Bivariate Generalized Rayleigh Distribution Based on Clayton Copula

El-Sayed A. El-Sherpieny[1], Ehab M. Almetwally[1]

[1]Department of Mathematical Statistics, Faculty of Graduate Studies for Statistical Research, Cairo University, Egypt.

Abstract: The bivariate generalized Rayleigh distribution is an important lifetime distribution in survival analysis. In this paper, Clayton copula and generalized Rayleigh marginal distribution are used for creating bivariate distribution which are called bivariate generalized Rayleigh based on Clayton copula (Clayton-BGR) distribution. The reliability function and hazard function were obtained for bivariate generalized Rayleigh distribution. Two different estimation methods for the unknown parameters of the Clayton-BGR model were discussed. To evaluate the performance of the estimators, a Monte Carlo simulation study was conducted to compare the efficiency between the Clayton-BGR and anther models. Also, a real data set is analyzed to investigate the models and useful results are obtained for illustrative purposes.

Keywords: Generalized Rayleigh Distribution; Clayton copula; Maximum Likelihood Estimation; Inference Function for Margins, Monte Carlo Simulations.

1. Introduction

Burr (1942) introduced twelve different forms of cumulative distribution functions for modeling lifetime data. Among those distributions, Burr Type-X which this called the two-parameter generalized Rayleigh (GR) distribution is the most popular ones. Several authors consider different aspects of the generalized Rayleigh distribution, see for example: In Kundu and Raqab (2005), different estimation procedures have been used to estimate the unknown parameter of GR distribution. A random variable X has the two-parameter GR distribution with shape and scale parameters α, λ respectively, the cumulative distribution function (cdf) of GR distribution is given by

$$F(x;\alpha,\lambda) = \left(1 - e^{-(\lambda x)^2}\right)^{\alpha}; \quad x > 0, \alpha, \lambda > 0, \tag{1.1}$$

and the probability density function (pdf) of GR distribution is given by

$$f(x;\alpha,\lambda) = 2\alpha\lambda^2 x e^{-(\lambda x)^2}\left(1 - e^{-(\lambda x)^2}\right)^{\alpha-1}. \tag{1.2}$$

Many authors introduced many applications for GR distribution as follows: Raqab and Madi (2011) discussed the maximum likelihood and Bayesian approache based on progressive Type-II censoring for GR distribution. Tomer et al. (2014) have discussed the maximum likelihood and Bayesian estimation of the parameters for GR distribution and reliability function under Type-I progressive hybrid censoring scheme. Kundu and Raqab (2015) considered the estimation of the stress–strength parameter $R = P[Y < X]$, when X and Y are both three-parameter generalized Rayleigh distributions with the same scale and locations parameters but different shape parameters. Almetwally et al. (2019_b) discussed parameter estimation for the GR distribution under the adaptive Type-II progressive censoring schemes based on maximum product spacing and maximum likelihood estimation. There are many uses and multiple applications for this distribution.

The copula is a convenient approach to describe a bivariate distribution with dependence structure. Nelsen (2006) showed the definition of copulas as following; copula is a function that join bivariate distribution functions with uniform [0, 1] margins. Sklar (1973) introduced the joint pdf and joint cdf for the two dimension copula as follows, He considered the two random variables X_1 and X_2, with distribution functions $F_1(x_1)$ and $F_2(x_2)$ respectively, then the joint cdf and joint pdf for bivariate copula are respectively given as

$$F(x_1, x_2) = C(F_1(x_1)\,, F_2(x_2)), \tag{1.3}$$

Where C is denoted the cdf of copula and c is denoted the pdf of copula.

$$f(x_1, x_2) = f_1(x_1) f_2(x) c(F_1(x_1)\,, F_2(x_2)). \tag{1.4}$$

In Archimedian copulas, there are three copulas in common use: the Clayton, Frank and Gumbel. Nadarajah et al. (2018) discussed Clayton in general form as

$$C(u_1, u_2, \dots, u_p) = \left[\sum_{i=1}^{p} u_i^{-\alpha} - p + 1\right]^{-1/\alpha}, \alpha > 0, p \text{ is a number of variable}$$

In this paper, we will be discussed the 2-dimintion of Clayton copula, this is an asymmetric Archimedean copula, exhibiting greater dependence in the negative tail than in the positive, that have been introduced by Clayton (1978). The joint cdf of Clayton copula is given by:

$$C(u, v) = \left(\mathrm{u}^{-\theta} + \mathrm{v}^{-\theta} - 1\right)^{-1/\theta}, \tag{1.5}$$

where $u, v \in [0, 1]$ and the joint pdf of Clayton copula is given by:

$$c(u, v) = \frac{\partial^2 C(u, v)}{\partial u \partial v} = (\theta + 1)(uv)^{-\theta-1}\left(\mathrm{u}^{-\theta} + \mathrm{v}^{-\theta} - 1\right)^{\frac{-2\theta-1}{\theta}}. \tag{1.6}$$

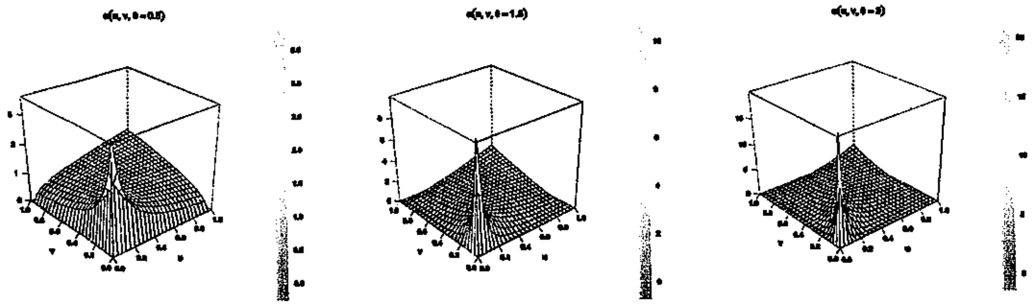


Figure 1. The pdf of Clayton copula with Various Value of the Copula Parameter (θ).

Flores (2009) presented six different bivariate Weibull distributions derived from different copula functions specially Bivariate Weibull based on Clayton copula can be denoted Clayton-BW distribution. Almetwally (2019) showed Clayton copula and introduced bivariate Weibull distribution based on Farlie–Gumbel–Morgenstern (FGM) copula. Fredricks and Nelsen (2007) derived the formula for Spearman's and Kendall's correlation coefficient. In case of Clayton copula, the Spearman's and Kendall's correlation as follows

$$\rho_{sperman} = \left(12 \int\int \left(\mathrm{u}^{-\theta} + \mathrm{v}^{-\theta} - 1\right)^{-1/\theta} du\, dv\right) - 3, \tag{1.7}$$

and

$$\rho_{\text{Kendall}} = 1 - 4 \int\int \frac{\partial C}{\partial u} C(u,v) \frac{\partial C}{\partial v} C(u,v)\, du\, dv = \frac{\theta}{\theta+2}, \tag{1.8}$$

where $\frac{\partial C}{\partial u} C(u,v) = \mathrm{u}^{-\theta-1}\left(\mathrm{u}^{-\theta} + \mathrm{v}^{-\theta} - 1\right)^{\frac{-\theta-1}{\theta}}$, $\frac{\partial C}{\partial v} C(u,v) = \mathrm{v}^{-\theta-1}\left(\mathrm{u}^{-\theta} + \mathrm{v}^{-\theta} - 1\right)^{\frac{-\theta-1}{\theta}}$.

