# Error analysis and uncertainty evaluation for the determination of stress intensity factor by caustic

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Abstract

The determination of the stress intensity factor (SIF) is a critical aspect of fracture mechanics, providing valuable information about crack growth behavior and structural integrity. Caustic curve calculations are subject to errors and uncertainties. This research paper aims to provide a comprehensive review of error analysis in caustic curve calculations to reduce the errors in calculating SIF and hence the uncertainty estimation of the SIF calculation by caustic method. The determination of the stress intensity factor (SIF) is a critical aspect of fracture mechanics, providing valuable information about crack growth behavior and structural integrity. Caustic curve calculations are subject to errors and uncertainties. This research paper aims to provide a comprehensive review of error analysis in caustic curve calculations to reduce the errors in calculating SIF and hence the uncertainty estimation of the SIF calculation by caustic method. The uncertainty estimation was done for two methods for calculating SIF: the first method is measuring the diameter of the caustic curve, and the second is measuring the whole area. Regarding the sources of errors, it was found that the most effective source of error is the magnification ratio, followed by the thickness of the specimen; they have the highest sensitivity coefficients in the uncertainty budget. The calculated SIF by area reduced uncertainty from 1.73 to 0.945 Mpa mm-1/2, i.e., approximately 50%. It means that the method of calculating SIF by area is more accurate and precise than the method of measuring the whole diameter.

*Keywords:* Stress intensity factor, caustic, uncertainty, fracture mechanics.

## 1.Introduction

Fracture mechanics and the determination of SIF are crucial in analysing the structural integrity of various materials and components. The accurate measurement of SIF allows engineers and researchers to evaluate the severity of damage, assess the remaining life of a structure, and make informed decisions regarding maintenance and repair. Caustic method is one of the experimental techniques that have shown promising results in recent years. The caustic method utilizes the principle of light refraction to quantify the stress intensity factor. The basic idea behind this technique is to utilize the deformation-induced caustic pattern that forms on the fracture surface of a loaded specimen. When light passes through this deformed region, it bends or refracts, resulting in a caustic pattern. By examining this pattern, engineers can calculate the SIF accurately [1-3]. The caustic patterns are formed by the interaction of light with the refractive index variation caused by the crack opening displacement near the crack tip. Various studies (or research) investigated the correct value of the stress intensity factor and the factors affecting its true value [4-6]. This work analyses the errors resulting from calculating the stress intensity factor by the experimental method of caustics and the estimated sources of uncertainty of the results. The objective of this error analysis is to find the most critical source of error, to reduce the error, and uncertainty resulting from the caustic method. The uncertainty budget for two methods of calculating SIF is estimated. The equations for calculating the error in determining the SIF in cracked specimens were determined as functions of the experimental parameters, and the relations for the influence of each parameter were studied. The analysis of caustic patterns requires the development of mathematical models and algorithms. The following section shows the mathematical framework used to calculate SIF from caustic pattern presented.

## 2. Theoretical Background:

The following equation gives the relationship between any general point on the caustic image with its locus and the stress intensification and optical magnification [7]:

$$\acute{x} = mr_0(\cos\emptyset + \frac{2}{3}\cos\frac{3}{2}\emptyset) \quad (1)$$

$$\acute{y} = mr_0(\sin\emptyset + \frac{2}{3}\sin\frac{3}{2}\emptyset) \quad (2)$$

**Where** $\acute{x}, \acute{y}$ are Cartesian coordinates of points on screen as shown in Fig. 1.

$\emptyset$ is the angle between r and the x axis

m is the optical magnification ratio = $\frac{Zi+Zo}{Zi}$ (3)

Where Zi is the distance between point light source and specimen plane and Zo is the distance between the specimen plane and the image plane as shown in Fig. 2.

