Estimation for Exponentiated Generalized Xgamma Distribution from New General Class Based on Dual Generalized Order Statistics

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Abstract

In this paper, the exponentiated generalized general class of distributions is introduced. Bayes estimators for the parameters, reliability and hazard rate functions are derived based on dual generalized order statistics. As a special model from exponentiated generalized general class distributions, Bayesian estimation for the unknown parameters, reliability and hazard rate functions of exponentiated generalized xgamma distribution based on dual generalized order statistics are considered. All results are specialized to lower records, also a numerical study is given to illustrate the theoretical procedures.

Keywords: *Exponentiated generalized distributions; Bayesian estimation; Dual generalized order statistics; Exponentiated generalized xgamma distribution.*

1. Introduction

The statistical literature contains many classes of distributions that have been constructed by extending the common families of continuous distributions providing more flexibility as far as applications is concerned. These new families have been used for modeling data in many applied areas such as engineering, economics, biological studies, environmental sciences lifetime analysis, finance and insurance. These *generalized* (G) distributions give more flexibility by adding new parameters to the baseline model and are useful in obtaining general results that could be applied to special cases to obtain new results.[Alzaatreh *et al.* (2013) and Alizadeh *et al.* (2017)].

There are several ways to generate new distributions, such as adding a positive parameter to a general survival function by Marshall and Olkin (1997). Also, Gupta *et al.* (1998) introduced the *exponentiated* (E) exponential distribution; by exponentiating a cumulative distribution function with a positive real number, beta generalized-G family by Eugene *et al.* (2002), G gamma generated distributions by Zografos and Balakrishnan (2009). Cordeiro and de Castro (2011) presented a class of Kumaraswamy G distribution and Alzaatreh *et al.* (2013) introduced a new method for generating families of continuous distributions.

Many authors focused on the E distributions and its applications; for example, Nadarajah and Kotz (2006), Ali *et al.* (2007), Silva *et al.* (2010), Lemonte *et al.* (2013), Elgarhy and Shawki (2017) and Rather and Subramanian (2018).

Cordeiro *et al.* (2013) proposed a class of distributions as an extension of the E type distribution which has greater flexibility of its tails and can be widely applied in many areas of biology and engineering. Given a non-negative continuous random variable X, the *cumulative distribution function* (cdf) of the *Exponentiated Generalized* (EG) general class of distribution is defined by

$$F(x;\alpha,\beta)=[1-(1-G(x))^{\alpha}]^{\beta} \quad , \alpha,\beta>0\,, \tag{1}$$

where α and β are additional shape parameters, the corresponding *probability density function* (pdf) for (1) is given by

$$f(x;\alpha,\beta)=\alpha\beta g(x)\big(1-G(x)\big)^{\alpha-1}[1-(1-G(x))^{\alpha}]^{\beta-1} \quad , \alpha,\beta>0. \tag{2}$$

By setting $\alpha = 1$ in (1) the E type distributions is derived by Gupta *et al.* (1998); further the E exponential and E gamma distributions can be obtained if G(x) is the exponential or gamma cumulative distributions, respectively. For $\beta = 1$ in (1) and if G(x) is the Gumbel or Fréhet cumulative distributions, then, one can get the E Gumbel and E Fréchet distributions, respectively, as defined by Nadarajah and Kotz (2006). Thus, the class of distributions (1) extends both E type distributions.

General class of distributions is important to obtain a general result that could be applied to special cases in obtaining new results. It is more flexible in dealing with statistical problems. Many authors focused on the G and EG distributions and its applications; for example, Oguntunde *et al.* (2014), Yousof *et al.* (2015), De Andrade *et al.* (2016), Mustafa *et al.* (2016) and Sindi *et al.* (2017).

AL-Hussaini (1999) introduced a general class of distributions with positive domain to be the underlying population model. This class of distributions includes, the Weibull, Pareto Type I, beta Type I, Gompertz and Compound Gompertz distribution, among others.

$$G_1(x|\underline{\theta})=1-exp[-\lambda(x)], \qquad x>0, \tag{3}$$

where $\lambda(x)\equiv\lambda(x,\,\underline{\theta})$ is a non-negative continuous function of x such that $\lambda(x,\,\underline{\theta})\to 0\ as\ x\to 0^{+}$ and $\lambda(x,\,\underline{\theta})\to\infty\ as\ x\to\infty$, and $\underline{\theta}=(\theta_1,\theta_2,\dots,\theta_s)$.

In this paper the new *exponentiated generalized general class* (EGGC) of distributions will be introduced; using (1) and (3), the cdf and pdf can be derived as follows:

$$F(x;\alpha,\beta)=\big[1-exp[-\alpha\lambda(x)]\big]^{\beta} \quad ,x>0,\ \alpha,\beta>0, \tag{4}$$

and

$$f(x;\alpha,\beta) = \alpha\beta\acute{\lambda}(x)exp[-\alpha\lambda(x)][1-exp[-\alpha\lambda(x)]]^{\beta-1} \quad ,x>0,\ \alpha,\beta>0\,, \qquad (5)$$

where $\acute{\lambda}(x)$ is the derivative of $\lambda(x)$ with respect to x.

The *reliability function* (rf), *hazard rate function* (hrf) and *reversed hazard rate function* (rhrf) are given, respectively, by

$$R(x) = 1-[1-exp[-\alpha\lambda(x)]]^{\beta} \quad , \alpha,\beta,x>0\,, \qquad (6)$$

$$h_1(x) = \frac{f(x)}{R(x)} = \frac{\alpha\beta\acute{\lambda}(x)exp[-\alpha\lambda(x)][1-exp[-\alpha\lambda(x)]]^{\beta-1}}{1-[1-exp[-\alpha\lambda(x)]]^{\beta}} \quad , \quad x>0; \quad \alpha,\beta>0\,, \qquad (7)$$

and

$$h_2(x) = \frac{f(x)}{F(x)} = \alpha\beta\acute{\lambda}(x)exp[-\alpha\lambda(x)][1-exp[-\alpha\lambda(x)]]^{-1} \quad ,x>0; \quad \alpha,\beta>0. \qquad (8)$$

Burkschat *et al.* (2003) studied the *dual generalized order statistics* (dgos) that enables a common approach to descending ordered random variables as reversed ordered order statistics, lower record models and lower Pfeifer records.

Let $X_{(1,n,m,k)}, X_{(2,n,m,k)}, \ldots X_{(n,n,m,k)}$ be n dgos from an absolutely cdf with corresponding pdf. Then, the joint pdf has the form

$$f_{X_{(1,n,m,k)},X_{(2,n,m,k)},\ldots X_{(n,n,m,k)}}\left(x_{(1)},\ldots,x_{(n)}\right) =$$
$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left[\prod_{i=1}^{n-1}\left(F\left(x_{(i)}\right)\right)^{m} f\left(x_{(i)}\right)\right]\left(F\left(x_{(n)}\right)\right)^{k-1} f\left(x_{(n)}\right), \qquad (9)$$

where $F^{-1}(1) \geq x_{(1)} \ldots \geq x_{(n)} \geq F^{-1}(0), n \in N,\ k \geq 1,\ m_1,\ldots,m_{n-1} = m,$
$m \in R$ is the parameters such that $\gamma_r = k+(n-r)(m+1) \geq 1\,, for\ all\ 1 \leq r \leq n.$

This paper is organized as follows: in Section 2, Bayes estimators for the parameters, rf and hrf of EGGC of distributions based on dgos under *squared error* (SE) and *linear exponential* (LINEX) loss functions are derived. The *exponentiated generalized xgamma* (EG-Xg) distribution is studied in details as an application for this general class in Section 3. Finally a numerical study is presented in Section 4, to illustrate the application procedures of the various results developed in this paper.

2. Bayesian Estimation Based on Dual Generalized Order Statistics

This section is devoted to estimate the parameters, rf and hrf based on dgos using Bayesian approach, under SE and LINEX loss functions. Also the credible intervals are obtained.