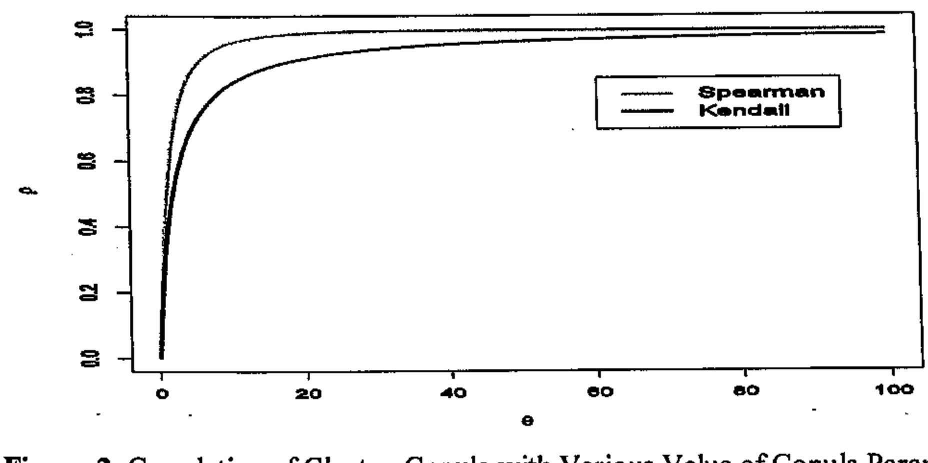


Figure 2. Correlation of Clayton Copula with Various Value of Copula Parameter.

In Bivariate Generalized Rayleigh Distribution, Abdel-Hady (2013) introduced Marshall and Olkin's bivariate exponential model to the generalized bivariate Rayleigh (GBR) Distribution. Sarhan (2019) introduced bivariate generalized Rayleigh distribution of type of

Marshall-Olkin (Shock model), this can be denoted BMOGR and the maximum likelihood and Bayes methods were applied to estimate the unknown parameters for BMOGR distribution. Shubhashree (2019) discussed parameter estimation of multicomponent stress-strength system reliability for a BGR distribution.

The main aim of this paper is to use the similar idea as of El-Sherpieny et al. (2018) to introduce a new extension for bivariate generalized Rayleigh distribution, using the GR distributions and Clayton copula function. The proposed distribution has five parameters. The joint cdf, the joint pdf, the joint survival function and the joint hazard rate function of the Clayton-BGR distribution are derived in closed forms. Parameter estimation for Clayton bivariate GR distribution is introduced by two estimation methods which are: the MLE and inference functions for margins (IFM). Therefore, numerical methods as Monte Carlo Simulation are required to calculate them.

The rest of this paper is organized as follows: Clayton bivariate GR distribution is obtained in Section 2. Parameter estimation methods for the Clayton bivariate GR distribution are obtained in Section 3. In Section 4, the potentiality of the new model is illustrated by Monte Carlo simulation study. In Section 5, Application of a real data set is discussed. Finally, Conclusions are addressed in Section 6.

2. Clayton Bivariate GR Distribution

According to Sklar theorem, using (1.1) and (1.2) in (1.3) and (1.4), the joint cdf of the bivariate GR distribution for any copula is as follows

$$F(x_1,x_2) = C\left(\left(1-e^{-(\lambda_1 x_1)^2}\right)^{\alpha_1}, \left(1-e^{-(\lambda_2 x_2)^2}\right)^{\alpha_2}\right), \tag{2.1}$$

and the joint pdf of the bivariate GR distribution for any copula is as follows

$$\begin{aligned} f(x_1,x_2) = {} & 4\alpha_1\lambda_1^2 x_1 e^{-(\lambda_1 x_1)^2}\left(1-e^{-(\lambda_1 x_1)^2}\right)^{\alpha_1-1}\alpha_2\lambda_2^2 x_2 e^{-(\lambda_2 x_2)^2}\Big(1 \\ & - e^{-(\lambda_2 x_2)^2}\Big)^{\alpha_2-1} \\ & c\left(\left(1-e^{-(\lambda_1 x_1)^2}\right)^{\alpha_1}, \left(1-e^{-(\lambda_2 x_2)^2}\right)^{\alpha_2}\right). \end{aligned} \tag{2.2}$$

In the following, The Equations (2.1) and (2.2) are used to define the Clayton bivariate GR distribution. The cdf, pdf, reliability or survival function and hazard function for Clayton-BGR distribution are obtained as follows:

$$F(x_1,x_2) = \left[\left(\left(1-e^{-(\lambda_1 x_1)^2}\right)^{-\theta\alpha_1} + \left(1-e^{-(\lambda_2 x_2)^2}\right)^{-\theta\alpha_2} - 1\right)^{-1/\theta}\right], \tag{2.3}$$

$$\begin{aligned} f(x_1,x_2) = {} & 4\alpha_1\lambda_1^2 x_1 e^{-(\lambda_1 x_1)^2}\left(1-e^{-(\lambda_1 x_1)^2}\right)^{\alpha_1-1}\alpha_2\lambda_2^2 x_2 e^{-(\lambda_2 x_2)^2}\left(1-e^{-(\lambda_2 x_2)^2}\right)^{\alpha_2-1}, \\ & (\theta+1)\left(\left(1-e^{-(\lambda_1 x_1)^2}\right)^{\alpha_1}\left(1-e^{-(\lambda_2 x_2)^2}\right)^{\alpha_2}\right)^{-\theta-1}\Big(\left(1-e^{-(\lambda_1 x_1)^2}\right)^{-\theta\alpha_1} \\ & + \left(1-e^{-(\lambda_2 x_2)^2}\right)^{-\theta\alpha_2} - 1\Big)^{\frac{-2\theta-1}{\theta}}, \end{aligned} \tag{2.4}$$

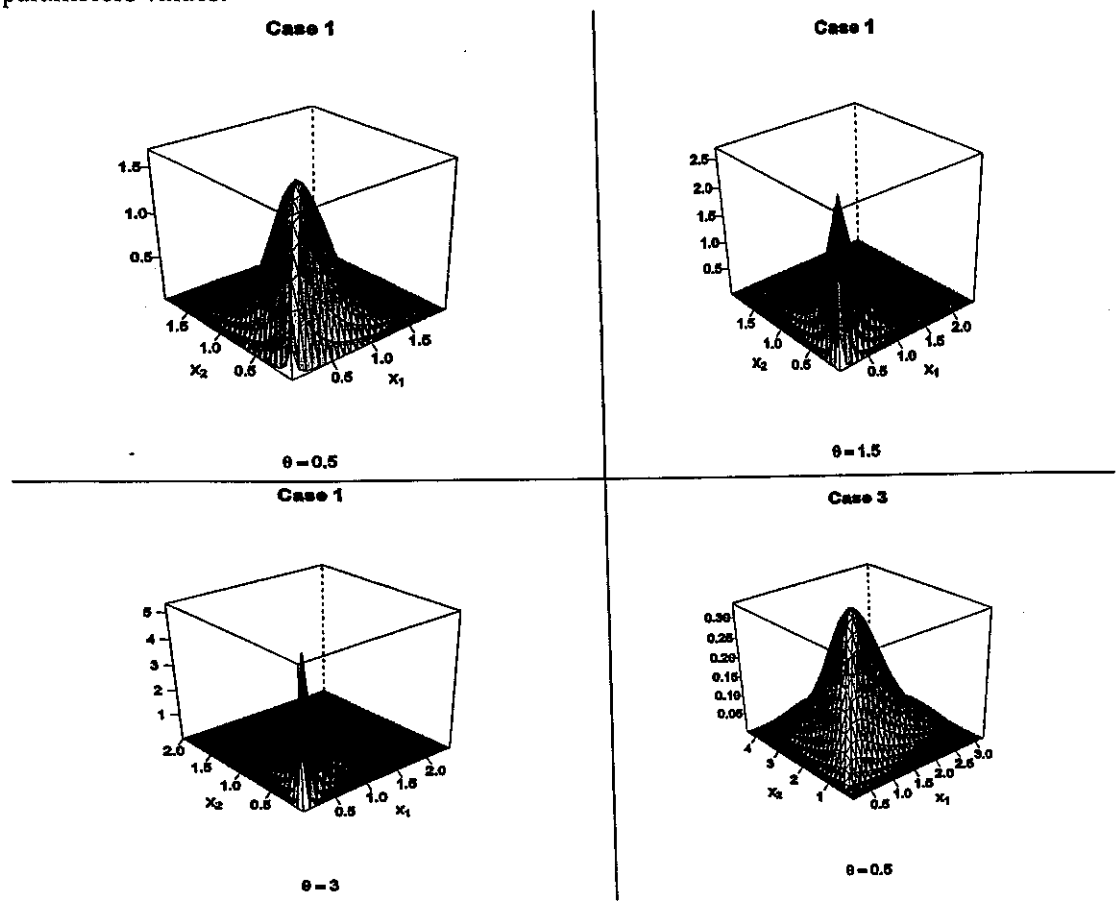
$$R(x_1,x_2)=\left(\left(1-\left(1-e^{-(\lambda_1 x_1)^2}\right)^{\alpha_1}\right)^{-\theta}+\left(1-\left(1-e^{-(\lambda_2 x_2)^2}\right)^{\alpha_2}\right)^{-\theta}-1\right)^{-1/\theta}, \quad (2.5)$$