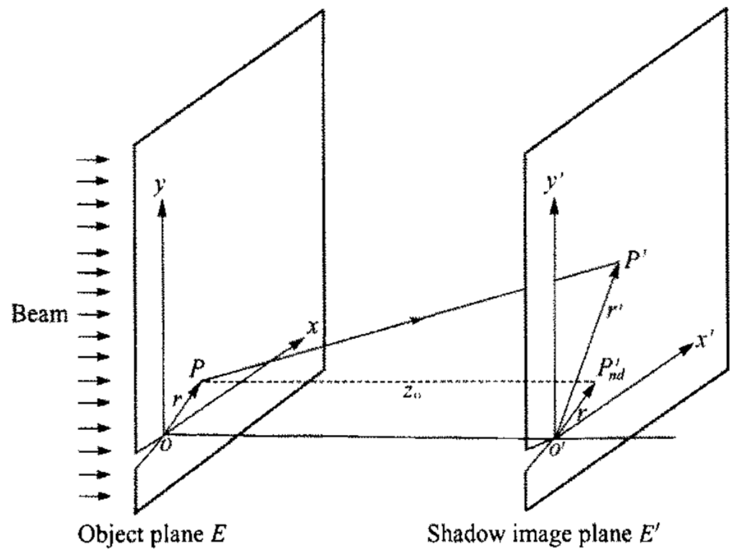


**Figure 1:** geometrical conditions of the shadow optical analysis**.**

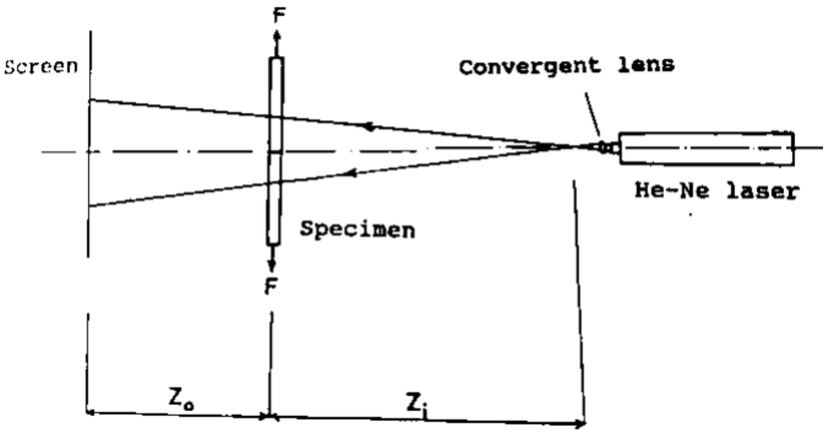


**Figure 2:** schematic view of the optical set-up .

The stress intensity factor $K_I$ is calculated from the equation obtained from the caustic method, which calculates the stress intensity factor through measuring the maximum transverse caustic diameter D in the following equation 4 as follows [8-11]:

$$K_I = \frac{2\sqrt{2\pi}D^{2.5}}{3\ (3.17)^{2.5} z_o d\ |c| m^{1.5}} \qquad (4)$$

Where d is the thickness of the specimen

C is the Optical constant

The following deduced equation determines the stress intensity factor using a new and more accurate method [12]. The new method presents the relationship between the stress intensity factor KI, and the area inside the caustic curve instead of the caustic diameter. Image processing software (ImageJ) was used to measure the area inside the curve. The stress intensity factor was calculated by measuring the inside area of the caustic curve.

Simply the area of the caustic curve is defined as, $A = \frac{1}{2}\int_{-\pi}^{\pi} r^2 \, d\theta$ (5)

A is the area of the caustic curve

In polar coordinates, $r^2 = x^2 + y^2$ (6)

$$r^2 = m^2 {r_0}^2 [\left(\cos \emptyset + \frac{2}{3}\cos\frac{3}{2}\emptyset\right)^2 + \left(sin\, \emptyset + \frac{2}{3} sin\frac{3}{2}\emptyset\right)^2]$$

$$= m^2 {r_0}^2 [(+\frac{4}{9}\cos^2\frac{3}{2}\emptyset + \frac{4}{3}\left(cos\, \emptyset\, cos\frac{3}{2}\emptyset\right) + (\sin^2 \emptyset + \frac{4}{9}\sin^2\frac{3}{2}\emptyset + \frac{4}{3}\left(sin\, \emptyset\, sin\frac{3}{2}\emptyset\right)] \quad (7)$$

Substitute, $\cos^2 \emptyset + \sin^2 \emptyset = 1, and \; \cos^2\frac{3}{2}\emptyset + \sin^2\frac{3}{2}\emptyset = 1$ in equation

$$\therefore r^2 = m^2 {r_0}^2 (1 + \frac{4}{9} + \frac{4}{3}\left(cos\, \emptyset\, cos\frac{3}{2}\emptyset + sin\, \emptyset\, sin\frac{3}{2}\emptyset\right) \quad (8)$$

$$cos\, \emptyset\, cos\frac{3}{2}\emptyset + sin\, \emptyset\, sin\frac{3}{2}\emptyset = \cos\left(\emptyset - \frac{3}{2}\emptyset\right) = cos\frac{\emptyset}{2} \quad (9)$$

$$\therefore r^2 = m^2 {r_0}^2 (\frac{13}{9} + \frac{4}{3} cos\frac{\emptyset}{2}) \quad (10)$$