Suppose that $X_{(1,n,m,k)}, X_{(2,n,m,k)}, \ldots, X_{(n,n,m,k)}$ are n dgos from EGGC distribution, the likelihood function can be obtained by substituting (4) and (5) in (9) as follows:

$$L(\alpha,\beta;\underline{x}) = k\left(\prod_{j=1}^{n-1}\gamma_j\right)\alpha^n\beta^n\prod_{i=1}^{n}\acute{\lambda}(x_i)\ exp\left[-\alpha\sum_{i=1}^{n}\lambda(x_i)\right]\prod_{i=1}^{n-1}[1-exp[-\alpha\lambda(x_i)]]^{\beta(m+1)-1}$$
$$\times\left[1-exp[-\alpha\lambda(x_n)]\right]^{\beta k-1}. \qquad (10)$$

Assuming that the parameters α and β of EGGC distributions are random variables with a joint bivariate prior density function that was used by AL-Hussaini and Jaheen (1992) as

$$g(\alpha,\beta) = g_1(\beta|\alpha)g_2(\alpha)\ , \qquad (11)$$

where $g_1(\beta|\alpha) = \frac{\alpha^{\tau+1}}{\Gamma(\tau+1)w^{\tau+1}}\beta^\tau e^{\frac{-\alpha\beta}{w}}$, $\tau > -1\ , w,\alpha,\beta > 0$, (12)

and the prior of α is

$$g_2(\alpha) = \frac{\alpha^{c-1}}{\Gamma(c)b^c}e^{-\frac{\alpha}{b}}\ , \qquad \alpha,b,c > 0\ , \qquad (13)$$

which is the gamma (c, b) density.

The joint prior pdf of α and β, will be obtained by substituting (12) and (13) in (11) and it's given by;

$$g(\alpha,\beta) \propto \alpha^{c+\tau}\beta^\tau e^{-\alpha\left(\frac{1}{b}+\frac{\beta}{w}\right)}, \qquad c,b,w > 0, \tau > -1. \qquad (14)$$

The joint posterior of α and β can be derived by using (10) and (14) as follows:

$$\pi(\alpha,\beta|\underline{x}) \propto L(\alpha,\beta|\underline{x})g(\alpha,\beta)$$

$$\pi(\alpha,\beta|\underline{x}) \propto \alpha^{n+\tau+c}\beta^{n+\tau}e^{-\alpha\left[\frac{1}{b}+\sum_{i=1}^{n}\lambda(x_i)\right]}e^{-\beta\left[\frac{\alpha}{w}-(m+1)\sum_{i=1}^{n-1}\ln u_i-k\ln u_n\right]}\prod_{i=1}^{n}(u_i)^{-1},$$

where $u_i = \left[1-exp[-\alpha\lambda(x_i)]\right]$ and $u_n = \left[1-exp[-\alpha\lambda(x_n)]\right]$, (15)

Hence, the joint posterior distribution of α and β is given by;

$$\pi(\alpha,\beta|\underline{x}) = \frac{\alpha^{n+\tau+c}\beta^{n+\tau}e^{-\alpha[\frac{1}{b}+\sum_{i=1}^{n}\lambda(x_i)]}e^{-\beta[\frac{\alpha}{w}-(m+1)\sum_{i=1}^{n-1}\ln u_i-k\ln u_n]}\prod_{i=1}^{n}(u_i)^{-1}}{\Gamma(n+\tau+1)\,\varphi}\ , \qquad (16)$$

where $\varphi = \int_0^\infty \frac{\alpha^{n+\tau+c} e^{-\alpha[\frac{1}{b}+\sum_{i=1}^{n}\lambda(x_i)]} \prod_{i=1}^{n}(u_i)^{-1}}{[\frac{\alpha}{w}-(m+1)\sum_{i=1}^{n-1}\ln u_i - k\ln u_n]^{n+\tau+1}} d\alpha.$ (17)

2.1 Point estimation

In this subsection the Bayes estimators for the parameters, rf and hrf based on dgos under SE and LINEX loss functions of the EGGC distribution are obtained.

a. Bayesian estimation of the parameters, rf and hrf under squared error loss function

Under SE loss function the Bayes estimators of the parameters α and β are given by their marginal posterior expectations using (16) as shown below

$$\alpha^*_{(SE)} = E(\alpha|\underline{x})$$

$$= \int_0^\infty \frac{1}{\varphi[\frac{\alpha}{w} - (m+1)\sum_{i=1}^{n-1}\ln u_i - k\ln u_n]^{n+\tau+1}} \alpha^{n+\tau+c+1} e^{-\alpha[\frac{1}{b}+\sum_{i=1}^{n}\lambda(x_i)]} \prod_{i=1}^{n}(u_i)^{-1}\, d\alpha, \tag{18}$$

and

$$\beta^*_{(SE)} = E(\beta|\underline{x})$$

$$= \int_0^\infty \frac{(n+\tau+1)\alpha^{n+\tau+c} e^{-\alpha[\frac{1}{b}+\sum_{i=1}^{n}\lambda(x_i)]} \prod_{i=1}^{n}(u_i)^{-1}}{\varphi\left(\frac{\alpha}{w}-(m+1)\sum_{i=1}^{n-1}\ln u_i - k\ln u_n]\right)^{n+\tau+2}}\, d\alpha. \tag{19}$$

The Bayes estimators of the rf and hrf under SE loss function are as follows:

$$R^*_{(SE)}(x) = E(R(x)|\underline{x})$$

$$= 1 - \int_0^\infty \frac{\alpha^{n+\tau+c} e^{-\alpha[\frac{1}{b}+\sum_{i=1}^{n}\lambda(x_i)]} \prod_{i=1}^{n}(u_i)^{-1}}{\varphi\left(\frac{\alpha}{w}-(m+1)\sum_{i=1}^{n-1}\ln u_i - k\ln u_n - \ln u\right)^{n+\tau+1}}\, d\alpha, \tag{20}$$

and

$$h^*_{1(SE)}(x) = E(h(x)|\underline{x}) =$$

$$\int_0^\infty\int_0^\infty \frac{\alpha^{n+\tau+c+1}\beta^{n+\tau+1}\lambda(x) exp[-\alpha\lambda(x)] u^{\beta-1} e^{-\alpha[\frac{1}{b}+\sum_{i=1}^{n}\lambda(x_i)]} e^{-\beta[\frac{\alpha}{w}-(m+1)\sum_{i=1}^{n-1}\ln u_i - k\ln u_n]} \prod_{i=1}^{n}(u_i)^{-1}}{(1-u^\beta)\Gamma(n+\tau+1)\,\varphi}\, d\alpha\, d\beta. \tag{21}$$

where $u = [1 - exp[-\alpha\lambda(x)]]$. (22)
and u_i and u_n are given by (15).

To obtain the Bayes estimates of the parameters, rf and hrf, the Equations (18) - (21) should be solved numerically.

b. Bayesian estimation of the parameters, rf and hrf under linear exponential loss function

Under the LINEX loss function, the Bayes estimators for the shape parameters α and β are given, respectively, by

$$\alpha^*_{(LNX)} = \frac{-1}{\vartheta} \ln E\left(e^{-\vartheta\alpha} \middle| \underline{x}\right), \tag{23}$$

where

$$E\left(e^{-\vartheta\alpha} \middle| \underline{x}\right) = \int_0^\infty \frac{\alpha^{n+\tau+c} e^{-\alpha\left[\vartheta+\frac{1}{b}+\sum_{i=1}^{n}\lambda(x_i)\right]} \prod_{i=1}^{n}(u_i)^{-1}}{\varphi\left[\frac{\alpha}{w}-(m+1)\sum_{i=1}^{n-1}\ln u_i - k\ln u_n\right]^{n+\tau+1}}\, d\alpha, \tag{24}$$

and

$$\beta^*_{(LNX)} = \frac{-1}{\vartheta} \ln E\left(e^{-\vartheta\beta} \middle| \underline{x}\right), \tag{25}$$

where

$$E\left(e^{-\vartheta\beta} \middle| \underline{x}\right) = \int_0^\infty \frac{\alpha^{n+\tau+c} e^{-\alpha\left[\frac{1}{b}+\sum_{i=1}^{n}\lambda(x_i)\right]} \prod_{i=1}^{n}(u_i)^{-1}}{\varphi\left(\vartheta+\frac{\alpha}{w}-(m+1)\sum_{i=1}^{n-1}\ln u_i - k\ln u_n]\right)^{n+\tau+1}}\, d\alpha. \tag{26}$$