and

$$h(x_1,x_2)=\frac{f(x_1,x_2)}{\left(\left(1-\left(1-e^{-(\lambda_1 x_1)^2}\right)^{\alpha_1}\right)^{-\theta}+\left(1-\left(1-e^{-(\lambda_2 x_2)^2}\right)^{\alpha_2}\right)^{-\theta}-1\right)^{-1/\theta}}, \quad (2.6)$$

respectively, where $\alpha_1,\lambda_1,\alpha_2,\lambda_2$, $\theta>0$ and $x_1,x_2>0$

Figure 3 shows the plot 3-dimension for the pdf of Clayton-BGR distribution with different parameters values.



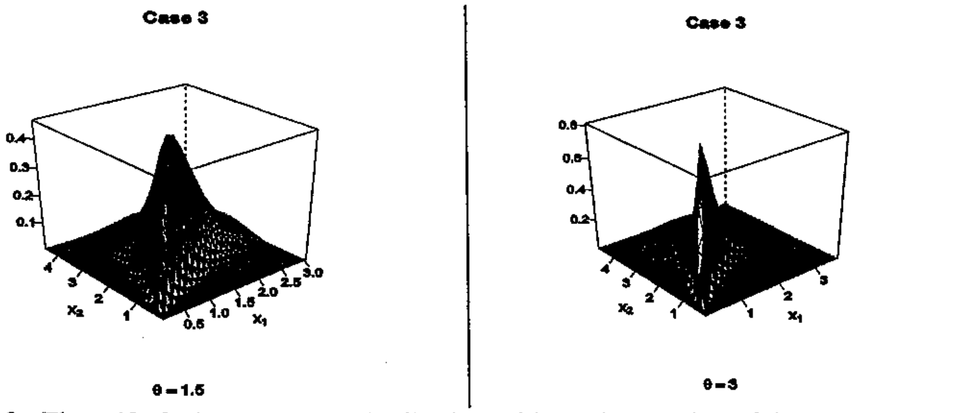
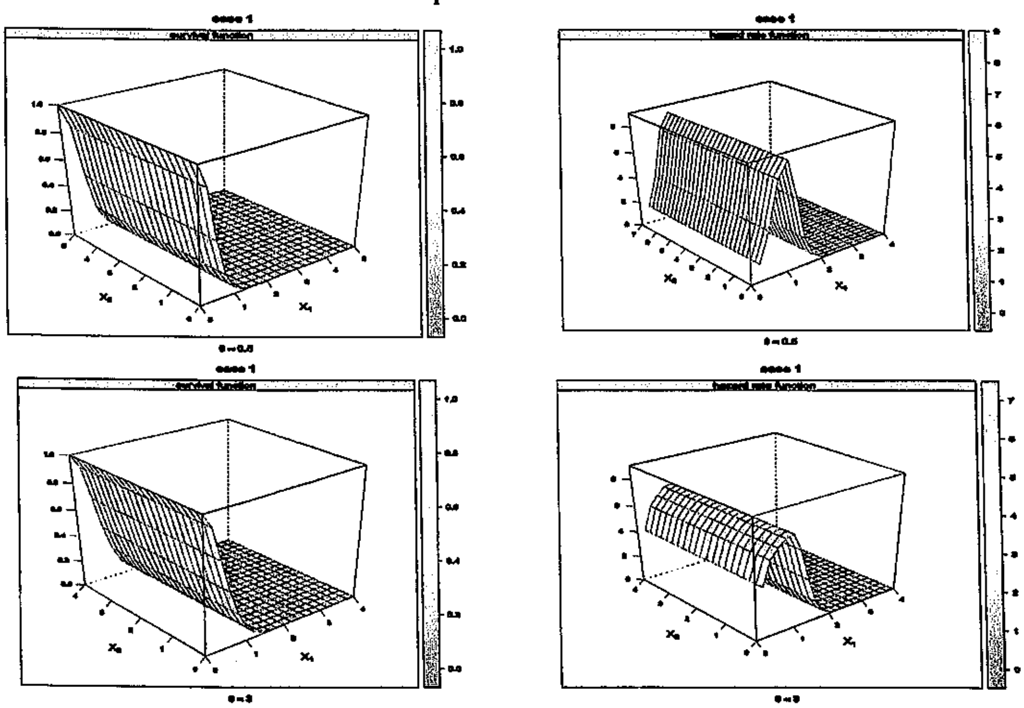


Figure 3. The pdf of Clayton-BGR Distribution with Various Value of the Parameters.

Figure 4 shows the plot 3-dimension for the survival function and hazard rate function of Clayton-BGR distribution with different parameters values.



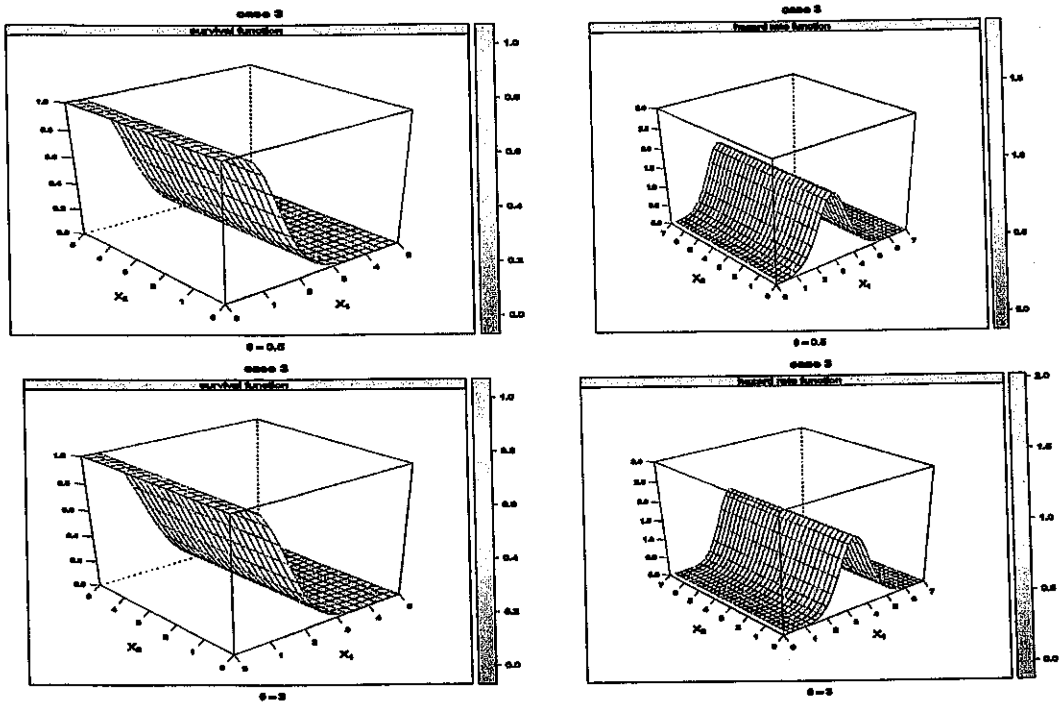


Figure 4. Survival Function and Hazard Rate Function of Clayton-BGR Distribution with Various Value of the Parameters.