$$\text{Area (A)} = \frac{1}{2}\int_{-\pi}^{\pi} m^2 {r_0}^2 (\frac{13}{9} + \frac{4}{3} cos\frac{\emptyset}{2})\, d\emptyset \quad (11)$$

$$= \frac{1}{2} m^2 {r_0}^2 (\frac{13}{9}\pi + \frac{8}{3} sin\frac{\pi}{2}) - \frac{1}{2} m^2 {r_0}^2 (\frac{13}{9}(-\pi) + \frac{8}{3} sin -\frac{\pi}{2}) = m^2 {r_0}^2 (\frac{13}{9}\pi + \frac{8}{3}) \quad (12)$$

$$Area\ (A) = m^2 (\frac{3KI|C|dZo}{2\sqrt{2\pi m}})^{\frac{4}{5}} (\frac{13}{9}\pi + \frac{8}{3}) \quad (13)$$

$$K_I = \frac{A^{1.25}}{7.1\, m^2\ |c|\ \mathrm{d}\, z_o} \quad (14)$$

To estimate the error resulting from each parameter influencing the calculating (SIF), partial differentiation for each parameter in the equations of calculating the (SIF) by two methods (by measuring caustic curve transverse diameter and by measuring area) was introduced in the following section.

## 3. Error analysis

Caustic curve error analysis refers to the process of examining and quantifying the uncertainties or errors associated with the construction or interpretation of caustic curves [13-15]. When analyzing errors in caustic curves, several factors need to be considered:

1. Measurement errors: These errors can occur during the process of measuring the position or intensity of the caustic points or patterns. Factors such as instrument resolution, calibration, and alignment can contribute to measurement uncertainties. Using precise and accurate measurement techniques and equipment can help minimize these errors.

2. Optical system errors: Errors can arise from imperfections or limitations in the optical system used to generate or observe the caustic curve. Factors such as aberrations, diffraction, or non-

uniform illumination can affect the shape and accuracy of the caustic curve. Understanding the limitations of the optical system and compensating for these errors can improve the reliability of the caustic curve analysis.

3. Data processing errors: Errors can occur during the process of analyzing and interpreting the raw data obtained from the caustic curve experiment. Factors such as noise, signal processing techniques, or algorithmic errors can introduce uncertainties or inaccuracies in the final analysis. Careful data processing and validation techniques can help reduce these errors.

4. Assumptions and simplifications: Caustic curves are often analyzed using mathematical models or approximations that make certain assumptions or simplifications about the light wave behavior or the surface geometry. Errors can arise if these assumptions do not accurately represent real-world conditions. Validating the models against experimental data or considering more sophisticated models can help mitigate these errors.

5. Environmental conditions: Environmental factors such as temperature, humidity, or air turbulence can affect the behavior of light waves and introduce errors in the caustic curve analysis. Controlling or monitoring these environmental conditions can help minimize the impact of such errors.

To perform error analysis of caustic curves, statistical techniques such as uncertainty analysis or error propagation can be used. These methods involve quantifying uncertainties or errors associated with different factors and propagating them through the caustic curve construction or interpretation process to determine the overall error or uncertainty in the final result.

By conducting error analysis of caustic curves, it is possible to identify and quantify the uncertainties or errors associated with the construction or interpretation of these curves. This information can help in improving the accuracy and reliability of caustic curve analysis and ensuring the validity of the conclusions drawn from these curves in various fields such as optics, fluid dynamics, or materials science.

### *3.1 Error analysis for measuring (SIF) by transverse diameter the caustic curve*

#### *3.1.1 Error resulting from the transverse diameter D*

The influence of the caustic curve diameter D on the (SIF) or sensitivity coefficient is obtained by partial differentiation of equation (4) with respect to D, the influence of the caustic diameter on the (SIF) is determined by the following equation

$$C_D = \frac{\partial K}{\partial D} = \frac{5\sqrt{2\pi}\, D^{1.5}}{3(3.17)^{2.5} z_o d c m^{1.5}} \tag{15}$$

#### *3.1.2 Error resulting from $z_o$ distance between the specimen plane and the image plane*

Also by partial differentiation of the equation (4) with respect to $z_o$, sensitivity coefficient of the parameter $z_o$ is obtained from the following equation

$$C_{z_o} = \frac{\partial K}{\partial z_o} = \frac{-2\sqrt{2\pi}\, D^{2.5}}{3(3.17)^{2.5} {z_o}^2 d|c| m^{1.5}} \tag{16}$$

