Estimation of P[Y<X] in the Case of Exponentiated Weibull Distribution

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Abstract

This paper deals with the estimation of reliability $R = P(Y < X)$ when X and Y are two independent exponentiated Weibull distribution with three parameters. For all unknown parameters, the maximum likelihood estimator of R was obtained by solving two nonlinear equations numerically using Newton-Raphson method. Assuming that the scale parameter is known, the maximum likelihood estimator, approximate Bayes estimator based on Lindley's approximation and empirical Bayes estimator are obtained. In addition, the confidence interval of R will be obtained. Numerical illustrations are carried out to illustrate theoretical results.

Keywords: *Bayes estimator; Empirical Bayes estimator; Exponentiated Weibull; Maximum likelihood estimator; Reliability; Stress-strength;*

1. Introduction

The Eponentiated Weibull (EW) distribution was introduced by Mudholkar and Srivastava (1993) as a simple generalization or modification of the well-known Weibull family by introducing one more shape parameter. The structural properties of EW have been discussed by Mudholkar and Huston (1996), and Nassar and Eissa

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Currently, there are little studies of the use in the EW distribution in reliability studies. Ashour and Afify (2007a), (2007b) investigated the estimation problem of the unknown parameters of the EW distribution using maximum likelihood under Type II and Type I progressive interval censoring with random removal, respectively. Elshat (2006) considered the Bayesian estimation problem of the unknown parameters of the EW distribution using type I progressive interval censoring with fixed removal. Singh et al (2002), (2005a), (2005b), (2006), obtained the bayes estimates of the distribution parameters, $R(t)$, $H(t)$ with type II censored sample under squared error as well as under LINEX loss functions.

The cumulative distribution function (c.d.f.) and probability density function (p.d.f) of a random variable having the EW distribution are given, respectively, by

$$F(x;\mu,\alpha,\theta)=[1-e^{-(\mu x)^{\alpha}}]^{\theta};\quad x>0,\mu>0,\alpha>0,\theta>0, \tag{1.1}$$

and,

$$f(x;\mu,\alpha,\theta)=\theta\alpha\mu^{\alpha}x^{\alpha-1}e^{-(\mu x)^{\alpha}}(1-e^{-(\mu x)^{\alpha}})^{\theta-1};\quad x>0,\mu>0,\alpha>0,\theta>0, \tag{1.2}$$

where, μ is the scale parameter and α,θ are the shape parameters.

There has been continuous interest in the problem of estimating the probability that one random variable exceeds another, that is, $R=P(Y<X)$ where X and Y are independent random variables. This problem arises in the context of mechanical reliability of a system and R is a chosen measure of system performance. The system fails if and only if at any time the applied stress is greater than its strength. This problem model was discussed by many authors [for example, Church and Harries (1970), Tong (1974) and (1977), Sathe and Shah (1981), Awad and Charraf (1986) Mohamoud (1996), Ahmed et al (1997), Surles and Padgett (1998), Ashour et al [illegible] pta (2005), Raqab and Kundu (2005) and Krishnamoorthy et al [illegible] 2008)].

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This article is concerned with the inference of $P(Y < X)$ when X and Y are two independently distributed as EW distribution. For all unknown parameters, the maximum likelihood estimator (MLE) of R is obtained numerically by solving two nonlinear equations using Newton-Raphson technique. Assuming the scale parameter (μ) and one of the shape parameters (α) are known, MLE, Bayes estimator based on Lindley's approximation and empirical Bayes of reliability R are proposed. In addition, the confidence interval of R is obtained. Monte Carlo simulation is performed to compare the performance of different methods of estimation using Mathcad ver. (2001).

The rest of the paper is organized as follows. In Section 2, the MLEs of R are obtained under two cases separately, parameters are known and unknown. Assuming the scale (μ) and only one of the shape parameters (α) are known, different methods of estimating R are discussed in Section 3. The distribution of R and its confidence interval is provided in Section 4. Simulation studies are carried out in Section 5. Finally, conclusions are included in Section 6. Tables are displayed at the end of this article.

2 Maximum Likelihood Estimator of R

This Section deals with the MLE of reliability $R = P(Y < X)$ when X and Y are two independent EW distribution with parameters (θ, α, μ) and (λ, α, μ) respectively. Considering two cases, in the first one, it is assumed that all the parameters are unknown. While in the second case it is assumed that the scale parameter μ and the shape parameter α are known.

The reliability R can be computed as follows

$$R = P[Y < X] = \int_0^\infty \int_0^x \theta\lambda\alpha^2 x^{\alpha-1} y^{\alpha-1} e^{-((\mu x)^\alpha + (\mu y)^\alpha)} (1-e^{-(\mu x)^\alpha})^{\theta-1} (1-e^{-(\mu y)^\alpha})^{\lambda-1} dydx$$

$$= \frac{\theta}{\theta+\lambda} \tag{2.1}$$

Now to compute the MLE $\hat{R}$ for R, first the MLEs $\hat{\theta}$ for θ and $\hat{\lambda}$ for λ must be obtained. Let $X_1,\ldots,X_n$ and $Y_1,\ldots,Y_m$ are random samples drawn from EW with parameters (θ,α,μ) and (λ,α,μ). The MLEs $\hat{\theta},\hat{\lambda},\hat{\alpha}$ and $\hat{\mu}$ of the parameters θ,λ,α and μ are the values which maximize the likelihood function

$$\ln L(\theta,\lambda,\alpha,\mu|\underline{x},\underline{y}) = n\ln\theta + m\ln\lambda + \alpha(n+m)\ln\mu + (n+m)\ln\alpha$$

$$+(\alpha-1)(\sum_{i=1}^{n}\ln x_i + \sum_{j=1}^{m}\ln y_j) - (\sum_{i=1}^{n}(\mu x_i)^\alpha + \sum_{j=1}^{m}(\mu y_j)^\alpha)$$

$$+(\theta-1)\sum_{i=1}^{n}\ln(1-e^{-(\mu x_i)^\alpha}) + (\lambda-1)\sum_{j=1}^{m}\ln(1-e^{-(\mu y_j)^\alpha}). \tag{2.2}$$