As shown in the Figure 3, 4, the curve of Clayton-BGR distribution behavior have more than one direction where it takes increasing and decreasing direction, which will have many applications in life testing. Sub-model can be derived from Clayton-BGR distribution as Clayton bivariate Rayleigh (Clayton-BR) distribution (new) when $\alpha_1 = 1, \lambda_1 = \lambda_1, \alpha_2 = 1, \lambda_2 = \lambda_2, \theta = \theta$.

3. Parameter Estimation Methods

In the section, we introduce two estimation methods that used to estimate the unknown parameters of Clayton-BGR distribution, such as: MLE and IFM. To more information about these methods see Kim et al. (2007) and to more example see Elaal and Jarwan (2017) and Almetwally et al. (2019_a).

3.1. Maximum Likelihood Estimation (MLE)

The log-likelihood function of Clayton-BGR distribution can be written as

$$l(\Omega) = 2n(\ln(\lambda_1) + ln(\lambda_2)) + n(\ln(4) + \ln(\alpha_1) + \ln(\alpha_2) + \ln(\theta + 1)) + \sum_{i=1}^{n}(\ln(x_{1i}) + \ln(x_{2i})) - \sum_{i=1}^{n}(\lambda_1 x_{1i})^2 - \sum_{i=1}^{n}(\lambda_2 x_{2i})^2 + (-1 - \theta\alpha_1)\sum_{i=1}^{n}\ln\left(1 - e^{-(\lambda_1 x_{1i})^2}\right) + (-1 - \theta\alpha_2)\sum_{i=1}^{n}\ln\left(1 - e^{-(\lambda_2 x_{2i})^2}\right) - \frac{2\theta + 1}{\theta}\sum_{i=1}^{n}\ln\left(\left(1 - e^{-(\lambda_1 x_{1i})^2}\right)^{-\theta\alpha_1} + \left(1 - e^{-(\lambda_2 x_{2i})^2}\right)^{-\theta\alpha_2} - 1\right) \tag{3.1}$$

where Ω is a vector of parameters.

To obtain the normal likelihood equations for the unknown parameters, we differentiate equation (3.1) partially with Ω vector of parameters and equate them to zero. The estimators for $\widehat{\Omega}_{MLE}$ can be obtained as in the solution of the following equations:

$$\frac{\partial l(\Omega)}{\partial \alpha_j} = \frac{n}{\alpha_j} - \theta\sum_{i=1}^{n}\ln\left(1 - e^{-(\lambda_j x_{ji})^2}\right) - \frac{2\theta+1}{\theta}\sum_{i=1}^{n}\frac{-\theta\left(1-e^{-(\lambda_j x_{ji})^2}\right)^{-\theta\alpha_j}\ln\left(1-e^{-(\lambda_j x_{ji})^2}\right)}{\left(1-e^{-(\lambda_1 x_{1i})^2}\right)^{-\theta\alpha_1}+\left(1-e^{-(\lambda_2 x_{2i})^2}\right)^{-\theta\alpha_2}-1} = 0,$$

$$\frac{\partial l(\Omega)}{\partial \lambda_j} = \frac{2n}{\lambda_j} + 2\lambda_j\sum_{i=1}^{n}x_{ji}^2 - \left(1 + \theta\alpha_j\right)\sum_{i=1}^{n}\frac{2\lambda_j x_{ji}^2 e^{-(\lambda_j x_{ji})^2}}{1-e^{-(\lambda_j x_{ji})^2}} - \frac{2\theta+1}{\theta}\sum_{i=1}^{n}\frac{-\theta\alpha_j 2\lambda_j x_{ji}^2\left(1-e^{-(\lambda_1 x_{ji})^2}\right)^{-\theta\alpha_j-1}}{\left(1-e^{-(\lambda_1 x_{1i})^2}\right)^{-\theta\alpha_1}+\left(1-e^{-(\lambda_2 x_{2i})^2}\right)^{-\theta\alpha_2}-1} = 0,$$

and

$$\frac{\partial l(\Omega)}{\partial \theta} = \frac{n}{\theta+1} - \alpha_j\sum_{i=1}^{n}\ln\left(1 - e^{-(\lambda_j x_{ji})^2}\right) + \frac{1}{\theta^2}\sum_{i=1}^{n}\ln\left(\left(1 - e^{-(\lambda_1 x_{1i})^2}\right)^{-\theta\alpha_1} + \left(1 - e^{-(\lambda_2 x_{2i})^2}\right)^{-\theta\alpha_2} - 1\right) - \frac{2\theta+1}{\theta}\sum_{i=1}^{n}\frac{-\alpha_j\left(1-e^{-(\lambda_j x_{ji})^2}\right)^{-\theta\alpha_j}\ln\left(1-e^{-(\lambda_j x_{ji})^2}\right)-\alpha_l\left(1-e^{-(\lambda_l x_{li})^2}\right)^{-\theta\alpha_l}\ln\left(1-e^{-(\lambda_l x_{li})^2}\right)}{\left(1-e^{-(\lambda_1 x_{1i})^2}\right)^{-\theta\alpha_1}+\left(1-e^{-(\lambda_2 x_{2i})^2}\right)^{-\theta\alpha_2}-1} = 0,$$

where $j = l = 1,2\ ; j \neq l$. The MLEs $\widehat{\Omega}_{MLE}$ are obtained by performing numerically using a nonlinear optimization algorithm.

3.2. Estimation by Inference Functions for Margins (IFM)

Kim et al. (2007) discussed this parametric method with two-step of estimation. Firstly, each marginal of GR distribution is estimated separately, by using Equation (1.2), the log-likelihood function of GR distribution can be written as

$$\ln L_j(\alpha_j,\lambda_j)=\sum_{i=1}^{n}\ln f_j(x_{ji},\alpha_j,\lambda_j)\,;\ j=1,2$$

$$\ln L_j(\alpha_j,\lambda_j)=n\left(\ln(2)+\ln(\alpha_j)+2\ln(\lambda_j)\right)+\sum_{i=1}^{n}\left(\ln(x_{ji})\right)-\sum_{i=1}^{n}\left(\lambda_j x_{ji}\right)^2 \qquad (3.2)$$
$$+(\alpha_j-1)\sum_{i=1}^{n}\ln\left(1-e^{-(\lambda_j x_{ji})^2}\right)$$

To obtain the normal likelihood equations for the unknown parameters, we differentiate equation (3.2) partially with α_j,λ_j vector of parameters and equate them to zero. The estimators for $\hat{\alpha}_j,\hat{\lambda}_j$ can be obtained as in the solution of the following equations:

$$\frac{\partial\ln L_j(\alpha_j,\lambda_j)}{\partial\alpha_j}=\frac{n}{\alpha_j}+\sum_{i=1}^{n}\ln\left(1-e^{-(\lambda_j x_{ji})^2}\right),$$

$$\frac{\partial\ln L_j(\alpha_j,\lambda_j)}{\partial\lambda_j}=\frac{2n}{\lambda_j}+2\lambda_j\sum_{i=1}^{n}x_{ji}^2+(\alpha_j-1)\sum_{i=1}^{n}\frac{2\lambda_j x_{ji}^2 e^{-(\lambda_j x_{ji})^2}}{1-e^{-(\lambda_j x_{ji})^2}}.$$

The MLEs$(\hat{\alpha}_j,\hat{\lambda}_j)$ can be obtained by solving simultaneously the likelihood equations

$$\left.\frac{\partial\ln L_j(\alpha_j,\lambda_j)}{\partial\lambda_j}\right|_{\lambda_j=\hat{\lambda}_j}=0,\qquad \left.\frac{\partial\ln L_j(\alpha_j,\lambda_j)}{\partial\alpha_j}\right|_{\alpha_j=\hat{\alpha}_j}=0,\qquad j=1,2$$