Also, the Bayes estimator for rf based on dgos can be obtained as follows:

$$R^*_{(LNX)}(x) = \frac{-1}{\vartheta} \ln E\left(e^{-\vartheta R(x)} \middle| \underline{x}\right), \tag{27}$$

where

$$E\left(e^{-\vartheta R(x)} \middle| \underline{x}\right) =$$
$$1 - \int_0^\infty \int_0^\infty \frac{\alpha^{n+\tau+c}\beta^{n+\tau} e^{-\vartheta(1-u)^\beta} e^{-\alpha\left[\frac{1}{b}+\sum_{i=1}^{n}\lambda(x_i)\right]} e^{-\beta\left[\frac{\alpha}{w}-(m+1)\sum_{i=1}^{n-1}\ln u_i - k\ln u_n\right]} \prod_{i=1}^{n}(u_i)^{-1}}{\Gamma(n+\tau+1)\,\varphi}\, d\alpha\, d\beta, \tag{28}$$

and the Bayes estimators for hrf and rhrf based on dgos can be derived as given below:

$$h^*_{1(LNX)}(x) = \frac{-1}{\vartheta} \ln E\left(e^{-\vartheta h_1(x)} \middle| \underline{x}\right). \tag{29}$$

where

$E\left(e^{-\vartheta h_1(\underline{x})}\middle|\underline{x}\right) =$

$$\int_0^\infty\int_0^\infty \frac{\alpha^{n+\tau+c}\beta^{n+\tau}e^{-\vartheta\left[\frac{\alpha\beta\lambda(x)exp[-\alpha\lambda(x)][u]^{\beta-1}}{1-[u]^\beta}\right]}e^{-\alpha[\frac{1}{b}+\sum_{i=1}^{n}\lambda(x_i)]}e^{-\beta[\frac{\alpha}{w}-(m+1)\sum_{i=1}^{n-1}\ln u_i-k\ln u_n]}\prod_{i=1}^{n}(u_i)^{-1}}{\Gamma(n+\tau+1)\,\varphi}\,d\alpha\, d\beta\,, \qquad (30)$$

and

$$h^*_{2(LNX)}(x) = \frac{-1}{\vartheta}\ln E\left(e^{-\vartheta h_2(x)}\middle|\underline{x}\right). \qquad (31)$$

where

$E\left(e^{-\vartheta h_2(\underline{x})}\middle|\underline{x}\right) =$

$$\int_0^\infty\int_0^\infty \frac{\alpha^{n+\tau+c}\beta^{n+\tau}e^{-\vartheta\left[\alpha\beta\lambda(x)exp[-\alpha\lambda(x)]u^{-1}\right]}e^{-\alpha[\frac{1}{b}+\sum_{i=1}^{n}\lambda(x_i)]}e^{-\beta[\frac{\alpha}{w}-(m+1)\sum_{i=1}^{n-1}\ln u_i-k\ln u_n]}\prod_{i=1}^{n}(u_i)^{-1}}{\Gamma(n+\tau+1)\,\varphi}\,d\alpha\, d\beta\,. \qquad (32)$$

To obtain the Bayes estimators of the parameters, rf and hrf, the Equations (23) - (32) should be solved numerically.

2.2 Credible intervals

In this subsection the Bayesian analog to the confidence interval which is called a credible intervals are introduced. In general, $(L(\underline{x}), U(\underline{x}))$ is $100(1-\omega)\%$ credible interval of $\underline{\theta}$ if

$$P\left[L(\underline{x}) < \underline{\theta} < U(\underline{x})\middle|\underline{x}\right] = \int_{L(\underline{x})}^{U(\underline{x})}\pi^*\left(\underline{\theta}\middle|\underline{x}\right)d\underline{\theta} = 1-\omega. \qquad (33)$$

Since, the posterior distribution is given by (16), then a 100 (1- ω) % credible interval for α is $\left(L(\underline{x}), U(\underline{x})\right)$,

where

$$P\left[\alpha > L(\underline{x})\middle|\underline{x}\right] = \int_{L(\underline{x})}^{\infty}\frac{\alpha^{n+\tau+c}e^{-\alpha[\frac{1}{b}+\sum_{i=1}^{n}\lambda(x_i)]}}{\varphi\prod_{i=1}^{n}u_i[\frac{\alpha}{w}-(m+1)\sum_{i=1}^{n-1}\ln u_i-k\ln u_n]^{n+\tau+1}}\,d\alpha = 1-\frac{\omega}{2}, \qquad (34)$$

and

$$P\left[\alpha > U(\underline{x})\middle|\underline{x}\right] = \int_{U(\underline{x})}^{\infty}\frac{\alpha^{n+\tau+c}e^{-\alpha[\frac{1}{b}+\sum_{i=1}^{n}\lambda(x_i)]}}{\varphi\prod_{i=1}^{n}u_i[\frac{\alpha}{w}-(m+1)\sum_{i=1}^{n-1}\ln u_i-k\ln u_n]^{n+\tau+1}}\,d\alpha = \frac{\omega}{2}. \qquad (35)$$

Also, 100 (1- ω) % credible interval for β is $(L(\underline{x}), U(\underline{x}))$,

where

$$P\left[\beta > L(\underline{x})\middle|\underline{x}\right]$$

$$= \int_{L(\underline{x})}^{\infty}\int_0^\infty \frac{\alpha^{n+\tau+c}\beta^{n+\tau}e^{-\alpha[\frac{1}{b}+\sum_{i=1}^{n}\lambda(x_i)]}e^{-\beta[\frac{\alpha}{w}-(m+1)\sum_{i=1}^{n-1}\ln u_i-k\ln u_n]}\prod_{i=1}^{n}(u_i)^{-1}}{\varphi\,\Gamma(n+\tau+1)}\,d\alpha d\beta$$

$$= 1-\frac{\omega}{2}, \qquad (36)$$

and

$$P[\beta > U(\underline{x})|\underline{x}] = \int_{U(\underline{x})}^{\infty}\int_{0}^{\infty}\frac{\alpha^{n+\tau+c}\beta^{n+\tau}e^{-\alpha\left[\frac{1}{b}+\sum_{i=1}^{n}\lambda(x_i)\right]}e^{-\beta[\frac{\alpha}{w}-(m+1)\sum_{i=1}^{n-1}\ln u_i-k\ln u_n]}\prod_{i=1}^{n}(u_i)^{-1}}{\varphi\,\Gamma(n+\tau+1)}\,d\alpha d\beta$$
$$=\frac{\omega}{2}. \qquad (37)$$

3. Exponentiated Generalized Xgamma Distribution Based on Dual Generalized Order Statistics

Sen *et al.* (2016) introduced the *xgamma* (xg) distribution which is generated as a special finite mixture of exponential(θ) and gamma($3,\theta$) distributions with mixing proportion $\pi_1 = \frac{\theta}{1+\theta}$ and $\pi_2 = 1-\pi_1 = \frac{1}{1+\theta}$. The cdf and pdf of the xgamma distribution are, respectively

$$F_{xg}(x;\theta) = 1 - \frac{1+\theta+\theta x+\frac{\theta^2x^2}{2}}{1+\theta}e^{-\theta x}\ , \qquad x>0,\ \theta>0, \qquad (38)$$

and

$$f_{xg}(x;\theta) = \frac{\theta^2}{1+\theta}\left(1+\frac{\theta x^2}{2}\right)e^{-\theta x}\ , \qquad x>0,\ \theta>0. \qquad (39)$$

Yadav *et al.* (2018) studied the *generalized xgamma* (G-xg) distribution by adding power shape parameter to the cdf; some statistical properties of this G-xg are discussed. They used many methods of estimation to estimate the rf and hrf of the G-xg distribution.

Assuming that X is a random variable distribution with EG-xg distribution with shape parameters, $\alpha,\ \beta > 0$ and scale parameter $\theta > 0$ denoted by X~EGxg (α,β,θ), hence the pdf and cdf are given, respectively, by

$$F_{EGxg}(x;\alpha,\beta,\theta) = \left[1-\frac{(1+\theta+\theta x+\frac{\theta^2x^2}{2})^{\alpha}}{(1+\theta)^{\alpha}}e^{-\alpha\theta x}\right]^{\beta} \quad ,x>0,\ \alpha,\beta,\theta>0, \qquad (40)$$

and

$$f_{EGxg}(x;\alpha,\beta,\theta) = \frac{\alpha\beta\theta^2e^{-\alpha\theta x}}{(1+\theta)^{\alpha}}\left(1+\frac{\theta x^2}{2}\right)\left(1+\theta+\theta x+\frac{\theta^2x^2}{2}\right)^{\alpha-1}$$
$$\times\left[1-\frac{\left(1+\theta+\theta x+\frac{\theta^2x^2}{2}\right)^{\alpha}}{(1+\theta)^{\alpha}}e^{-\alpha\theta x}\right]^{\beta-1} \quad ,x>0,\ \alpha,\beta,\theta>0.$$

(41)