The first derivatives of the natural logarithm of the function with respect to θ,λ,α and μ are given by

$$\frac{\partial \ln L(\theta,\lambda,\alpha,\mu|\underline{x},\underline{y})}{\partial\theta} = \frac{n}{\hat{\theta}} + \sum_{i=1}^{n}\ln(1-e^{-(\hat{\mu}x_i)^{\hat{\alpha}}}) = 0, \tag{2.3}$$

$$\frac{\partial \ln L(\theta,\lambda,\alpha,\mu|\underline{x},\underline{y})}{\partial\lambda} = \frac{m}{\hat{\lambda}} + \sum_{j=1}^{m}\ln(1-e^{-(\hat{\mu}y_j)^{\hat{\alpha}}}) = 0, \tag{2.4}$$

$$\frac{\partial \ln L(\theta,\lambda,\alpha,\mu|\underline{x},\underline{y})}{\partial\alpha} = \frac{n+m}{\hat{\alpha}} + (n+m)\ln\hat{\mu} + \sum_{i=1}^{n}\ln x_i + \sum_{j=1}^{m}\ln y_j - \sum_{i=1}^{n}(\hat{\mu}x_i)^{\hat{\alpha}}\ln(\hat{\mu}x_i)$$

$$-\sum_{j=1}^{m}(\hat{\mu}y_j)^{\hat{\alpha}}\ln(\hat{\mu}y_j) + (\hat{\theta}-1)\sum_{i=1}^{n}\frac{e^{-(\hat{\mu}x_i)^{\hat{\alpha}}}(\mu x_i)^{\hat{\alpha}}\ln(\mu x_i)}{(1-e^{-(\hat{\mu}x_i)^{\hat{\alpha}}})} + (\hat{\lambda}-1)\sum_{j=1}^{m}\frac{e^{-(\hat{\mu}y_j)^{\hat{\alpha}}}(\mu y_j)^{\hat{\alpha}}\ln(\hat{\mu}y_j)}{(1-e^{-(\hat{\mu}y_j)^{\hat{\alpha}}})} = 0, \tag{2.5}$$

and,

$$\frac{\partial \ln L(\theta,\lambda,\alpha,\mu|\underline{x},\underline{y})}{\partial \mu}=\frac{\hat{\alpha}(n+m)}{\hat{\mu}}-\hat{\alpha}\hat{\mu}^{\hat{\alpha}-1}(\sum_{i=1}^{n}x_i^{\hat{\alpha}}+\sum_{j=1}^{m}y_j^{\hat{\alpha}})$$

$$+(\hat{\theta}-1)\sum_{i=1}^{n}\frac{\hat{\alpha}\hat{\mu}^{\hat{\alpha}-1}x_i^{\hat{\alpha}}e^{-(\hat{\mu}x_i)^{\hat{\alpha}}}}{(1-e^{-(\hat{\mu}x_i)^{\hat{\alpha}}})}+(\hat{\lambda}-1)\sum_{j=1}^{m}\frac{\hat{\alpha}\hat{\mu}^{\hat{\alpha}-1}y_j^{\hat{\alpha}}e^{-(\hat{\mu}y_j)^{\hat{\alpha}}}}{(1-e^{-(\hat{\mu}y_j)^{\hat{\alpha}}})}=0. \tag{2.6}$$

From equations (2.3) and (2.4) the MLEs of θ and λ can be obtained in terms of $\hat{\alpha}$ and $\hat{\mu}$, as

$$\hat{\theta}=-\frac{n}{\sum_{i=1}^{n}\ln(1-e^{-(\hat{\mu}X_i)^{\hat{\alpha}}})}, \tag{2.7}$$

and,

$$\hat{\lambda}=-\frac{m}{\sum_{j=1}^{m}\ln(1-e^{-(\hat{\mu}Y_j)^{\hat{\alpha}}})}. \tag{2.8}$$

(i) Case I:

If all parameters are unknown, μ and α must be firstly estimated by substituting equations (2.7) and (2.8) into equations (2.5) and (2.6), therefore,

$$\frac{n+m}{\hat{\alpha}}+(n+m)\ln\hat{\mu}+\sum_{i=1}^{n}\ln x_i+\sum_{j=1}^{m}\ln y_j-\sum_{i=1}^{n}(\hat{\mu}x_i)^{\hat{\alpha}}\ln(\hat{\mu}x_i)-\sum_{j=1}^{m}(\hat{\mu}y_j)^{\hat{\alpha}}\ln(\hat{\mu}y_j)$$