In the second step, the copula parameter is estimated by maximizing the log-likelihood function of the copula density using the MLE estimates of the marginal $\hat{F}_{1i}(x_{1i},\hat{\alpha}_1,\hat{\lambda}_1)$ and $\hat{F}_{2i}(x_{2i},\hat{\alpha}_2,\hat{\lambda}_2)$. The log-likelihood function by using IFM method estimation of any copula function is defined as

$$l(\theta)=\ln L_{IFM}(\theta)=\sum_{i=1}^{n}\ln\left(c\left(\hat{F}_{1i}(x_{1i},\hat{\alpha}_1,\hat{\lambda}_1),\hat{F}_{2i}(x_{2i},\hat{\alpha}_2,\hat{\lambda}_2)\right)\right).$$

By using Equation (1.6) then, the log-likelihood function of Clayton copula function can be written as

$$l(\theta)=n\ln(\theta+1)-(\theta+1)\sum_{i=1}^{n}\ln\left(\hat{F}_{1i}(x_{1i},\hat{\alpha}_1,\hat{\lambda}_1)\hat{F}_{2i}(x_{2i},\hat{\alpha}_2,\hat{\lambda}_2)\right)$$
$$-\frac{2\theta+1}{\theta}\sum_{i=1}^{n}\ln\left(\hat{F}_{1i}(x_{1i},\hat{\alpha}_1,\hat{\lambda}_1)^{-\theta}+\hat{F}_{2i}(x_{2i},\hat{\alpha}_2,\hat{\lambda}_2)^{-\theta}-1\right). \qquad (3.3)$$

Basing on this, differentiating the log-likelihood function with respect to θ for Clayton-BGR distribution is given as

$$\frac{\partial l(\theta)}{\partial \theta} = \frac{n}{\theta+1} - \sum_{i=1}^{n} \ln\left(\hat{F}_{1i}(x_{1i}, \hat{\alpha}_1, \hat{\lambda}_1)\hat{F}_{2i}(x_{2i}, \hat{\alpha}_2, \hat{\lambda}_2)\right) + \frac{1}{\theta^2}\sum_{i=1}^{n} \ln\left(\hat{F}_{1i}(x_{1i}, \hat{\alpha}_1, \hat{\lambda}_1)^{-\theta} + \hat{F}_{2i}(x_{2i}, \hat{\alpha}_2, \hat{\lambda}_2)^{-\theta} - 1\right) - \frac{2\theta+1}{\theta}\sum_{i=1}^{n} \frac{-\hat{F}_{1i}(x_{1i}, \hat{\alpha}_1, \hat{\lambda}_1)^{-\theta}\ln\left(\hat{F}_{1i}(x_{1i}, \hat{\alpha}_1, \hat{\lambda}_1)\right) - \hat{F}_{2i}(x_{2i}, \hat{\alpha}_2, \hat{\lambda}_2)^{-\theta}\ln\left(\hat{F}_{2i}(x_{2i}, \hat{\alpha}_2, \hat{\lambda}_2)\right)}{\hat{F}_{1i}(x_{1i}, \hat{\alpha}_1, \hat{\lambda}_1)^{-\theta} + \hat{F}_{2i}(x_{2i}, \hat{\alpha}_2, \hat{\lambda}_2)^{-\theta} - 1}$$

The estimates of copula parameter is handled numerically simultaneously of the likelihood equations

$$\left.\frac{\partial l(\theta)}{\partial \theta}\right|_{\theta=\hat{\theta}} = 0$$

There is no closed-form expression for the MLE $\hat{\theta}$ and its computation has to be performed numerically using a nonlinear optimization algorithm.

4. Simulation Study

In this section; a Monte Carlo simulation is done for comparison between estimation methods based on copula such as: MLE and IFM. For estimating Clayton-BGR distribution parameters by R packages.

To generate random variables: Nelsen (2006) discussed generating a sample from a specified joint distribution. By conditional distribution method, the joint distribution function is as follows

$$f(x_1, x_2) = f(x_1)f(x_2|x_1)$$

We used the R packages in copula package to generate the random variables of Clayton-BGR distribution.

Simulation Algorithm: A simulation experiments were carried out based on the following data-generated form Clayton-BGR distributions, the values of the parameters $\alpha_1, \lambda_1, \alpha_2, \lambda_2 and\ \theta$ is chosen as the following cases for the random variables generating:

Case 1: $\left(\alpha_1 = 1.2, \lambda_1 = 1.4, \alpha_2 = 1.1, \lambda_2 = 1.5, \theta = (0.5 \text{ or } 1.5 \text{ or } 3 \text{ or } 6)\right)$,

Case 2: $\left(\alpha_1 = 0.5, \lambda_1 = 1.4, \alpha_2 = 0.6, \lambda_2 = 1.5, \theta = (0.5 \text{ or } 1.5 \text{ or } 3 \text{ or } 6)\right)$,

Case 3: $\left(\alpha_1 = 2, \lambda_1 = 0.7, \alpha_2 = 2.5, \lambda_2 = 0.5, \theta = (0.5 \text{ or } 1.5 \text{ or } 3 \text{ or } 6)\right)$.

For different sample size $n = 50, 100, 200$. The simulation methods are compared using the criteria of parameters estimation, the comparison is performed by calculate in the Bias and the Root mean square error (RMSE) for each method of estimation as following

$$Bias = (\widehat{\Omega} - \Omega)\,. \tag{4.1}$$

where $\widehat{\Omega}$ is the estimated value of Ω, and RMSE is as follows

$$RMSE = mean\sqrt{(\widehat{\Omega} - \Omega)^2}. \tag{4.2}$$

We restricted the number of repeated-samples to 1000.

From Table (1) to Table(9) we can conclude the following :

i. By increasing the sample size, the RMSE and Bias of the considered parameters of Clayton-BGR distribution are decreases.

ii. If the copula parameter θ increases for any cases, then the value of the Bias and RMSE increase for the parameters especially θ of Clayton-BGR distribution.

iii. In IFM method, we note that the values of the $\hat{\alpha}_1, \hat{\lambda}_1, \hat{\alpha}_2, \hat{\lambda}_2$ parameters estimation are repeated with changing of θ due to the properties of the first-step in IFM method, which parameters estimated of marginal of GR distribution. They are not affected by the change of θ.

iv. The IFM estimates has more efficiency than MLE for copula parameter of Clayton-BGR distribution.

The IFM method is better than MLE method, it is noted that result, IFM method is the best method because it is a two steps of estimation, first, the marginal distribution parameters estimated and second the copula parameter is estimated, taking into consideration of previous parameter estimates of marginal distribution.

Table 1: Estimation of the Parameters of Clayton-BGR Distribution: Case 1, n=50

θ		MLE			IFM		
		Mean	Bias	RMSE	Mean	Bias	RMSE
0.5	$\hat{\alpha}_1$	1.2658	0.0658	0.2736	1.2686	0.0686	0.2792
	$\hat{\lambda}_1$	1.4299	0.0299	0.1340	1.4307	0.0307	0.1361
	$\hat{\alpha}_2$	1.1545	0.0545	0.2303	1.1543	0.0543	0.2318
	$\hat{\lambda}_2$	1.5296	0.0296	0.1422	1.5294	0.0294	0.1422
	$\hat{\theta}$	0.5368	0.0368	0.2548	0.5292	0.0292	0.2485
1.5	$\hat{\alpha}_1$	1.2644	0.0644	0.2639	1.2686	0.0686	0.2792
	$\hat{\lambda}_1$	1.4301	0.0301	0.1286	1.4307	0.0307	0.1361
	$\hat{\alpha}_2$	1.1506	0.0506	0.2195	1.1518	0.0518	0.2291
	$\hat{\lambda}_2$	1.5305	0.0305	0.1377	1.5297	0.0297	0.1433
	$\hat{\theta}$	1.5787	0.0787	0.4506	1.5446	0.0446	0.4392
3	$\hat{\alpha}_1$	1.2644	0.0644	0.2595	1.2686	0.0686	0.2792
	$\hat{\lambda}_1$	1.4305	0.0305	0.1226	1.4307	0.0307	0.1361
	$\hat{\alpha}_2$	1.1526	0.0526	0.2220	1.1545	0.0545	0.2384
	$\hat{\lambda}_2$	1.5323	0.0323	0.1331	1.5313	0.0313	0.1465
	$\hat{\theta}$	3.1345	0.1345	0.7527	3.0370	0.0370	0.7268
6	$\hat{\alpha}_1$	1.2639	0.0639	0.2554	1.2686	0.0686	0.2792
	$\hat{\lambda}_1$	1.4308	0.0308	0.1173	1.4307	0.0307	0.1361
	$\hat{\alpha}_2$	1.1548	0.0548	0.2247	1.1564	0.0564	0.2440
	$\hat{\lambda}_2$	1.5339	0.0339	0.1290	1.5320	0.0320	0.1489
	$\hat{\theta}$	6.2542	0.2542	1.3608	5.9452	-0.0548	1.2999