The rf, hrf and rhrf are, respectively, given by

$$R_{EGxg}(x;\alpha,\beta,\theta)=1-\left[1-\frac{\left(1+\theta+\theta x+\frac{\theta^2x^2}{2}\right)^{\alpha}}{(1+\theta)^{\alpha}}e^{-\alpha\theta x}\right]^{\beta}, \qquad x>0,\ \alpha,\beta,\theta>0, \quad (42)$$

$$h_{1EGxg}(x;\alpha,\beta,\theta)$$
$$=\frac{\frac{\alpha\beta\theta^2e^{-\alpha\theta x}}{(1+\theta)^{\alpha}}\left(1+\frac{\theta x^2}{2}\right)\left(1+\theta+\theta x+\frac{\theta^2x^2}{2}\right)^{\alpha-1}\left[1-\frac{\left(1+\theta+\theta x+\frac{\theta^2x^2}{2}\right)^{\alpha}}{(1+\theta)^{\alpha}}e^{-\alpha\theta x}\right]^{\beta-1}}{1-\left[1-\frac{\left(1+\theta+\theta x+\frac{\theta^2x^2}{2}\right)^{\alpha}}{(1+\theta)^{\alpha}}e^{-\alpha\theta x}\right]^{\beta}},$$

$$x>0,\ \alpha,\beta,\theta>0, \quad (43)$$

and

$$h_{2EGxg}(x;\alpha,\beta,\theta)=\frac{\alpha\beta\theta^2e^{-\alpha\theta x}}{(1+\theta)^{\alpha}}\left(1+\frac{\theta x^2}{2}\right)\left(1+\theta+\theta x+\frac{\theta^2x^2}{2}\right)^{\alpha-1}$$
$$\times\left[1-\frac{\left(1+\theta+\theta x+\frac{\theta^2x^2}{2}\right)^{\alpha}}{(1+\theta)^{\alpha}}e^{-\alpha\theta x}\right]^{-1}, \quad x,\alpha,\beta,\theta>0.$$

(44)

Plots of the pdf, hrf and rhrf of EG-xg are, respectively, given in Figures 1-3. The plots, in Figure 1 indicate the behavior of the density function and explain the flexibility of the model graphically with its sub-families. From Figure 1, one can observe that the curves of the pdf are monotone decreasing, increasing and bathtub, with different values of shape and scale parameters. In Figure 2, the curves of the hrf are increasing, decreasing and monotone decreasing with different values of shape and scale parameters. The curves of the rhrf at the all values are decreasing and then constant in Figure 3.

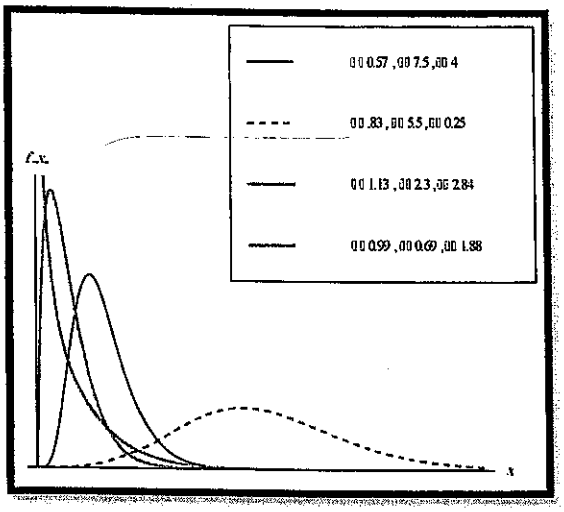

Figure 1

Plots of the probability density function

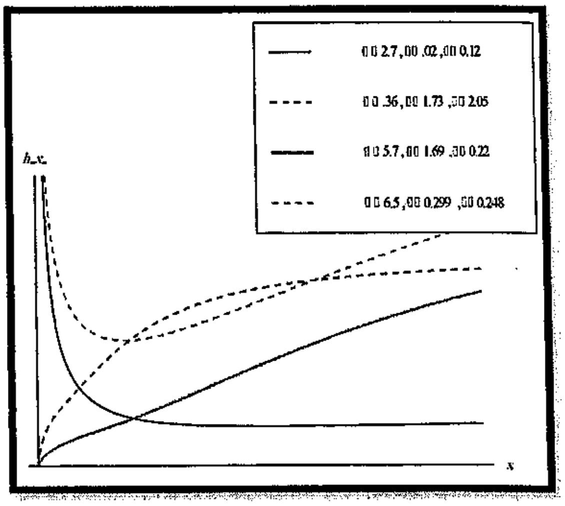

Figure 2

Plots of the hazard rate function

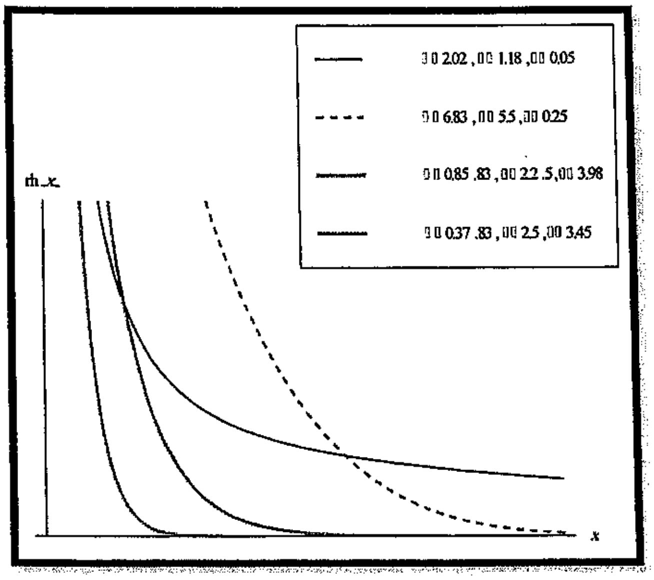

Figure 3

Plots of the reversed hazard rate function

3.1 Bayesian estimation for exponentiated generalized xgamma distribution

This section derived the estimation of the parameters, rf and hrf based on dgos using Bayesian approach. Also the credible intervals of the parameters, rf and hrf are obtained.

Suppose that $X_{(1,n,m,k)}, X_{(2,n,m,k)},\ldots, X_{(n,n,m,k)}$ are n dgos from EG-Xg distribution, the likelihood function can be derived by substituting (40) and (41) in (9) as follows:

$$L_{EGxg}(\alpha,\beta,\theta;\underline{x}) = k\left(\prod_{j=1}^{n-1}\gamma_j\right)\frac{\alpha^n\beta^n\theta^{2n}e^{-\alpha\theta\sum_{i=1}^{n}x_i}}{(1+\theta)^{n\alpha}}\prod_{i=1}^{n}\left(1+\frac{\theta x_i^2}{2}\right)\delta_i^{\alpha-1}$$

$$\times\prod_{i=1}^{n-1}\left[1-\left(\frac{\delta_i}{(1+\theta)}\right)^{\alpha}e^{-\alpha\theta x_i}\right]^{\beta(m+1)-1}\left[1-\left(\frac{\delta_n}{(1+\theta)}\right)^{\alpha}e^{-\alpha\theta x_n}\right]^{\beta k-1}, \quad (45)$$

where $\delta_i = \left(1+\theta+\theta x_i+\frac{\theta^2 x_i^2}{2}\right)$ and $\delta_n = \left(1+\theta+\theta x_n+\frac{\theta^2 x_n^2}{2}\right)$. (46)

3.1.1 Point estimation

In this subsection, the Bayesian approach is considered under SE and LINEX loss functions to estimate the parameters, rf and hrf of the EG-xg distribution based on dgos. Also credible intervals for the parameters are obtained.
Let $\vartheta_1 = \alpha, \vartheta_2 = \beta$ and $\vartheta_3 = \theta$ are independent random variables with gamma prior distribution with the pdf as follows:

$\pi(\vartheta_j) = \frac{d_j^{c_j}}{\Gamma(c_j)}\vartheta_j^{c_j-1}e^{-d_j\vartheta_j}, \qquad \vartheta_j, d_j, c_j > 0,\ j = 1,2,3,$ where c_j, d_j are the hyper parameters.

A joint prior density function of $\underline{\vartheta} = (\vartheta_1,\vartheta_2,\vartheta_3)'$ is then given by

$$\pi(\underline{\vartheta}) \propto \prod_{j=1}^{3}\vartheta_j^{c_j-1}e^{-d_j\vartheta_j}.$$

$$\propto \alpha^{c_1-1}\beta^{c_2-1}\theta^{c_3-1}e^{-[d_1\alpha+d_2\beta+d_3\theta]}. \quad (47)$$

The likelihood function given by (45) can be rewritten as

$$L_{EGxg}(\alpha,\beta,\theta;\underline{x}) = k\left(\prod_{j=1}^{n-1}\gamma_j\right)\frac{\alpha^n\beta^n\theta^{2n}e^{-\alpha\theta\sum_{i=1}^{n}x_i}}{(1+\theta)^{n\alpha}}\prod_{i=1}^{n}\rho_i\,(u_i^*)^{-1}\prod_{i=1}^{n-1}u_i^{*[\beta(m+1)]}\ u_n^{*[\beta k]}, \quad (48)$$

where $u_i^* = \left[1-\left(\frac{\delta_i}{(1+\theta)}\right)^{\alpha}e^{-\alpha\theta x_i}\right]$, $u_n^* = \left[1-\left(\frac{\delta_n}{(1+\theta)}\right)^{\alpha}e^{-\alpha\theta x_n}\right]$ and $\rho_i = \left(1+\frac{\theta x_i^2}{2}\right)\delta_i^{\alpha-1}$.