$$-(\frac{n}{\sum_{i=1}^{n}\ln(1-e^{-(\hat{\mu}x_i)^{\hat{\alpha}}})}+1)\sum_{i=1}^{n}\frac{e^{-(\hat{\mu}x_i)^{\hat{\alpha}}}(\hat{\mu}x_i)^{\hat{\alpha}}\ln(\hat{\mu}x_i)}{(1-e^{-(\hat{\mu}x_i)^{\hat{\alpha}}})}-(\frac{m}{\sum_{j=1}^{m}\ln(1-e^{-(\hat{\mu}y_j)^{\hat{\alpha}}})}+1)\sum_{j=1}^{m}\frac{e^{-(\hat{\mu}y_j)^{\hat{\alpha}}}(\hat{\mu}y_j)^{\hat{\alpha}}\ln(\hat{\mu}y_j)}{(1-e^{-(\hat{\mu}y_j)^{\hat{\alpha}}})}=0, \tag{2.9}$$

and,

$$\hat{\alpha}(n+m)-\hat{\alpha}(\sum_{i=1}^{n}(\hat{\mu}x_i)^{\hat{\alpha}}+\sum_{j=1}^{m}(\hat{\mu}y_j)^{\hat{\alpha}})-(\frac{n}{\sum_{i=1}^{n}\ln(1-e^{-(\hat{\mu}x_i)^{\hat{\alpha}}})}+1)\sum_{i=1}^{n}[\frac{\hat{\alpha}e^{-(\hat{\mu}x_i)^{\hat{\alpha}}}\,\hat{\mu}x_i)^{\hat{\alpha}}}{(1-e^{-(\hat{\mu}x_i)^{\hat{\alpha}}})}]$$

$$-(\frac{m}{\sum_{j=1}^{m}\ln(1-e^{-(\hat{\mu}y_j)^{\hat{\alpha}}})}+1)\sum_{i=1}^{n}[\frac{\hat{\alpha}e^{-(\hat{\mu}y_j)^{\hat{\alpha}}}\,(\hat{\mu}y_j)^{\hat{\alpha}}}{(1-e^{-(\hat{\mu}y_j)^{\hat{\alpha}}})}]=0. \tag{2.10}$$

Obviously, it is not easy to obtain a closed form solution for the two non-linear equations (2.9) and (2.10). Therefore, Newton Raphson method must be applied to solve these equations numerically. Once the values of α and μ are determined, estimates of θ and λ are obtained from equations (2.7) and (2.8). Therefore, the MLE of R can be obtained by substituting the estimates of θ and λ into equation (2.1).

(ii) Case II

Assume the scale parameter ($\mu = 1$) and shape parameter (α) are known, the MLEs $\hat{\theta}$ for θ and $\hat{\lambda}$ for λ can be easily obtained by using equations (2.7) and (2.8). Therefore the MLE of R becomes

$$\hat{R}=\frac{\hat{\theta}}{\hat{\theta}+\hat{\lambda}}=\frac{n\sum_{j=1}^{m}\ln(1-e^{-Y_j^{\alpha}})}{n\sum_{j=1}^{m}\ln(1-e^{-Y_j^{\alpha}})+m\sum_{i=1}^{n}\ln(1-e^{-X_j^{\alpha}})}. \tag{2.11}$$

3. Other Estimators of R for Known μ and α

In this Section, different method of estimation of reliability R will be proposed under the assumption that μ and α are known. The proposed method are, namely; MLE, Bayes estimates based on Lindley's approximation and the empirical Bayes estimator. The MLE method is discussed in Section 2 while the other proposed methods will be obtained in this Section.

3.1 Bayesian Estimation of R

The Bayesian estimate of R is obtained under the assumptions that the shape parameters θ and λ are random variables for both populations and have independent gamma priors with p.d.f.'s;

$$\pi(\theta)=\frac{b_1^{a_1}}{\Gamma(a_1)}\theta^{a_1-1}e^{-b_1\theta}; \qquad \theta>0, a_1, b_1>0, \tag{3.1}$$

and

$$\pi(\lambda)=\frac{b_2^{a_2}}{\Gamma(a_2)}\lambda^{a_2-1}e^{-b_2\lambda}; \qquad \lambda>0, a_2, b_2>0. \tag{3.2}$$

That is, θ and λ follow gamma distributions with parameters (a_1, b_1) and (a_2, b_2) respectively. Let $X_1,...X_n$ be a random sample drawn from EW with parameter (θ,α) and $Y_1,...,Y_m$ be another random sample drawn from EW with parameter (λ,α), then the posterior density of θ and λ are given by,

$$\pi(\theta\mid \underline{x}) \propto \theta^{n+a_1-1}e^{-\theta(b_1-\sum_{i=1}^{n}\ln(1-e^{-X_i^{\alpha}})}\,,$$

and,

$$\pi(\lambda\mid \underline{y}) \propto \lambda^{m+a_2-1}e^{-\lambda(b_2-\sum_{j=1}^{m}\ln(1-e^{-Y_j^{\alpha}}))}\,. \tag{3.3}$$

Assume that θ and λ are independent, then the joint bivariate posterior density of θ and λ will be

$$\pi(\theta,\lambda\mid \underline{x},\underline{y}) \propto \theta^{n+a_1-1}\lambda^{m+a_2-1}e^{-[\theta(b_1+H_1)+\lambda(b_2+H_2)]}; \quad \theta,\lambda>0, \tag{3.4}$$

where, $H_1=-\sum_{i=1}^{n}\ln(1-e^{-X_i^{\alpha}})$ and $H_2=-\sum_{j=1}^{m}\ln(1-e^{-Y_j^{\alpha}})$.