Table 2: Estimation of the Parameters of Clayton-BGR Distribution: Case 1, n=100

		MLE			IFM		
		Mean	Bias	RMSE	Mean	Bias	RMSE
0.5	$\hat{\alpha}_1$	1.2315	0.0315	0.1794	1.2315	0.0315	0.1804
	$\hat{\lambda}_1$	1.4109	0.0109	0.0900	1.4110	0.0110	0.0907
	$\hat{\alpha}_2$	1.1246	0.0246	0.1519	1.1258	0.0258	0.1535
	$\hat{\lambda}_2$	1.5101	0.0101	0.0953	1.5106	0.0106	0.0967
	$\hat{\theta}$	0.5155	0.0155	0.1808	0.5120	0.0120	0.1790
1.5	$\hat{\alpha}_1$	1.2298	0.0298	0.1735	1.2315	0.0315	0.1804
	$\hat{\lambda}_1$	1.4103	0.0103	0.0859	1.4110	0.0110	0.0907
	$\hat{\alpha}_2$	1.1248	0.0248	0.1521	1.1267	0.0267	0.1590
	$\hat{\lambda}_2$	1.5102	0.0102	0.0922	1.5109	0.0109	0.0986
	$\hat{\theta}$	1.5303	0.0303	0.3173	1.5125	0.0125	0.3140
3	$\hat{\alpha}_1$	1.2283	0.0283	0.1698	1.2315	0.0315	0.1804
	$\hat{\lambda}_1$	1.4097	0.0097	0.0817	1.4110	0.0110	0.0907
	$\hat{\alpha}_2$	1.1245	0.0245	0.1502	1.1261	0.0261	0.1603
	$\hat{\lambda}_2$	1.5101	0.0101	0.0883	1.5104	0.0104	0.0991
	$\hat{\theta}$	3.0555	0.0555	0.5205	3.0043	0.0043	0.5132
6	$\hat{\alpha}_1$	1.2277	0.0277	0.1668	1.2315	0.0315	0.1804
	$\hat{\lambda}_1$	1.4096	0.0096	0.0780	1.4110	0.0110	0.0907
	$\hat{\alpha}_2$	1.1247	0.0247	0.1491	1.1256	0.0256	0.1620
	$\hat{\lambda}_2$	1.5106	0.0106	0.0855	1.5098	0.0098	0.0995
	$\hat{\theta}$	6.1031	0.1031	0.9186	5.9378	-0.0622	0.9039

Table 3: Estimation of the Parameters of Clayton-BGR Distribution: Case 1, n=200

θ		MLE			IFM		
		Mean	Bias	RMSE	Mean	Bias	RMSE
0.5	$\hat{\alpha}_1$	1.2137	0.0137	0.1157	1.2142	0.0142	0.1166
	$\hat{\lambda}_1$	1.4066	0.0066	0.0614	1.4068	0.0068	0.0621
	$\hat{\alpha}_2$	1.1127	0.0127	0.1046	1.1131	0.0131	0.1055
	$\hat{\lambda}_2$	1.5073	0.0073	0.0676	1.5074	0.0074	0.0684
	$\hat{\theta}$	0.5161	0.0161	0.1292	0.5143	0.0143	0.1284
1.5	$\hat{\alpha}_1$	1.2139	0.0139	0.1127	1.2142	0.0142	0.1166
	$\hat{\lambda}_1$	1.4068	0.0068	0.0587	1.4068	0.0068	0.0621
	$\hat{\alpha}_2$	1.1108	0.0108	0.1021	1.1118	0.0118	0.1049
	$\hat{\lambda}_2$	1.5065	0.0065	0.0649	1.5070	0.0070	0.0686
	$\hat{\theta}$	1.5273	0.0273	0.2218	1.5186	0.0186	0.2199
3	$\hat{\alpha}_1$	1.2137	0.0137	0.1111	1.2142	0.0142	0.1166
	$\hat{\lambda}_1$	1.4067	0.0067	0.0558	1.4068	0.0068	0.0621
	$\hat{\alpha}_2$	1.1110	0.0110	0.1005	1.1118	0.0118	0.1043
	$\hat{\lambda}_2$	1.5068	0.0068	0.0620	1.5071	0.0071	0.0683
	$\hat{\theta}$	3.0434	0.0434	0.3638	3.0186	0.0186	0.3590
6	$\hat{\alpha}_1$	1.2137	0.0137	0.1098	1.2142	0.0142	0.1166
	$\hat{\lambda}_1$	1.4068	0.0068	0.0530	1.4068	0.0068	0.0621
	$\hat{\alpha}_2$	1.1116	0.0116	0.0996	1.1124	0.0124	0.1046
	$\hat{\lambda}_2$	1.5073	0.0073	0.0594	1.5076	0.0076	0.0683
	$\hat{\theta}$	6.0741	0.0741	0.6461	5.9911	-0.0089	0.6330

Table 4: Estimation of the Parameters of Clayton-BGR Distribution: Case 2, n=50

θ	n=50	MLE			IFM		
		Mean	Bias	RMSE	Mean	Bias	RMSE
0.5	$\hat{\alpha}_1$	0.5191	0.0191	0.0937	0.5199	0.0199	0.0952
	$\hat{\lambda}_1$	1.4433	0.0433	0.1772	1.4444	0.0444	0.1794
	$\hat{\alpha}_2$	0.6232	0.0232	0.1098	0.6233	0.0233	0.1104
	$\hat{\lambda}_2$	1.5375	0.0375	0.1707	1.5375	0.0375	0.1708
	$\hat{\theta}$	0.5367	0.0367	0.2530	0.5299	0.0299	0.2470
1.5	$\hat{\alpha}_1$	0.5188	0.0188	0.0914	0.5199	0.0199	0.0952
	$\hat{\lambda}_1$	1.4430	0.0430	0.1719	1.4444	0.0444	0.1794
	$\hat{\alpha}_2$	0.6212	0.0212	0.1058	0.6220	0.0220	0.1100
	$\hat{\lambda}_2$	1.5383	0.0383	0.1662	1.5383	0.0383	0.1729
	$\hat{\theta}$	1.5776	0.0776	0.4455	1.5466	0.0466	0.4350
3	$\hat{\alpha}_1$	0.5190	0.0190	0.0911	0.5199	0.0199	0.0952
	$\hat{\lambda}_1$	1.4431	0.0431	0.1657	1.4444	0.0444	0.1794
	$\hat{\alpha}_2$	0.6218	0.0218	0.1067	0.6231	0.0231	0.1138
	$\hat{\lambda}_2$	1.5396	0.0396	0.1599	1.5404	0.0404	0.1770
	$\hat{\theta}$	3.1321	0.1321	0.7454	3.0437	0.0437	0.7217
6	$\hat{\alpha}_1$	0.5191	0.0191	0.0909	0.5199	0.0199	0.0952
	$\hat{\lambda}_1$	1.4433	0.0433	0.1607	1.4444	0.0444	0.1794
	$\hat{\alpha}_2$	0.6226	0.0226	0.1080	0.6238	0.0238	0.1157
	$\hat{\lambda}_2$	1.5406	0.0406	0.1534	1.5411	0.0411	0.1797
	$\hat{\theta}$	6.2489	0.2489	1.3529	5.9675	-0.0325	1.2948