The joint posterior density can be derived by using (47) and (48) as follows:

$$\pi^{*}_{EGxg}(\underline{\vartheta}|\underline{x})$$
$$= T(\alpha^{c_1+n-1}\beta^{c_2+n-1}\theta^{c_3+2n-1})e^{-\alpha(d_1+\theta\sum_{i=1}^{n}x_i+nln(1+\theta))}$$
$$\times e^{-\beta[d_2-(m+1)\sum_{i=1}^{n-1}ln(u_i^*)-kln(u_n^*)]}\; e^{-\theta d_3}\prod_{i=1}^{n}\rho_i\,(u_i^*)^{-1}, \qquad (49)$$

where T is the normalizing constant defined by

$$T^{-1}$$
$$= \int_0^\infty\int_0^\infty\left(\alpha^{c_1+n-1}\theta^{c_3+2n-1}e^{-\alpha(d_1+\theta\sum_{i=1}^{n}x_i+nln(1+\theta))}\,e^{-\theta d_3}\prod_{i=1}^{n}\rho_i\,(u_i^*)^{-1}\right)$$
$$\times\left[\int_0^\infty\beta^{c_2+n-1}e^{-\beta[d_2-(m+1)\sum_{i=1}^{n-1}ln(u_i^*)-kln(u_n^*)]}d\beta\right]d\alpha d\theta.$$
$$= \Gamma(c_2+n)\int_0^\infty\int_0^\infty\left(\alpha^{c_1+n-1}\theta^{c_3+2n-1}\,e^{-\alpha(d_1+\theta\sum_{i=1}^{n}x_i+nln(1+\theta))}\,e^{-\theta d_3}\prod_{i=1}^{n}\rho_i\,(u_i^*)^{-1}\right)$$
$$\times\left[\frac{1}{[d_2-(m+1)\sum_{i=1}^{n-1}ln(u_i^*)-kln(u_n^*)]^{c_2+n}}\right]d\alpha d\theta$$

Let

$$\varphi^* = \int_0^\infty\int_0^\infty\left(\alpha^{c_1+n-1}\theta^{c_3+2n-1}\,e^{-\alpha(d_1+\theta\sum_{i=1}^{n}x_i+nln(1+\theta))}\,e^{-\theta d_3}\prod_{i=1}^{n}\rho_i\,(u_i^*)^{-1}\right)$$
$$\times\left[\frac{1}{[d_2-(m+1)\sum_{i=1}^{n-1}ln(u_i^*)-kln(u_n^*)]^{c_2+n}}\right]d\alpha d\theta$$

Hence, the joint posterior distribution of α,β and θ given $\underline{x}$ can be written as;

$$\pi^{*}_{EGxg}(\alpha,\beta,\theta|\underline{x}) =$$
$$\frac{(\alpha^{c_1+n-1}\beta^{c_2+n-1}\theta^{c_3+2n-1})e^{-\alpha(d_1+\theta\sum_{i=1}^{n}x_i+nln(1+\theta))}e^{-\theta d_3}\prod_{i=1}^{n}\rho_i\,(u_i^*)^{-1}e^{-\beta[d_2-(m+1)\sum_{i=1}^{n-1}ln(u_i^*)-kln(u_n^*)]}}{\varphi^*\,\Gamma(c_2+n)}, \qquad (50)$$

a. Bayesian estimation of the parameters, rf and hrf under squared error loss function

Under SE loss function the Bayes estimators of the parameters α , β and θ are given by their marginal posterior expectations using (50) as shown below

$$\alpha^{*}_{(SE)EGxg} = E(\alpha|\underline{x})$$
$$= \int_0^\infty\int_0^\infty\int_0^\infty\frac{1}{\varphi^*\,\Gamma(c_2+n)}(\alpha^{c_1+n}\beta^{c_2+n-1}\theta^{c_3+2n-1})e^{-\alpha(d_1+\theta\sum_{i=1}^{n}x_i+nln(1+\theta))}$$
$$\times e^{-\theta d_3}\prod_{i=1}^{n}\rho_i\,(u_i^*)^{-1}\,e^{-\beta[d_2-(m+1)\sum_{i=1}^{n-1}ln(u_i^*)-kln(u_n^*)]}\;d\beta d\theta d\alpha,$$

$$= \int_0^\infty \int_0^\infty \frac{(\alpha^{c_1+n}\,\theta^{c_3+2n-1})}{\varphi^*[d_2-(m+1)\sum_{i=1}^{n-1} ln(u_i^*) - kln(u_n^*)]^{c_2+n}} e^{-\alpha(d_1+\theta\sum_{i=1}^{n} x_i + nln(1+\theta))}\, e^{-\theta d_3} \prod_{i=1}^{n} \rho_i\,(u_i^*)^{-1}\, d\theta d\alpha, \tag{51}$$

$$\beta^*_{(SE)EGxg} = E(\beta|\underline{x})$$
$$= \int_0^\infty \int_0^\infty \int_0^\infty \frac{1}{\varphi^*\Gamma(c_2+n)} \left(\alpha^{c_1+n-1}\beta^{c_2+n}\theta^{c_3+2n-1}\right) e^{-\alpha(d_1+\theta\sum_{i=1}^{n} x_i+nln(1+\theta))}\, e^{-\theta d_3} \prod_{i=1}^{n} \rho_i\,(u_i^*)^{-1}$$
$$\times\, e^{-\beta\left[d_2-(m+1)\sum_{i=1}^{n-1} ln(u_i^*)-kln(u_n^*)\right]}\, d\beta d\alpha d\theta,$$

$$= \int_0^\infty \int_0^\infty \frac{(c_2+n)(\alpha^{c_1+n-1}\theta^{c_3+2n})e^{-\alpha(d_1+\theta\sum_{i=1}^{n} x_i+nln(1+\theta))}\, e^{-\theta d_3} \prod_{i=1}^{n}\rho_i\,(u_i^*)^{-1}}{\varphi^*[d_2-(m+1)\sum_{i=1}^{n-1} ln(u_i^*) - kln(u_n^*)]^{c_2+n+1}}\, d\alpha d\theta, \tag{52}$$

and

$$\theta^*_{(SE)EGxg} = E(\theta|\underline{x})$$
$$= \int_0^\infty \int_0^\infty \int_0^\infty \frac{(\alpha^{c_1+n-1}\beta^{c_2+n-1}\theta^{c_3+2n})}{\varphi^*\Gamma(c_2+n)} e^{-\alpha(d_1+\theta\sum_{i=1}^{n} x_i+nln(1+\theta))}\, e^{-\theta d_3} \prod_{i=1}^{n} \rho_i\,(u_i^*)^{-1}$$
$$\times\, e^{-\beta[d_2-(m+1)\sum_{i=1}^{n-1} ln(u_i^*)-kln(u_n^*)]}\, d\beta d\alpha d\theta,$$

$$= \int_0^\infty \int_0^\infty \frac{(\alpha^{c_1+n-1}\theta^{c_3+2n})e^{-\alpha(d_1+\theta\sum_{i=1}^{n} x_i+nln(1+\theta))}\, e^{-\theta d_3} \prod_{i=1}^{n}\rho_i\,(u_i^*)^{-1}}{\varphi^*[d_2-(m+1)\sum_{i=1}^{n-1} ln(u_i^*) - kln(u_n^*)]^{c_2+n}}\, d\alpha d\theta, \tag{53}$$

The Bayes estimators of the rf, hrf and rhrf under SE loss function can be obtained using (42), (43) and (50) as follows:

$$R^*_{(SE)EGxg}(x) = E(R_{EGxg}(x)|\underline{x}) = 1 - \int_{\underline{\vartheta}} (u^*)^{\beta}\, \pi^*_{EGxg}(\underline{\vartheta}|\underline{x})\, d\underline{\vartheta}$$

$$= 1$$
$$- \int_0^\infty \int_0^\infty \int_0^\infty \frac{\left(\alpha^{c_1+n-1}\beta^{c_2+n-1}\theta^{c_3+2n-1}\right)}{\varphi^*\Gamma(c_2+n)} e^{-\alpha(d_1+\theta\sum_{i=1}^{n} x_i+nln(1+\theta))}\, e^{-\theta d_3} \prod_{i=1}^{n} \rho_i\,(u_i^*)^{-1}$$
$$\times\, e^{-\beta[d_2-(m+1)\sum_{i=1}^{n-1} ln(u_i^*)-kln(u_n^*)-lnu^*]}\, d\beta d\theta d\alpha,$$
$$= 1 - \int_0^\infty \int_0^\infty \frac{(\alpha^{c_1+n-1}\theta^{c_3+2n-1})e^{-\alpha(d_1+\theta\sum_{i=1}^{n} x_i+nln(1+\theta))}\, e^{-\theta d_3} \prod_{i=1}^{n}\rho_i\,(u_i^*)^{-1}}{\varphi^*[d_2-(m+1)\sum_{i=1}^{n-1} ln(u_i^*) - kln(u_n^*) - lnu^*]^{c_2+n}}\, d\theta d\alpha,$$