Applying the transformation technique of random variables, let

$$r=\frac{\theta}{\theta+\lambda} \text{ and } t=\theta+\lambda,\ t>0, 0<r<1.$$

Then,

$$\pi(r,t\mid \underline{x},\underline{y}) \propto r^{n+a_1-1}(1-r)^{m+a_2-1}\,t^{n+m+a_1+a_2-1}e^{-t[r(b_1+H_1)+(1-r)(b_2+H_2)]}. \tag{3.5}$$

So,

$$\pi(r\mid \underline{x},\underline{y}) \propto r^{n+a_1-1}(1-r)^{m+a_2-1}\int_0^\infty t^{n+m+a_1+a_2-1}e^{-t[r(b_1+H_1)+(1-r)(b_2+H_2)]}dt.$$

Hence, the posterior probability density function of R, over the interval $(0,1)$ is given by,

$$f_R(r|\underline{x},\underline{y})=E\frac{r^{n+a_1-1}(1-r)^{m+a_2-1}}{[r(b_1+H_1)+(1-r)(b_2+H_2)]^{n+m+a_1+a_2}};\qquad 0<r<1, \tag{3.6}$$

where,

$$E=\frac{\Gamma(n+m+a_1+a_2)}{\Gamma(m+a_2)\Gamma(n+a_1)}(b_1+H_1)^{n+a_1}(b_2+H_2)^{m+a_2}.$$

Since the Bayes estimate of R under squared error loss can not be computed analytically. Alternatively, using the approximation of Lindley (1980) and following the approach of Ahmad et al (1997). Therefore approximate Bayes estimate of R, say $\tilde{\tilde{R}}_{BS}$ is given by

$$\tilde{\tilde{R}}_{BS}=\tilde{R}[1+\frac{\tilde{\lambda}\tilde{R}^2}{\tilde{\theta}^2(n+a_1-1)(m+b_2-1)}\times(\tilde{\lambda}(m+a_2-1)-\tilde{\theta}(n+a_1-1))], \tag{3.7}$$

where, $\tilde{\theta}=\dfrac{n+a_1-1}{b_1+H_1}$, $\tilde{\lambda}=\dfrac{m+a_2-1}{b_2+H_2}$ and $\tilde{R}=\dfrac{\tilde{\theta}}{\tilde{\theta}+\tilde{\lambda}}$.

3.2 Empirical Bayes Estimator of R

In the empirical Bayes technique, the unknown parameters of the prior density a_i and b_i for $i=1,2$ will be estimated using the past estimates of the parameters θ and λ. The resulting estimates of a_i and b_i will be used in the Bayes estimate of $R=P(Y<X)$ which gives the empirical Bayes estimate of R.

Suppose that $X_{N+1,1}, X_{N+1,2},\dots,X_{N+1,n}$ is the current sample having EW (θ,α) distribution with random variable θ has the value θ_{N+1}. When the current sample is observed, there are available past samples $X_{l,1}, X_{l,2},\dots,X_{l,n}, l=1,2,\dots,N$ for which the MLE of θ is given by equation (2.7) and can be written as

$$\hat{\theta}_l \equiv S_l = \frac{n}{H_{1l}}, \tag{3.8}$$

where, $H_{1i} = -\ln(1-e^{-X_i^\alpha}), i=1,2,\dots,n.$ but for current sample $X_{N+1,1}, X_{N+1,2},\dots,X_{N+1,n}$.

For a given θ_l, the conditional probability density function of H_{1l} is gamma and then S_l has inverted gamma with the following probability density function

$$u(s_l \mid \theta_l) = \frac{(n\theta_l)^n}{\Gamma(n)} \frac{1}{s_l^{n+1}} e^{\frac{-n\theta}{s_l}}, \quad s_l > 0. \tag{3.9}$$

The marginal probability density function of S_l is given by

$$u(s_l) = \frac{b_1 n^n}{\beta(a_1,n)} \frac{(b_1 s_l)^{a_1-1}}{[n+b_1 s_l]^{n+a_1}}, \quad s_l > 0. \tag{3.10}$$

The moments estimate of the parameters a_1 and b_1 can be obtained and take the form

$$\hat{a}_1 = \frac{w_1^2}{w_2 - w_1^2} \text{ and } \hat{b}_1 = \frac{w_1}{w_2 - w_1^2}, \tag{3.11}$$

where

$$w_1 = \frac{(n-1)}{nN}\sum_{l=1}^{N}\hat{\theta}_l \text{ and } w_2 = \frac{(n-1)(n-2)}{n^2 N}\sum_{l=1}^{N}\hat{\theta}_l^2 .$$

Similarly, for random variables Y and λ, the prior parameters a_2 and b_2 can be estimated using the past estimates $\hat{\lambda}_k$, $k = 1,...,M$ from the past samples $Y_{k,1}, Y_{k,2},..., Y_{k,m}$ and written in the form

$$\hat{a}_2 = \frac{w_1^{*2}}{w_2^* - w_1^{*2}} \text{ and } \hat{b}_2 = \frac{w_1^*}{w_2^* - w_1^{*2}}, \tag{3.12}$$

where,

$$w_1^* = \frac{(m-1)}{mM}\sum_{k=1}^{M}\hat{\lambda}_k \text{ and } w_2^* = \frac{(m-1)(m-2)}{m^2 M}\sum_{k=1}^{M}\hat{\lambda}_k^2 .$$

Substituting $\hat{a}_1, \hat{b}_1, \hat{a}_2$ and $\hat{b}_2$ from equations (3.11) and (3.12) in (3.7) yield the empirical Bayes (EB) estimate of R in the forms

$$\hat{R}_{EB} = \tilde{R}[1 + \frac{\tilde{\lambda}\tilde{R}^2}{\tilde{\theta}^2(n+\hat{a}_1-1)(m+\hat{b}_2-1)} \times (\tilde{\lambda}(m+\hat{a}_2-1) - \tilde{\theta}(n+\hat{a}_1-1))], \tag{3.13}$$

where $(\tilde{\theta}, \tilde{\lambda})$ are the posterior modes from the current samples (the samples of order $N+1$ and $M+1$).