Table 5: Estimation of the Parameters of Clayton-BGR Distribution: Case 2, n=100

θ		MLE			IFM		
		Mean	Bias	RMSE	Mean	Bias	RMSE
0.5	$\hat{\alpha}_1$	0.5101	0.0101	0.0634	0.5099	0.0099	0.0634
	$\hat{\lambda}_1$	1.4165	0.0165	0.1178	1.4166	0.0166	0.1182
	$\hat{\alpha}_2$	0.6114	0.0114	0.0745	0.6118	0.0118	0.0751
	$\hat{\lambda}_2$	1.5140	0.0140	0.1150	1.5144	0.0144	0.1164
	$\hat{\theta}$	0.5148	0.0148	0.1791	0.5118	0.0118	0.1774
1.5	$\hat{\alpha}_1$	0.5097	0.0097	0.0620	0.5099	0.0099	0.0634
	$\hat{\lambda}_1$	1.4159	0.0159	0.1138	1.4166	0.0166	0.1182
	$\hat{\alpha}_2$	0.6114	0.0114	0.0748	0.6121	0.0121	0.0777
	$\hat{\lambda}_2$	1.5138	0.0138	0.1111	1.5145	0.0145	0.1180
	$\hat{\theta}$	1.5286	0.0286	0.3131	1.5132	0.0132	0.3102
3	$\hat{\alpha}_1$	0.5093	0.0093	0.0613	0.5099	0.0099	0.0634
	$\hat{\lambda}_1$	1.4150	0.0150	0.1093	1.4166	0.0166	0.1182
	$\hat{\alpha}_2$	0.6112	0.0112	0.0741	0.6117	0.0117	0.0783
	$\hat{\lambda}_2$	1.5132	0.0132	0.1053	1.5136	0.0136	0.1182
	$\hat{\theta}$	3.0523	0.0523	0.5162	3.0081	0.0081	0.5092
6	$\hat{\alpha}_1$	0.5092	0.0092	0.0609	0.5099	0.0099	0.0634
	$\hat{\lambda}_1$	1.4147	0.0147	0.1055	1.4166	0.0166	0.1182
	$\hat{\alpha}_2$	0.6112	0.0112	0.0735	0.6114	0.0114	0.0786
	$\hat{\lambda}_2$	1.5135	0.0135	0.1007	1.5128	0.0128	0.1187
	$\hat{\theta}$	6.0985	0.0985	0.9164	5.9511	-0.0489	0.9009

Table 6: Estimation of the Parameters of Clayton-BGR Distribution: Case 2, n=200

θ		MLE			IFM		
		Mean	Bias	RMSE	Mean	Bias	RMSE
0.5	$\hat{\alpha}_1$	0.5041	0.0041	0.0420	0.5043	0.0043	0.0423
	$\hat{\lambda}_1$	1.4095	0.0095	0.0804	1.4098	0.0098	0.0810
	$\hat{\alpha}_2$	0.6056	0.0056	0.0517	0.6058	0.0058	0.0520
	$\hat{\lambda}_2$	1.5095	0.0095	0.0816	1.5097	0.0097	0.0823
	$\hat{\theta}$	0.5161	0.0161	0.1282	0.5145	0.0145	0.1275
1.5	$\hat{\alpha}_1$	0.5043	0.0043	0.0414	0.5043	0.0043	0.0423
	$\hat{\lambda}_1$	1.4098	0.0098	0.0781	1.4098	0.0098	0.0810
	$\hat{\alpha}_2$	0.6046	0.0046	0.0508	0.6052	0.0052	0.0518
	$\hat{\lambda}_2$	1.5084	0.0084	0.0784	1.5092	0.0092	0.0822
	$\hat{\theta}$	1.5270	0.0270	0.2194	1.5191	0.0191	0.2175
3	$\hat{\alpha}_1$	0.5042	0.0042	0.0413	0.5043	0.0043	0.0423
	$\hat{\lambda}_1$	1.4096	0.0096	0.0753	1.4098	0.0098	0.0810
	$\hat{\alpha}_2$	0.6047	0.0047	0.0502	0.6051	0.0051	0.0516
	$\hat{\lambda}_2$	1.5084	0.0084	0.0743	1.5093	0.0093	0.0818
	$\hat{\theta}$	3.0428	0.0428	0.3612	3.0204	0.0204	0.3564
6	$\hat{\alpha}_1$	0.5043	0.0043	0.0413	0.5043	0.0043	0.0423
	$\hat{\lambda}_1$	1.4097	0.0097	0.0725	1.4098	0.0098	0.0810
	$\hat{\alpha}_2$	0.6049	0.0049	0.0499	0.6054	0.0054	0.0517
	$\hat{\lambda}_2$	1.5088	0.0088	0.0704	1.5098	0.0098	0.0818
	$\hat{\theta}$	6.0729	0.0729	0.6448	5.9980	-0.0020	0.6314

Table 7: Estimation of the Parameters of Clayton-BGR Distribution: Case 3, n=50

θ		MLE			IFM		
		Mean	Bias	RMSE	Mean	Bias	RMSE
0.5	$\hat{\alpha}_1$	2.1379	0.1379	0.5268	2.1435	0.1435	0.5383
	$\hat{\lambda}_1$	0.7104	0.0104	0.0847	0.7133	0.0133	0.0605
	$\hat{\alpha}_2$	2.6776	0.1776	0.6593	2.6770	0.1770	0.6652
	$\hat{\lambda}_2$	0.5071	0.0071	0.0504	0.5079	0.0079	0.0396
	$\hat{\theta}$	0.5371	0.0371	0.2559	0.5291	0.0291	0.2497
1.5	$\hat{\alpha}_1$	2.1362	0.1362	0.5073	2.1435	0.1435	0.5383
	$\hat{\lambda}_1$	0.7121	0.0121	0.0710	0.7133	0.0133	0.0605
	$\hat{\alpha}_2$	2.6654	0.1654	0.6145	2.6691	0.1691	0.6487
	$\hat{\lambda}_2$	0.5065	0.0065	0.0573	0.5079	0.0079	0.0397
	$\hat{\theta}$	1.5791	0.0791	0.4532	1.5436	0.0436	0.4421
3	$\hat{\alpha}_1$	2.1370	0.1370	0.4971	2.1435	0.1435	0.5383
	$\hat{\lambda}_1$	0.7138	0.0138	0.0548	0.7133	0.0133	0.0605
	$\hat{\alpha}_2$	2.6708	0.1708	0.6162	2.6787	0.1787	0.6792
	$\hat{\lambda}_2$	0.5089	0.0089	0.0363	0.5084	0.0084	0.0405
	$\hat{\theta}$	3.1354	0.1354	0.7559	3.0334	0.0334	0.7303
6	$\hat{\alpha}_1$	2.1359	0.1359	0.4865	2.1435	0.1435	0.5383
	$\hat{\lambda}_1$	0.7140	0.0140	0.0526	0.7133	0.0133	0.0605
	$\hat{\alpha}_2$	2.6757	0.1757	0.6182	2.6863	0.1863	0.7025
	$\hat{\lambda}_2$	0.5093	0.0093	0.0349	0.5086	0.0086	0.0413
	$\hat{\theta}$	6.2577	0.2577	1.3655	5.9348	-0.0652	1.3033

