\{a_{ij}^{\frac{1}{2}} V_0^{\frac{1}{2}} \frac{\partial}{\partial x_j} y_m\right\}, \left\{b_i \frac{\partial}{\partial x_i} y_m\right\}$ and $\{L_0 y_m\}$ are bounded sequences in $H^{\sim}$. This implies for $m \longrightarrow \infty$ that

$$y_m \longrightarrow V^{-1}\Psi_0 = y, \frac{\partial}{\partial x_i} y_m \longrightarrow \frac{\partial}{\partial x_i} y, \frac{\partial}{\partial x_i} a_{ij} \frac{\partial}{\partial x_j} y_m \longrightarrow \frac{\partial}{\partial x_i} a_{ij} \frac{\partial}{\partial x_j} y,$$

and

$$L_0 y_m \longrightarrow L_0 y, i, j = 1, \dots, n.$$

Hence for a given $W \in H^{\sim}$ there exist $y \in H^{\sim} \cap W^2_{2,k,loc}(\mathbb{R}^n, H)$ such that $Ly = W$. Suppose $y^{\sim}$ is another solution of the differential equation $Lf = W$, then $L(y - y^{\sim}) = 0$, but $Lf = 0$ has only zero solution, implies $y - y^{\sim} = 0$ so $y = y^{\sim}$, then the uniqueness is proved. Hence the proof is completed.

REFERENCES

Atia, H.A. and Mahmoud, R.A., Separation of the two dimensional Grushin operator by the disconjugacy property. Applicable Analysis: An International Journal. 91 (2012) 2133 − 2143.

Atia, H.A. and Mahmoud, R.A., Separation of the two dimensional Laplace operator by the disconjugacy property. Iranian Journal of Science and Technology. A1(2012)1 − 6.

Biomatov, K. Kh., Separation theorems, weighted spaces and their applications to boundary value problems. Dokl. Acad. Nauk. SSSR, 247, 532-536; English transl. in Soviet Math. Dokl. 20 (1979).

Biomatov, K.Kh., Coercive estimates and separation for second order elliptic differential equations. Sov. Math. Dokl. 38 (1989). English transl. in Am. Math. Soc. (1989) 157 − 160.

Brown, R.C., Hinton, D.B. and Shaw, M.F., Some separation criteria and inequalities associated with linear second order differential operators, in: D.E. Edmunds et al. (Eds.), Function Spaces and Applications, Narosa Publishing House, New Delhi, 2000, pp. 7 − 35.

Everitt, W. N. and Giertz, M., Some properties of the domains of certain differential operators. Proceedings of the London Mathematical Society, 23 (3) (1971) 301 − 324..

Everitt, W. N. and Giertz, M., On some properties of the powers of a family self-adjoint differential expressions. Proceedings of the London Mathematiical Society, 24(3)(1972)149 − 170.

Everitt, W. N. and Giertz, M., On some properties of the domains of powers of certain differential operators. Proceedings of the London Mathematical Society, 24 (3) (1972) 756 − 768.

Everitt, W. N. and Giertz, M., Inequalities and separation for Schrodingertype operators in $L_2(R^n)$ Proceedings of the Royal Society of Edinburgh. Section A. Mathematics, 79A (1977) 257 − 265.

Mohamed, A.S. and Atia, H.A., Separation of the Schrodinger operator with an operator potential in the Hilbert spaces. Appl. Anal. 84 (1) (2005) 103 − 110.

Mohamed, A.S. and Atia, H.A., Separation of the general second order elliptic differential operator with an operator potential in the weighted Hilbert spaces. Appl. Math. Comput. 162 (2005) 155 - 163.

Mohamed, A.S. and El-Gendi, B.A., Separation for ordinary differential equation with matrix coefficient. Universitat de Barcelona. Collectanea Mathematica, 48 (3) (1997) 243 − 252.

Zayed, E.M.E., Mohamed, A.S. and Atia, H.A., Separation of LaplaceBeltrami differential operator with an operator potential. J. Math. Anal. Appl. 336 (2007) 81 − 92.

Zettle, A., Separation for differential operators and the L_P spaces. proceedings of the American Mathematical Society. 55 (1976) 44 − 46.