(54)

where $u^* = \left[1-\left(\frac{\delta}{(1+\theta)}\right)^{\alpha} e^{-\theta\alpha x}\right]$

and

$$h^*_{1(SE)EGxg}(x) = E\left(h_{1\,EGxg}(x)\middle|\underline{x}\right)$$
$$= \int_{\underline{\vartheta}} h_{1\,EGxg}(x)\pi^*_{EGxg}\left(\underline{\vartheta}\middle|\underline{x}\right)\,d\underline{\vartheta}, \tag{55}$$

To obtain the Bayes estimators of the parameters, rf, hrf and rhrf, the Equations (51) - (55) should be solved numerically.

b. Bayesian estimation of the parameters, rf and hrf under linear exponential loss function

Under the LINEX loss function, the Bayes estimators for the shape parameters α, β and θ are given, respectively, by

$$\alpha^*_{(LNX)EGxg} = \frac{-1}{v}\ln E\left(e^{-v\alpha}\middle|\underline{x}\right), \tag{56}$$

where

$$E\left(e^{-v\alpha}\middle|\underline{x}\right)$$
$$= \int_0^\infty\int_0^\infty\int_0^\infty \frac{1}{\varphi^*\Gamma(c_2+n)}\left(\alpha^{c_1+n-1}\beta^{c_2+n-1}\theta^{c_3+2n-1}\right)e^{-\alpha\left(d_1+\theta\sum_{i=1}^n x_i+n\ln(1+\theta)+v\right)}\,e^{-\theta d_3}$$
$$\times \prod_{i=1}^n \rho_i\,(u_i^*)^{-1}\,e^{-\beta\left[d_2-(m+1)\sum_{i=1}^{n-1}\ln(u_i^*)-k\ln(u_n^*)\right]}\;d\beta d\theta d\alpha,$$
$$= \int_0^\infty\int_0^\infty \frac{\left(\alpha^{c_1+n-1}\theta^{c_3+2n-1}\right)e^{-\alpha\left(d_1+\theta\sum_{i=1}^n x_i+n\ln(1+\theta)+v\right)}\,e^{-\theta d_3}}{\varphi^*\left[d_2-(m+1)\sum_{i=1}^{n-1}\ln(u_i^*)-k\ln(u_n^*)\right]^{c_2+n}}\prod_{i=1}^n \rho_i\,(u_i^*)^{-1}\;d\theta d\alpha,$$

and

$$\beta^*_{(LNX)EGxg} = \frac{-1}{v}\ln E\left(e^{-v\beta}\middle|\underline{x}\right), \tag{57}$$

where

$$E\left(e^{-\nu\beta}|\underline{x}\right)$$
$$=\int_0^\infty\int_0^\infty\int_0^\infty \frac{\left(\alpha^{c_1+n-1}\beta^{c_2+n-1}\theta^{c_3+2n-1}\right)}{\varphi^*\Gamma(c_2+n)} e^{-\alpha(d_1+\theta\sum_{i=1}^{n}x_i+nln(1+\theta))}\, e^{-\theta d_3}\prod_{i=1}^{n}\rho_i\,(u_i^*)^{-1}$$
$$\times\, e^{-\beta[\nu+d_2-(m+1)\sum_{i=1}^{n-1}ln(u_i^*)-kln(u_n^*)]}\;d\beta d\alpha d\theta,$$

$$=\int_0^\infty\int_0^\infty \frac{(c_2+n)\left(\alpha^{c_1+n-1}\theta^{c_3+2n}\right)e^{-\alpha(d_1+\theta\sum_{i=1}^{n}x_i+nln(1+\theta))}\,e^{-\theta d_3}}{\varphi^*\left[\nu+d_2-(m+1)\sum_{i=1}^{n-1}ln(u_i^*)-kln(u_n^*)\right]^{c_2+n}}\prod_{i=1}^{n}\rho_i\,(u_i^*)^{-1}\;d\alpha d\theta,$$

Also

$$\theta^*_{(LNX)EGxg}=\frac{-1}{\nu}\ln E\left(e^{-\nu\theta}|\underline{x}\right), \tag{58}$$

where

$$E\left(e^{-\nu\theta}|\underline{x}\right)$$
$$=\int_0^\infty\int_0^\infty\int_0^\infty \frac{\left(\alpha^{c_1+n-1}\beta^{c_2+n-1}\theta^{c_3+2n-1}\right)e^{-\alpha(d_1+\theta\sum_{i=1}^{n}x_i+nln(1+\theta))}\,e^{-\theta(d_3+\nu)}}{\varphi^*\Gamma(c_2+n)}\prod_{i=1}^{n}\rho_i\,(u_i^*)^{-1}$$
$$\times\, e^{-\beta[d_2-(m+1)\sum_{i=1}^{n-1}ln(u_i^*)-kln(u_n^*)]}\;d\beta d\alpha d\theta,$$
$$=\int_0^\infty\int_0^\infty \frac{\left(\alpha^{c_1+n-1}\theta^{c_3+2n-1}\right)e^{-\alpha(d_1+\theta\sum_{i=1}^{n}x_i+nln(1+\theta))}\,e^{-\theta(d_3+\nu)}}{\varphi^*[d_2-(m+1)\sum_{i=1}^{n-1}ln(u_i^*)-kln(u_n^*)]^{c_2+n}}\prod_{i=1}^{n}\rho_i\,(u_i^*)^{-1}\,d\alpha d\theta.$$

Also, the Bayes estimator for rf based on dgos can be obtained as follows:

$$R^*_{(LNX)EGxg}(x)=\frac{-1}{\nu}\ln E\left(e^{-\nu R_{EGxg}(x)}|\underline{x}\right), \tag{59}$$

where

$$E\left(e^{-\nu R(x)}|\underline{x}\right)=$$
$$=$$
$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{\left(\alpha^{c_1+n-1}\beta^{c_2+n-1}\theta^{c_3+2n-1}\right)e^{-\alpha\left(d_1+\theta\sum_{i=1}^{n}x_i+nln(1+\theta)\right)}\,e^{-\theta d_3-\nu\left(1-(u^*)^{\beta}\right)}}{\varphi^*\,\Gamma(c_2+n)} e^{-\beta\left[d_2-(m+1)\sum_{i=1}^{n-1}ln(u_i^*)-kln(u_n^*)-lnu^*\right]}\times$$
$$\prod_{i=1}^{n}\rho_i\,(u_i^*)^{-1}\,d\beta d\theta d\alpha,$$

and the Bayes estimator for hrf and rhrf based on dgos can be derived as follows:

$$h^*_{1(LNX)EGxg}(x)=\frac{-1}{\nu}\ln E\left(e^{-\nu h_1(x)}|\underline{x}\right). \tag{60}$$

where

$$E\left(e^{-\nu h_1(x)}|\underline{x}\right)=\int_0^\infty\int_0^\infty\int_0^\infty e^{-\nu h_1(x)}\pi^*_{EGxg}\left(\alpha,\beta,\theta|\underline{x}\right)d\beta d\theta d\alpha.$$

3.1.2 Credible interval

Since, the posterior distribution is given by (51), then a 100 (1- ω) % credible interval for α is $\left(L(\underline{x}), U(\underline{x})\right)$,

where

$$P\left[\alpha_{EGxg} > L(\underline{x})|\underline{x}\right] = \int_{L(\underline{x})}^{\infty}\int_{0}^{\infty}\frac{\left(\alpha^{c_1+n-1}\theta^{c_3+2n-1}\right)e^{-\alpha(d_1+\theta\sum_{i=1}^{n}x_i+nln(1+\theta))}\, e^{-\theta d_3}}{\varphi^{*}\left[d_2-(m+1)\sum_{i=1}^{n-1}ln(u_i^{*})-kln(u_n^{*})\right]^{c_2+n}}\prod_{i=1}^{n}\rho_i\,(u_i^{*})^{-1}\, d\theta d\alpha$$
$$= 1-\frac{\omega}{2}. \quad (61)$$

and

$$P\left[\alpha_{EGxg} > U(\underline{x})|\underline{x}\right] = \int_{U(\underline{x})}^{\infty}\int_{0}^{\infty}\frac{\left(\alpha^{c_1+n-1}\theta^{c_3+2n-1}\right)e^{-\alpha(d_1+\theta\sum_{i=1}^{n}x_i+nln(1+\theta))}\, e^{-\theta d_3}}{\varphi^{*}\left[d_2-(m+1)\sum_{i=1}^{n-1}ln(u_i^{*})-kln(u_n^{*})\right]^{c_2+n}}\prod_{i=1}^{n}\rho_i\,(u_i^{*})^{-1}\, d\theta d\alpha$$
$$= \frac{\omega}{2}. \quad (62)$$