4. Interval Estimation of R

To obtain the confidence interval of R, firstly the asymptotic distribution of $\hat{R}_{MLE}$ will be obtained. For identical known shape parameter α and scale parameter $\mu = 1$, the MLE R is given by the equation (2.11) as

$$\hat{R} = \frac{n\sum_{j=1}^{m}\ln(1-e^{-Y_j^\alpha})}{n\sum_{j=1}^{m}\ln(1-e^{-Y_j^\alpha}) + m\sum_{i=1}^{n}\ln(1-e^{-X_i^\alpha})}, \tag{4.1}$$

where, $O_i = -\ln(1-e^{-X_i^{\alpha}})$ and $V_j = -\ln(1-e^{-Y_i^{\alpha}})$ are independent exponentially distributed as indicated before. Also, it is known that $-2\theta\sum_{i=1}^{n}\ln(1-e^{-X_i^{\alpha}}) \sim \chi^2_{2n}$ and $-2\lambda\sum_{j=1}^{m}\ln(1-e^{-Y_j^{\alpha}}) \sim \chi^2_{2m}$.

Thus $\hat{R}$ in equation (4.1) can be written as

$$\hat{R} \overset{d}{=} \frac{1}{1+\frac{\lambda}{\theta}Z}. \tag{4.2}$$

Here $\overset{d}{=}$ indicates equivalence in distribution and Z has F distribution with $2n$ and $2m$ degrees of freedom. So the probability density function of $\hat{R}$ can be obtained as follows

$$\phi(\hat{r}) = \frac{\Gamma(n+m)}{\Gamma(n)+\Gamma(m)}\left(\frac{n}{m}\right)^n\left(\frac{\theta}{\lambda}\right)^n\left(\frac{1-\hat{r}}{\hat{r}}\right)^{n-1}\left(1+\frac{n\theta}{m\lambda}\left(\frac{1-\hat{r}}{\hat{r}}\right)\right)^{-(n+m)}\frac{1}{\hat{r}^2};\quad 0<\hat{r}<1. \tag{4.3}$$

Since the value of Z can be written as

$$Z = \frac{\theta}{\lambda}\left(\frac{1-\hat{R}}{\hat{R}}\right),\quad \text{therefore } Z = \frac{(1-\hat{R})}{\hat{R}}\cdot\frac{R}{(1-R)} \tag{4.4}$$

Using Z as a pivotal quantity, then

$$\frac{1}{1+\frac{1}{a}\left(\frac{1}{\hat{R}}-1\right)} < R < \frac{1}{1+\frac{1}{b}\left(\frac{1}{\hat{R}}-1\right)}.$$

Since (a,b) are the lower and upper $\frac{\alpha}{2}^{th}$ percentiles points of $F_{(2n,2m)}$, then $(1-\alpha)100\%$ confidence interval for R can be obtained as follows

$$\left[\frac{1}{1+F_{2m,2n;1-\frac{\alpha}{2}}\times\left(\frac{1}{\hat{R}}-1\right)},\frac{1}{1+F_{2m,2n;\frac{\alpha}{2}}\times\left(\frac{1}{\hat{R}}-1\right)}\right], \tag{4.5}$$

where $F_{2m,2n;\frac{\alpha}{2}}$ and $F_{2m,2n;1-\frac{\alpha}{2}}$ are the lower and upper limit of F distribution with $2m$ and $2n$ degrees of freedom.

5. Numerical Illustration

In this Section, an extensive numerical investigation will be carried out to compare the performance of the different methods of estimation mainly in terms of their biases and MSEs for different sample sizes and for different parameter values. Two cases will be considered, separately, to draw inference on R, when (i) α and μ are unknown and (ii) α and μ are known. The simulation procedures are described through the following steps :

Step (1): Following Raqab and Kundu (2005), 1000 random samples $X_1,...,X_n$ and $Y_1,...,Y_m$ of sizes $(m,n)=(10,10)$, $(15,15)$, $(20,20)$, $(25,25)$ and $(30,30)$ were generated from EW lifetime distribution. This can be achieved by firstly generating a random samples from the Uniform (0,1) distribution, $U_1,...U_n$ and $C_1,...,C_m$. Then the uniform random numbers can be transformed to EW random numbers by using the following transformations

$$G_i=\frac{1}{\mu}[-\ln(1-(U_i)^{\frac{1}{\theta}})]^{\frac{1}{\alpha}}, \qquad i=1,...,n,$$

and,

$$D_j=\frac{1}{\mu}[-\ln(1-(C_j)^{\frac{1}{\lambda}})]^{\frac{1}{\alpha}}, \qquad j=1,...,m.$$

Step (2): The parameter selected values for θ and λ are: $\theta=0.5\,(0.5)\,1.5$ and $\lambda=0.5\ (0.5)\ 3$. Without loss of generality α and μ were taken to be 1 in all experiments.

Step (3): Case (I): α and μ are unknown.

From the sample, the estimates of α and μ is computed by Netwon Raphson method using equations (2.9) and (2.10). Once the estimates of α and μ are computed, the estimates of θ and λ are obtained using equations (2.7) and (2.8) respectively. Finally the MLE of R will be obtained using equation (2.1).