Table 8: Estimation of the Parameters of Clayton-BGR Distribution: Case 3, n=100

θ		MLE			IFM		
		Mean	Bias	RMSE	Mean	Bias	RMSE
0.5	$\hat{\alpha}_1$	2.0631	0.0631	0.3377	2.0635	0.0635	0.3405
	$\hat{\lambda}_1$	0.7047	0.0047	0.0400	0.7048	0.0048	0.0404
	$\hat{\alpha}_2$	2.5736	0.0736	0.4144	2.5775	0.0775	0.4210
	$\hat{\lambda}_2$	0.5026	0.0026	0.0263	0.5027	0.0027	0.0268
	$\hat{\theta}$	0.5160	0.0160	0.1818	0.5122	0.0122	0.1800
1.5	$\hat{\alpha}_1$	2.0595	0.0595	0.3256	2.0635	0.0635	0.3405
	$\hat{\lambda}_1$	0.7044	0.0044	0.0382	0.7048	0.0048	0.0404
	$\hat{\alpha}_2$	2.5742	0.0742	0.4102	2.5814	0.0814	0.4366
	$\hat{\lambda}_2$	0.5027	0.0027	0.0253	0.5029	0.0029	0.0274
	$\hat{\theta}$	1.5313	0.0313	0.3194	1.5118	0.0118	0.3163
3	$\hat{\alpha}_1$	2.0570	0.0570	0.3185	2.0635	0.0635	0.3405
	$\hat{\lambda}_1$	0.7043	0.0043	0.0365	0.7048	0.0048	0.0404
	$\hat{\alpha}_2$	2.5733	0.0733	0.4007	2.5806	0.0806	0.4405
	$\hat{\lambda}_2$	0.5027	0.0027	0.0241	0.5028	0.0028	0.0277
	$\hat{\theta}$	3.0566	0.0566	0.5220	3.0015	0.0015	0.5149
6	$\hat{\alpha}_1$	2.0565	0.0565	0.3120	2.0635	0.0635	0.3405
	$\hat{\lambda}_1$	0.7044	0.0044	0.0351	0.7048	0.0048	0.0404
	$\hat{\alpha}_2$	2.5741	0.0741	0.3958	2.5795	0.0795	0.4499
	$\hat{\lambda}_2$	0.5029	0.0029	0.0232	0.5026	0.0026	0.0278
	$\hat{\theta}$	6.1043	0.1043	0.9188	5.9299	-0.0701	0.9050

Table 9: Estimation of the Parameters of Clayton-BGR Distribution: Case 3, n=200

θ		MLE			IFM		
		Mean	Bias	RMSE	Mean	Bias	RMSE
0.5	$\hat{\alpha}_1$	2.0282	0.0282	0.2139	2.0292	0.0292	0.2162
	$\hat{\lambda}_1$	0.7029	0.0029	0.0273	0.7030	0.0030	0.0277
	$\hat{\alpha}_2$	2.5396	0.0396	0.2811	2.5408	0.0408	0.2850
	$\hat{\lambda}_2$	0.5019	0.0019	0.0186	0.5019	0.0019	0.0189
	$\hat{\theta}$	0.5162	0.0162	0.1298	0.5142	0.0142	0.1290
1.5	$\hat{\alpha}_1$	2.0285	0.0285	0.2076	2.0292	0.0292	0.2162
	$\hat{\lambda}_1$	0.7030	0.0030	0.0260	0.7030	0.0030	0.0277
	$\hat{\alpha}_2$	2.5344	0.0344	0.2710	2.5376	0.0376	0.2837
	$\hat{\lambda}_2$	0.5008	0.0008	0.0360	0.5018	0.0018	0.0191
	$\hat{\theta}$	1.5275	0.0275	0.2234	1.5182	0.0182	0.2216
3	$\hat{\alpha}_1$	2.0283	0.0283	0.2038	2.0292	0.0292	0.2162
	$\hat{\lambda}_1$	0.7030	0.0030	0.0248	0.7030	0.0030	0.0277
	$\hat{\alpha}_2$	2.5350	0.0350	0.2637	2.5375	0.0375	0.2817
	$\hat{\lambda}_2$	0.5019	0.0019	0.0169	0.5019	0.0019	0.0190
	$\hat{\theta}$	3.0436	0.0436	0.3652	3.0174	0.0174	0.3604
6	$\hat{\alpha}_1$	2.0286	0.0286	0.2005	2.0292	0.0292	0.2162
	$\hat{\lambda}_1$	0.7031	0.0031	0.0237	0.7030	0.0030	0.0277
	$\hat{\alpha}_2$	2.5366	0.0366	0.2585	2.5394	0.0394	0.2824
	$\hat{\lambda}_2$	0.5020	0.0020	0.0160	0.5020	0.0020	0.0190
	$\hat{\theta}$	6.0743	0.0743	0.6463	5.9872	-0.0128	0.6326

5. Application of Real Data

In this section, we analyze a real data set of economic data in Egypt for World Bank national accounts data that analyzed by El-Sherpieny et al. (2018). We study the parameter estimation of the appropriate distribution of each data, where the correlation between the two variables (bivariate data) is positive correlation. And through this access to a fit model specialized in the study of weak relations and the extent of their impact and effectiveness. The economic data set: Exports of goods and services (x_1) and GDP growth (x_2). In this data, we cannot used this data in Shock model, because $x_1 > x_2$.

Genest et al. (2013) introduced Multiplier bootstrap-based goodness-of-fit test, this consideration leads naturally to Anderson–Darling-type statistics such as

$$R_n = n\int_0^1 \left[\frac{\hat{C}_n(u_1,u_2) - C_{\theta_n}(u_1,u_2)}{\left[C_{\theta_n}(u_1,u_2)\left(1 - C_{\theta_n}(u_1,u_2)\right) + \delta_m\right]^m}\right]^2 d\hat{C}_n(u_1,u_2)$$

Involving a consistent, rank-based estimator θ_n of θ, and tuning parameters $m \geq 0$ and $\delta_m \geq 0$.

We use the concludes of Genest to fit of Clayton copulas by R package, then $R_n = 0.2658$, $\hat{\theta} = 0.3912$ and $p - value = 0.2257$, while $p - value > 0.05$, then the data fit of the Clayton copula. This by using a parametric bootstrap N=10000 time and the empirical copula estimate.

Table 10: parameter estimation of marginal, standard error and goodness of fit

	x_1	x_2
$\hat{\alpha}$	5.47579 (1.8474)	1.7395 (0.4537)
$\hat{\lambda}$	0.06494 (0.0056)	0.2134 (0.0219)
K-S (p-value)	0.0979 (0.8993)	0.0547 (0.9999)

Table 11: The Estimates and the Corresponding Standard Deviation of Parameters of Clayton-BGR Distribution

	MLE		IFM	
	Coef	std	coef	Std
$\hat{\alpha}_1$	5.4509	1.847	5.47579	(1.847)
$\hat{\lambda}_1$	0.065	0.0055	0.06494	(0.0056)
$\hat{\alpha}_2$	1.7186	0.4487	1.7395	(0.4537)
$\hat{\lambda}_2$	0.2118	0.0220	0.2134	(0.0219)
$\hat{\theta}$	0.2458	0.2903	0.2407	0.2716

In the Table (11), the best method for of Estimation of the Clayton-BGR distribution is IFM method since it has the least standard deviation of the copula parameter.

Table 12: The Estimates of Parameters of Clayton-BGR and anther Distributions

	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\alpha}_2$	$\hat{\beta}_2$	$\hat{\theta}$	AIC	BIC	HQIC	CAIC
Clayton-BGR	5.4509	0.065	1.7186	0.2118	0.2458	335.5796	342.7495	337.9168	337.9796
Clayton-BR	-	0.04285	-	0.19398	1.2567	355.0116	359.3136	356.4139	355.9005
Clayton-BW	4.4445	0.0397	2.6535	0.1726	0.3523	336.1476	343.3176	338.4848	338.5476
FGMBW	4.5225	0.0395	2.6953	0.1712	0.6712	335.617	342.789	337.9545	338.0172

The new distribution Clayton-BGR is more appropriate as compared to the Clayton-BR, Clayton-BW and FGM bivariate Weibull (FGMBW) distributions, where the Clayton-BGR distribution gives the lowest value for the AIC, BIC, AICC and H AIC statistics compared other models.

6. Conclusion

In this paper, we have proposed a class of bivariate GR distributions based on Clayton copula function. Moreover, we obtained the reliability functions for Clayton-BGR distributions; therefore, it can be used quite effectively in life testing data. Additionally, the new BGR model based on Clayton copula can be used as an alternative to different distributions as Clayton-BR, Clayton-BW and FGMBW. A comparison between different estimation methods of the bivariate GR distributions are concluded. The result show that the best method of estimation is IFM method.

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