Also, 100 (1- ω) % credible interval for β is $(L(\underline{x}), U(\underline{x}))$,

where

$$P\left[\beta_{EGxg} > L(\underline{x})|\underline{x}\right]$$
$$= \int_{L(\underline{x})}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}\frac{\left(\alpha^{c_1+n-1}\beta^{c_2+n-1}\theta^{c_3+2n-1}\right)}{\Gamma(c_2+n)\,\varphi^{*}}e^{-\alpha(d_1+\theta\sum_{i=1}^{n}x_i+nln(1+\theta))}\, e^{-\theta d_3}e^{-\beta\left[d_2-(m+1)\sum_{i=1}^{n-1}ln(u_i^{*})-kln(u_n^{*})\right]}$$
$$\times\prod_{i=1}^{n}\rho_i\,(u_i^{*})^{-1}\, d\alpha d\theta d\beta$$
$$= 1-\frac{\omega}{2}, \quad (63)$$

and

$$P\left[\beta_{EGxg} > U(\underline{x})|\underline{x}\right]$$
$$= \int_{U(\underline{x})}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}\frac{\left(\alpha^{c_1+n-1}\beta^{c_2+n-1}\theta^{c_3+2n-1}\right)}{\Gamma(c_2+n)\,\varphi^{*}}e^{-\alpha(d_1+\theta\sum_{i=1}^{n}x_i+nln(1+\theta))}\, e^{-\theta d_3}e^{-\beta\left[d_2-(m+1)\sum_{i=1}^{n-1}ln(u_i^{*})-kln(u_n^{*})\right]}$$
$$\times\prod_{i=1}^{n}\rho_i\,(u_i^{*})^{-1}\, d\alpha d\theta d\beta = \frac{\omega}{2}\,. \quad (64)$$

Also, 100 (1- ω) % credible interval for θ is $(L(\underline{x}), U(\underline{x}))$,

$$P\left[\theta_{EGxg} > L(\underline{x})|\underline{x}\right]$$
$$= \int_{L(\underline{x})}^{\infty}\int_{0}^{\infty}\frac{\left(\alpha^{c_1+n-1}\theta^{c_3+2n-1}\right)e^{-\alpha(d_1+\theta\sum_{i=1}^{n}x_i+nln(1+\theta))}\, e^{-\theta d_3}}{\varphi^{*}\left[d_2-(m+1)\sum_{i=1}^{n-1}ln(u_i^{*})-kln(u_n^{*})\right]^{c_2+n}}\prod_{i=1}^{n}\rho_i\,(u_i^{*})^{-1}\, d\alpha d\theta$$
$$= 1-\frac{\omega}{2}, \quad (65)$$

and

$$P\left[\theta_{EGxg} > U(\underline{x}) \middle| \underline{x}\right] = \int_{U(\underline{x})}^{\infty}\int_{0}^{\infty} \frac{\left(\alpha^{c_1+n-1}\theta^{c_3+2n-1}\right)e^{-\alpha(d_1+\theta\sum_{i=1}^{n}x_i+n\ln(1+\theta))}\, e^{-\theta d_3}}{\varphi^*\left[d_2-(m+1)\sum_{i=1}^{n-1}\ln(u_i^*)-k\ln(u_n^*)\right]^{c_2+n}} \prod_{i=1}^{n}\rho_i\,(u_i^*)^{-1}\, d\alpha d\theta$$

$$= \frac{\omega}{2}. \tag{66}$$

4. Numerical Results

This section aims to illustrate the theoretical results of Bayesian estimation under SE and LINEX loss functions. Numerical results are presented for EG-xg distribution based on lower record values through a simulation study and some applications.

4.1 Simulation study

The lower record values can be obtained as a special case from dogs by setting $m = -1$, $k = 1$; the estimation results obtained in Sections 2 and 3 can be specialized to lower records. The Bayes estimates of α, β, θ, rf and hrf and their average estimates, *estimated risks* (ERs) are computed based on lower record values through Monte Carlo simulation study according to the following steps:

a. To generate random number from EG-xg with shape parameters α , β and scale parameter θ, the following steps may be used.

   - Specified the values α, β, θ and n.
   - Generate U_i from uniform (0, 1) distribution ($i = 1,2,3, \dots, n$).
   - Generate V_i from gamma $(1,\theta)$ distribution($i = 1,2,3, \dots, n$).
   - Generate W_i from gamma $(3,\theta)$ distribution ($i = 1,2,3, \dots, n$).
   - If $U_i \leq \frac{\theta}{1+\theta}$, set $X_i = V_i$,otherwise set $X_i = W_i$.

b. For each sample size n, consider the first observation is the first lower record value x_1 denote it by R_1 and the second observation x_2 denote it by R_2 ; which is smaller than the maximum $(x_1 > x_2)$ record and if $x_1 \leq x_2$ ignore it and repeat until you get sample of *record values* (Rv) records.

c. At the number of the surviving units, the population parameter values α, β, θ and the hyper parameters of the prior distribution, the Bayes estimates of the parameters, rf and hrf under SE and LINEX loss functions are computed. The computations are performed using R programming language.

d. Tables 1 and 2 present the Bayes estimates under SE and LINEX loss functions of the parameters and their ERs, *relative absolute biases* (RAB) and credible intervals based on lower record values for different population parameter values for $\alpha = (0.2, 2)$, $\beta =$

(1.3, 3) and $\theta = (0.9,0.3)$, based on size of records Rv=5, 7, 9 and replications NR = 10000.

e. Table 3 displays the Bayes averages and 95% confidence intervals of the rf and hrf at $t_0 = 0.5, 1, 1.5$ from EG-xg distribution based on lower record values for different sample size of records Rv= 5,9, and replications NR = 10000.

4.2Applications

In this subsection, the application to real data set is provided to illustrate the importance of the EG-xg distribution based on lower records. Table 4 displays Bayes estimates of rf, hrf and *standard deviation* (sd) from EG-xg distribution for the real data based on lower records. Bayes estimates of the parameters, sds and ERs for the real data based on lower records in Table 5.

To check the validity of the fitted model, Kolmogorov-Smirnov goodness of fit test is performed for each data set. The p values are given, respectively, 0.35 and 0.204. The p values in each case indicate that the model fits the data very well.

I. The application is the vinyl chloride data obtained from clean upgrading, monitoring wells in mg/L; this data set was used for Bhaumik *et al.* (2009). The data are: 5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4, 0.2.

II. The second data set are service times of 63 aircraft windshield from Tahir *et al.* (2015). The data are: 0.046, 1.436, 2.592, 0.140, 1.492, 2.600, 0.150, 1.580, 2.670, 0.248, 1.719, 2.717, 0.280, 1.794, 2.819, 0.313, 1.915, 2.820, 0.389, 1.920, 2.878, 0.487, 1.963, 2.950, 0.622,1.978, 3.003, 0.900, 2.053, 3.102, 0.952, 2.065, 3.304, 0.996, 2.117, 3.483, 1.003, 2.137, 3.500,1.010, 2.141, 3.622, 1.085, 2.163, 3.665, 1.092, 2.183, 3.695, 1.152, 2.240, 4.015,1.183, 2.341, 4.628, 1.244, 2.435, 4.806, 1.249, 2.464, 4.881, 1.262, 2.543, 5.140.

4.3 Concluding remarks

- It is clear from Tables 1-3 that the ERs of the Bayes estimates of the parameters, rf and hrf performs better and the length of the credible intervals get shorter when the sample size of Rv increases.
- One can notice that the ERs for the estimates of the parameters, rf and hrf under LINEX loss function have the less values than the corresponding ERs of the estimates under SE loss function.
- The results obtained in this paper can be modified to obtain special results for sub-models of EG-xg distribution as follows:

i. The exponentiated xgamma distribution, if $\alpha = 1$.
ii. The xgamma distribution, if $\alpha = 1$ and $\beta = 1$.
iii.