Case (II): α and μ are known.

The MLE of R are obtained using equations (2.11). Different Bayes estimates are computed using the non-informative prior namely, $a_1 = a_2 = b_1 = b_2 = 0$. Using the same prior distributions, the approximate Bayes estimate using Lindley's approximation method and empirical Bayes are obtained.

Step (4): The average biases and the MSEs over 1000 replication under two cases are reported separately in Tables (1) and (2).

Step (5): The approximate confidence intervals of R using formula (4.5) are constructed with confidence level at $\gamma = 0.95$ and 0.99. The evaluated upper and lower confidence intervals are reported in Table (3).

All simulated studies presented here are obtained via Mathcad ver. (2001).

6. Conclusion

In this article different proposed methods of estimating $R = P(Y < X)$ are compared when X and Y follow EW distribution. When all parameters are unknown, it is observed that the MLE can be obtained by solving two non-linear equations. When the scale parameter and only one of the shape parameters are known different estimators are compared namely; MLE, Bayes estimates based on Lindley's approximation and the empirical Bayes estimators. Also, the asymptotic distribution of R is obtained and is used to compute the asymptotic confidence intervals. The biases and MSEs of different estimators are reported in Table (1) and (2).

Table (3) contains the evaluated upper and lower confidence intervals. From these tables many conclusions can be made on the performance of all methods. These conclusions are summarized as follows:

Case I: unknown parameters

1. For small sample sizes, the performance of the MLEs is quite satisfactory in terms of biases and MSEs (see Table (1)).

2. It is observed that as the sample sizes m and n increase, the biases and MSEs decrease (see Table (1)).

Case II: μ and α are known

3. For all the methods as the sample sizes m and n increase, the biases and the MSEs decrease.
4. Comparing the biases of different estimators, it is clear that the MLE yields the minimum biases in almost all cases.
5. Comparing the MSEs of the different estimators, it is clear from Table (2) that the approximate Bayes estimates using Lindley's approximation have the minimum MSE in almost all cases. In few cases, the empirical Bayes estimates are probably the second minimum MSE.
6. When the value of λ increases and for the smallest value of θ, the biases and MSEs decrease for all different sample estimators. (See Table (2))
7. When the value of λ increases and for the largest value of θ, the biases and MSEs increase for all different sample estimators. (See Table (2))
8. Comparison study revealed that the approximate Bayes estimate using Lindley's approximation works the best.
9. The approximate confidence intervals decrease when the sample size is increased. Also, the interval of reliability at 95% is smaller than the interval of reliability at 99% (see Table (3)).

Table (1): Case I all parameters are unknown; Biases and MSEs for the MLE of reliability R

		λ					
(n,m)	θ	0.5	1	1.5	2	2.5	3
(10,10)	0.5	0.0110(0.0160)	0.0057(0.0130)	-0.0059(0.0081)	-0.0076(0.0057)	-0.0005(0.0041)	0.0006(0.0028)
	1	0.0039(0.0110)	-0.0053(0.0140)	-0.0047(0.0110)	0.0024(0.0095)	0.0027(0.0096)	0.0048(0.0075)
	1.5	0.0065(0.0068)	0.0005(0.0140)	-0.0037(0.0130)	0.0030(0.0120)	0.0075(0.0110)	0.0180(0.0095)
(15,15)	0.5	-0.0040(0.0049)	-0.0093(0.0073)	-0.0032(0.0054)	-0.0020(0.0036)	-0.0004(0.0072)	0.0039(0.0022)
	1	0.0089(0.0077)	0.0056(0.0096)	0.0026(0.0088)	0.0028(0.0075)	0.0009(0.0058)	0.0045(0.0051)
	1.5	0.0100(0.0051)	0.0079(0.0085)	0.0062(0.0086)	0.0010(0.0078)	0.0110(0.0077)	0.0180(0.0070)
(20,20)	0.5	0.0015(0.0075)	-0.0044(0.0060)	-0.0024(0.0040)	-0.0022(0.0028)	-0.0016(0.0020)	0.0032(0.0016)
	1	0.0130(0.0055)	-0.0005(0.0068)	-0.0028(0.0061)	0.0020(0.0055)	0.0019(0.0048)	0.0021(0.0037)
	1.5	0.0110(0.0037)	0.0086(0.0062)	-0.0005(0.0066)	-0.0004(0.0060)	0.0077(0.0057)	0.0170(0.0052)
(25,25)	0.5	0.0045(0.0061)	-0.0068(0.0051)	-0.0060(0.0035)	-0.0035(0.0024)	-0.0033(0.0013)	0.0006(0.0013)
	1	0.0070(0.0050)	-0.0043(0.0050)	-0.0040(0.0055)	0.0018(0.0040)	-0.0005(0.0040)	-0.0022(0.0026)
	1.5	0.0062(0.0035)	0.0015(0.0048)	0.0026(0.0055)	-0.0028(0.0048)	0.0031(0.0046)	0.0120(0.0042)
(30,30)	0.5	0.0010(0.0050)	-0.0006(0.0042)	-0.0013(0.0023)	-0.0030(0.0017)	0.0004(0.0012)	0.0001(0.0010)
	1	0.0065(0.0030)	-0.0013(0.0046)	-0.0041(0.0049)	-0.0017(0.0030)	-0.0072(0.0025)	0.0022(0.0032)
	1.5	-0.0003(0.0034)	0.0015(0.0047)	-0.0033(0.0044)	-0.0035(0.0033)	0.0078(0.0031)	0.0130(0.0037)

Each rows represent the average biases of the MLE and the corresponding MSEs are reported within brackets.