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Table 1

Bayes averages of the parameters and their estimated risks, relative absolute biases and credible intervals based on lower records ($\alpha = 2, \beta = 1.3, \theta = 0.9, \nu = 0.5$ and NR = 10000)

Rv	Loss functions	Estimators	Average	ER	RAB	LL	UL	Length
5	SE	α^*	1.9986	2.85e-06	0.0006	1.9968	2.0003	0.0035
		β^*	1.2988	2.25e-06	0.0024	1.2967	1.3003	0.0035
		θ^*	0.9021	5.38e-06	0.0009	0.8999	0.9037	0.0037
	LINEX	α^*	2.0010	2.57e-06	0.0005	1.9994	2.0032	0.0037
		β^*	1.3007	1.08e-06	0.0017	1.2992	1.3019	0.0027
		θ^*	0.9016	3.31e-06	0.0006	0.8995	0.9028	0.0033
7	SE	α^*	1.9996	1.23e-06	0.0002	1.9978	2.0008	0.0029
		β^*	1.3010	1.00e-06	0.0008	1.2996	1.3019	0.0023
		θ^*	0.9004	1.08e-06	0.0005	0.8990	0.9011	0.0021
	LINEX	α^*	2.0006	7.56e-07	0.0003	1.9992	2.0015	0.0023
		β^*	1.2996	5.33e-07	0.0003	1.2982	1.3006	0.0025
		θ^*	0.8995	5.17e-07	0.0005	0.8984	0.9004	0.0020
9	SE	α^*	2.0003	6.50e-07	0.0001	1.9990	2.0014	0.0023
		β^*	1.2997	1.66e-07	0.0002	1.2989	1.3004	0.0014
		θ^*	0.8997	3.30e-07	0.0003	0.8987	0.9007	0.0019
	LINEX	α^*	1.9999	3.03e-07	4.52e-05	1.9986	2.0009	0.0022
		β^*	1.3001	2.56e-07	5.48e-05	1.2988	1.3007	0.0019
		θ^*	0.8999	1.54-07	9.09e-05	0.8988	0.9005	0.0017

Table 2

Bayes averages for the parameters, their estimated risks, relative absolute biases and credible intervals based on lower records ($\alpha = 0.2, \beta = 3, \theta = 0.3, \nu = 0.5$, NR = 10000)

Rv	Loss functions	Estimators	Average	ER	RAB	LL	UL	Length
5	SE	α^*	0.1985	3.78e-06	0.0071	0.1962	0.2004	0.0042
		β^*	2.9992	1.41e-06	0.0002	2.9971	3.0002	0.0031
		θ^*	0.3011	3.79e-06	0.0036	0.2985	0.3036	0.0051
	LINEX	α^*	0.1979	4.75e-06	0.0101	0.1962	0.1990	0.0027
		β^*	3.0021	5.90e-06	0.0007	2.9996	3.0040	0.0044
		θ^*	0.2979	5.17e-06	0.0068	0.2964	0.2998	0.0034
7	SE	α^*	0.2008	1.13e-06	4.41e-03	0.1996	0.2018	0.0021
		β^*	2.9997	3.09e-07	8.23e-05	2.9985	3.0004	0.0019
		θ^*	0.2993	7.67e-07	2.18e-03	0.2981	0.3002	0.0021
	LINEX	α^*	0.1985	2.42e-06	0.0073	0.1975	0.1996	0.0021
		β^*	2.9990	1.10e-06	0.0003	2.9975	2.9998	0.0022
		θ^*	0.2995	4.52e-07	0.0015	0.2980	0.3001	0.0021
9	SE	α^*	0.1992	7.99e-07	3.84e-03	0.1980	0.1999	0.0019
		β^*	2.9998	1.27e-07	6.11e-05	2.9988	3.0003	0.0014
		θ^*	0.3006	5.59e-07	2.06e-03	0.2996	0.3013	0.0015
	LINEX	α^*	0.2002	3.76e-07	0.0013	0.1992	0.2011	0.0019
		β^*	2.9995	4.15e-07	0.0001	2.9983	3.0005	0.0021
		θ^*	0.30008	2.35e-07	0.0002	0.2988	0.30077	0.0019

Table 3

Bayes averages, their estimated risks, relative absolute biases and credible intervals of the rf and hrf at $t_0 = 0.5, 1, 1.5$ from EG-xg distribution based on SE and LENIX loss functions for different sample size of records Rv, and repetitions NR = 10000

Rv	t_0	Loss functions	Estimators	Average	ER	RAB	LL	UL	Length
5	0.5	SE	$R^*_{EGxg}(t_0)$	0.6235	1.63e-06	0.0010	0.6216	0.6248	0.0032
			$h^*_{1EGxg}(t_0)$	0.3315	4.58e-06	0.0057	0.3297	0.3331	0.0033
		LINEX	$R^*_{EGxg}(t_0)$	0.6254	1.83e-06	0.0019	0.6239	0.6262	0.0023
			$h^*_{1EGxg}(t_0)$	0.3327	2.07e-06	0.0022	0.3304	0.3342	0.0038
	1	SE	$R^*_{EGxg}(t_0)$	0.5425	6.52e-06	0.0039	0.5399	0.5446	0.0046
			$h^*_{1EGxg}(t_0)$	0.2351	8.73e-07	0.0021	0.2334	0.2368	0.0033
		LINEX	$R^*_{EGxg}(t_0)$	0.5467	5.57e-06	0.0039	0.5439	0.5478	0.0039
			$h^*_{1EGxg}(t_0)$	0.2343	4.05e-07	0.0014	0.2334	0.2352	0.0018
	1.5	SE	$R^*_{EGxg}(t_0)$	0.4894	1.24e-06	0.0020	0.4883	0.4903	0.0020
			$h^*_{1EGxg}(t_0)$	0.2054	4.03e-07	0.0001	0.2036	0.2063	0.0026
		LINEX	$R^*_{EGxg}(t_0)$	0.4881	8.47e-07	0.0007	0.4859	0.4891	0.0032
			$h^*_{1EGxg}(t_0)$	0.2040	3.84e-06	0.0063	0.2013	0.2061	0.0049
9	0.5	SE	$R^*_{EGxg}(t_0)$	0.6242	1.54e-07	7.39e-05	0.6234	0.6249	0.0015
			$h^*_{1EGxg}(t_0)$	0.3326	1.00e-06	2.59e-03	0.3316	0.3332	0.0016
		LINEX	$R^*_{EGxg}(t_0)$	0.6235	5.09e-07	0.0010	0.6228	0.6240	0.0011
			$h^*_{1EGxg}(t_0)$	0.3337	3.63e-07	0.0009	0.3326	0.3345	0.0019
	1	SE	$R^*_{EGxg}(t_0)$	0.5442	3.40e-07	0.0006	0.5430	0.5451	0.0021
			$h^*_{1EGxg}(t_0)$	0.2351	8.02e-07	0.0019	0.2339	0.2367	0.0027
		LINEX	$R^*_{EGxg}(t_0)$	0.5449	3.96e-07	0.0006	0.5436	0.5458	0.0022
			$h^*_{1EGxg}(t_0)$	0.2347	3.07e-07	0.0002	0.2334	0.2355	0.0021
	1.5	SE	$R^*_{EGxg}(t_0)$	0.4886	2.87e-07	4.35e-04	0.4874	0.4894	0.0019
			$h^*_{1EGxg}(t_0)$	0.2053	8.17e-08	3.29e-05	0.2045	0.2057	0.0012
		LINEX	$R^*_{EGxg}(t_0)$	0.4891	8.92e-07	8.92e-07	0.4878	0.4899	0.0020
			$h^*_{1EGxg}(t_0)$	0.2056	2.64e-07	2.64e-07	0.2048	0.2064	0.0015

Table 5

Bayes estimates of the parameters, rf, hrf and standard error
from EG-xg distribution for the real data based on lower records

	Rv		Estimators	Estimates	s.e
I	3	SE	$R^{*}_{EGxg}(t_0)$	0.0017	0.0012
			$h^{*}_{1EGxg}(t_0)$	10.282	0.0014
		LINEX	$R^{*}_{EGxg}(t_0)$	0.0001	0.0004
			$h^{*}_{1EGxg}(t_0)$	10.285	0.0004
II	7	SE	$R^{*}_{EGxg}(t_0)$	0.0035	0.0005
			$h^{*}_{1EGxg}(t_0)$	6.9606	0.0007
		LINEX	$R^{*}_{EGxg}(t_0)$	0.0034	0.0004
			$h^{*}_{1EGxg}(t_0)$	6.9607	0.0006

Table 5

Bayes estimates of the parameters, standard error and estimated risks
for the real data based on lower records

	Rv	Loss functions	Estimators	Estimates	s.e
I	3	SE	α^*	2.0017	0.0007
			β^*	1.0024	0.0008
			θ^*	6.0042	0.0013
		LINEX	α^*	2.0001	0.0012
			β^*	0.9994	0.0007
			θ^*	5.9986	0.0010
II	7	SE	α^*	1.5015	0.0009
			β^*	3.0014	0.0014
			θ^*	6.0008	0.0005
		LINEX	α^*	1.5011	0.0005
			β^*	3.0024	0.0008
			θ^*	5.9992	0.0004

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