Table (2): Case II μ and α are known; Biases and MSEs of reliability R for MLE, Approximate Bayes (Lindley's approximation) and empirical Bayes.

		λ					
(n,m)	θ	0.5	1	1.5	2	2.5	3
(10,10)	0.5	0.0041(0.0116) 0.0038(0.0105) 0.0137(0.0124)	0.0044(0.0093) 0.0116(0.0088) 0.0311(0.0299)	0.0121(0.0072) 0.0212(0.0072) 0.1917(0.0627)	0.0120(0.0055) 0.0217(0.0058) 0.2602(0.0794)	0.0074(0.0041) 0.0170(0.0044) 0.0012(0.0037)	0.0072(0.0033) 0.0164(0.0036) 0.0259(0.0046)
	1	-0.0086(0.0098) -0.0155(0.0093) 0.0025(0.0101)	-0.0101(0.0122) -0.0096(0.0118) 0.0069(0.0499)	-0.0050(0.0111) -0.0018(0.0101) 0.0541(0.0351)	-0.0017(0.0096 0.0054(0.0090) 0.1239(0.0257)	0.0119(0.0084) 0.0203(0.0083) -0.0092(0.0063)	0.0123(0.0073) 0.0215(0.0074) 0.0289(0.0073)
	1.5	-0.0114(0.0078) -0.0205(0.0078) 0.0103(0.0095)	-0.0053(0.0109) -0.0098(0.0100) -0.0396(0.0505)	-0.0033(0.0118) -0.0032(0.0107) -0.0797(0.0360)	0.0005(0.0114) 0.0033(0.0104) 0.0176(0.0139)	0.0009(0.0107) 0.0066(0.0098) -0.0208(0.0095)	0.0027(0.0097) 0.0098(0.0091) 0.0244(0.0092)
(15,15)	0.5	0.0002(0.0081) 0.0002(0.0076) 0.0101(0.0086)	0.0021(0.0067) 0.0069(0.0064) 0.0128(0.0848)	0.0064(0.0049) 0.0125(0.0049) 00005(0.0393)	0.0060(0.0035) 0.0124(0.0036) -0.0009(0.0029)	0.0058(0.0028) 0.0121(0.0029) 0.0004(0.0023)	0.0036(0.0022) 0.0096(0.0023) 0.0032(0.0020)
	1	-0.0069(0.0068) -0.0116(0.0066) 0.0076(0.0074)	-0.0008(0.0080) -0.0008(0.0075) 0.0429(0.0366)	0.0047(0.0076) 0.0077(0.0072) -0.0155(0.0606)	0.0063(0.0076) 0.0110(0.0065) -0.0038(0.0073)	0.0072(0.0058) 0.0128(0.0057) 0.0025(0.0053)	0.0058(0.0050) 0.0119(0.0050) 0.0022(0.0044)
	1.5	-0.0096(0.0054) -0.0157(0.0054) 0.0043(0.0053)	-0.0032(0.0072) -0.0063(0.0068) -0.0369(0.0410)	-0.0002(0.0077) -0.0002(0.0072) -0.0513(0.0182)	0.00180.0074) 0.0041(0.0070) -0.0079(0.0080)	0.0033(0.0069) 0.0071(0.0066) -0.0021(0.0063)	0.0052(0.0067) 0.0099(0.0064) -0.0049(0.0063)
(20,20)	0.5	0.0049(0.0061) 0.0048(0.0058) 0.0053(0.0059)	0.0064(0.0050) 0.0099(0.0049) 0.0245(0.0199)	0.0070(0.0037) 0.0116(0.0037) -0.0054(0.0068)	0.0068(0.0028) 0.0117(0.0028) 0.0005(0.0024)	0.0064(0.0021) 0.0112(0.0022) 0.0025(0.0019)	0.0060(0.0017) 0.0105(0.0018) 0.0030(0.0015)
	1	-0.0020(0.0046) -0.0057(0.0045) 0.0074(0.0045)	0.0017(0.0064) 0.0016(0.0061) 0.0216(0.0313)	0.0041(0.0060) 0.0063(0.0057) -0.0236(0.0082)	0.0053(0.0052) 0.0089(0.0050) -0.0062(0.0042)	0.0059(0.0044) 0.0102(0.0044) -0.0016(0.0038)	0.0062(0.0038) 0.0108(0.0038) 0.0003(0.0033)
	1.5	-0.0036(0.0038) -0.0083(0.0038) 0.0038(0.0031)	-0.0008(0.0060) -0.0031(0.0057) 0.0208(0.0377)	0.0017(0.0064) 0.0016(0.0061) -0.0318(0.0100)	0.0035(0.0062) 0.0051(0.0059) -0.0116(0.0051)	0.0046(0.0057) 0.0074(0.0055) -0.0057(0.0049)	0.0053(0.0052) 0.0089(0.0050) -0.0028(0.0045)
(25,25)	0.5	-0.0030(0.0049) -0.0030(0.0047) 0.0050(0.0054)	0.0001(0.0038) 0.0030(0.0038) -0.0206(0.0173)	0.0013(0.0028) 0.0051(0.0028) -0.0114(0.0036)	0.0018(0.0020) 0.0056(0.0021) 0.0063(0.0021)	0.0019(0.0015) 0.0057(0.0016) 0.0020(0.0017)	0.0019(0.0012) 0.0055(0.0012) 0.0025(0.0013)
	1	-0.0057(0.0040) -0.0085(0.0039) 0.0030(0.0041)	-0.0030(0.0049) -0.0030(0.0047) -0.0381(0.0284)	-0.0010(0.0045) 0.0082(0.0043) -0.0247(0.0071)	0.0015(0.0038) 0.0030(0.0038) -0.0044(0.0040)	0.0009(0.0033) 0.0043(0.0032) -0.0009(0.0035)	0.0013(0.0028) 0.0051(0.0028) 0.0006(0.0030)
	1.5	-0.0058(0.0030) -0.0095(0.0030) 0.0050(0.0026)	-0.0044(0.0046) -0.0062(0.0045) -0.0424(0.0347)	-0.0026(0.0050) -0.0025(0.0048) -0.0208(0.0060)	-0.0011(0.0048) 0.0002(0.0046) -0.0071(0.0042)	-0.0001(0.0044) 0.0021(0.0042) -0.0028(0.0040)	0.0006(0.0039) 0.0035(0.0038) -0.0005(0.0037)
(30,30)	0.5	0.0030(0.0043) 0.0030(0.0042) 0.0016(0.0040)	0.0053(0.0035) 0.0077(0.0035) -0.0044(0.0165)	0.0056(0.0026) 0.0087(0.0026) -0.0103(0.0034)	0.0054(0.0019) 0.0086(0.0019) -0.0012(0.0016)	0.0050(0.0014) 0.0082(0.0015) 0.0002(0.0012)	0.0047(0.0011) 0.0076(0.0012) 0.0005(0.0010)
	1	0.0014(0.0034) -0.0022(0.0034) 0.0005(0.0031)	0.0030(0.0043) 0.0030(0.0042) -0.0129(0.0257)	0.0046(0.0040) 0.0061(0.0039) -0.0208(0.0051)	0.0035(0.0035) 0.0077(0.0035) -0.0058(0.0030)	0.0056(0.0030) 0.0084(0.0030) -0.0030(0.0026)	0.0056(0.0026) 0.0087(0.0026) -0.0015(0.0022)
	1.5	-0.0098(0.0025) -0.0040(0.0025) -0.0828(0.0099)	0.0012(0.0040) -0.0003(0.0039) -0.0178(0.0299)	0.0030(0.0043) 0.0030(0.0042) -0.0265(0.0060)	0.0042(0.0042) 0.0053(0.0041) -0.0095(0.0037)	0.0049(0.0039) 0.0068(0.0038) -0.0060(0.0034)	0.0053(0.0035) 0.0077(0.0035) -0.0040(0.0031)

The first, second and third rows represent the average biases and MSEs of reliability R by MLE, Approximate Bayes (Lindley's approximation) and empirical Bayes for only two unknown parameters θ and λ.

Table (3): Confidence interval of R

		λ					
(n,m)	θ	0.5	1	1.5	2	2.5	3
(10,10)	0.5	0.5367 0.4226	0.4953 0.3854	0.4442 0.3408	0.3966 0.3003	0.3518 0.2631	0.3189 0.2363
	1	0.5974 0.3874	0.5366 0.4225	0.5197 0.4072	0.4919 0.3824	0.4711 0.3641	0.4445 0.3910
	1.5	0.4436 0.3403	0.5229 0.4102	0.5363 0.4226	0.5289 0.4156	0.5128 0.4011	0.4944 0.3846
(15,15)	0.5	0.3493 0.1999	0.4090 0.3157	0.3594 0.2744	0.3155 0.2386	0.3318 0.2105	0.2495 0.1860
	1	0.4113 0.3176	0.4487 0.3493	0.4355 0.3381	0.4774 0.3174	0.5049 0.2956	0.3589 0.2740
	1.5	0.3618 0.2764	0.4351 0.3378	0.5200 0.3493	0.5716 0.3434	0.6073 0.3310	0.4105 0.3170
(20,20)	0.5	0.2680 0.3043	0.3584 0.2757	0.4106 0.2377	0.4395 0.2063	0.4580 0.1815	0.4707 0.1617
	1	0.3565 0.2741	0.3931 0.3043	0.3808 0.2942	0.3580 0.2753	0.3338 0.2556	0.3111 0.2372
	1.5	0.3093 0.2358	0.3800 0.2934	0.3931 0.3043	0.3870 0.2992	0.3695 0.2883	0.3580 0.2753
(25,25)	0.5	0.3541 0.2732	0.3193 0.2449	0.2751 0.2095	0.2388 0.1808	0.2101 0.1584	0.1873 0.1406
	1	0.3216 0.2486	0.3541 0.2732	0.3414 0.2628	0.3193 0.2449	0.2964 0.2265	0.2751 0.2095
	1.5	0.2780 0.2118	0.3428 0.2639	0.3541 0.2732	0.3476 0.2679	0.3346 0.2753	0.3195 0.2451
(30,30)	0.5	0.3248 0.2500	0.2942 0.2254	0.2536 0.1932	0.2201 0.1668	0.1936 0.1462	0.1724 0.1298
	1	0.2920 0.2237	0.3248 0.2500	0.3142 0.2414	0.2942 0.2254	0.2732 0.2087	0.2536 0.1932
	1.5	0.2508 0.1909	0.3128 0.2403	0.3248 0.2500	0.3195 0.2458	0.3079 0.2364	0.2942 0.2254

The first row represents the average width of 99% confidence interval and the second row represent the average width of 95% confidence interval using the estimate of R as the MLE.

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