#### *3.1.3 Error resulting from d (the thickness of the specimen)*

The following equation (19) expressed the sensitivity coefficient of the thickness d

$$C_d = \frac{\partial K}{\partial d} = \frac{-2\sqrt{2\pi}\,D^{2.5}}{3(3.17)^{2.5} z_o d^2 |c| m^{1.5}} \tag{17}$$

*3.1.4 Error resulting from m*

Also by partial differentiation of the equation (4), the sensitivity coefficient of magnification m is as follow

$$C_m = \frac{\partial K}{\partial m} = \frac{-\sqrt{2\pi}\,D^{2.5}}{(3.17)^{2.5} z_o d |c| m^{2.5}} \tag{18}$$

*3.2 Error analysis for measuring (SIF) by area of the caustic curve*

*3.2.1 Error resulting from measuring Area*

The influence of area of caustic curve on the (SIF) or sensitivity coefficient is obtained by partial differentiation of the equation (14) with respect to A; the influence of the area on the (SIF) is determined by the following equation

$$C_A = \frac{\partial K}{\partial A} = \frac{1.25A^{0.25}}{7.1m^2 d z_o |c|} \tag{19}$$

*3.2.2 Error resulting from measuring $z_o$*

Also by partial differentiation of the equation (14) with respect to zo, sensitivity coefficient of the parameter zo is obtained from the following equation

$$C_{zo} = \frac{\partial K}{\partial zo} = \frac{-A^{1.25}}{7.1m^2 {z_o}^2 d |c|} \tag{20}$$

3.2.3 *Error resulting from d (the thickness of the specimen*)

The following equation (23) expressed the sensitivity coefficient of the thickness d

$$C_d = \frac{\partial K}{\partial d} = \frac{-A^{1.25}}{7.1m^2 z_o d^2 |c|} \tag{21}$$

*3.1.4 Error resulting from m*

Also by partial differentiation of the equation (14), the sensitivity coefficient of magnification m is as follow

$$C_m = \frac{\partial K}{\partial m} = \frac{-2A^{1.25}}{7.1m^3 z_o d |c|} \tag{22}$$

*4. The uncertainty budget*

The uncertainty in caustic curve calculations can arise from various sources. It is important to assess and quantify these uncertainties to understand the reliability and validity of the calculated caustic curve. Uncertainty analysis involves evaluating the sensitivity of the caustic curve calculations to different sources of error and estimating the resulting uncertainties. The following is a numerical example for estimating the uncertainty budget.

An example of a caustic curve with the analyzed caustic diameter and area processed by imageJ software is shown in Fig 3

The specimen was artificially cracked by impact notcher to produce sharp cracks in the specimen. Sharp cracked specimens from Polymethyl methacrylate (PMMA) material were loaded in tension with an 8 kg weight.

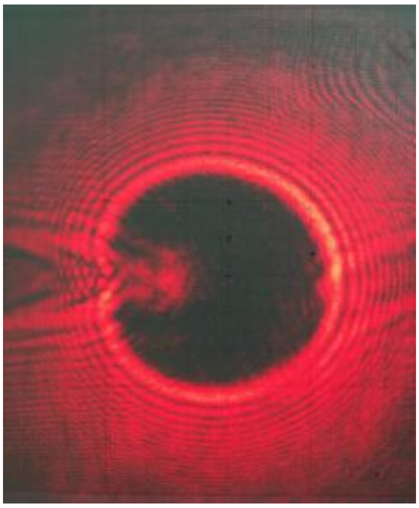

**Figure 3:** caustic curve.

Where

$Z_o$ =2700 mm

$Z_i$= 400 mm

d= 5.3 mm

The uncertainty budget are shown in tables 1,2

**Table 1** *Uncertainty budget for measuring SIF by diameter*

| Source | Error Value (+/-) | unit | Devisor | Sensitivity coefficient | |
|---|---|---|---|---|---|
| A | 0.5 | mm | √3 | 1.187 | 0.353 |
| Zo | 0.5 | mm | √3 | 0.006 | 3.08025E-09 |
| d | 0.3 | mm | 1 | 2.65 | 0.793 |
| m | 0.02 | | √3 | 2.71 | 0.838 |

The combined uncertainty = $\sqrt{u_D^2 + u_{zo}^2 + u_d^2 + u_m^2}$

Expanded uncertainty = combined uncertainty *coverage factor

Using confidence level 95% with coverage factor =2 =0.866 *2=1.73Mpa mm$^{-1/2}$

The calculated SIF by diameter D= 28.6±1.73 Mpa mm$^{-1/2}$

**Table 2** *Uncertainty budget for measuring SIF by diameter*

| Source | Error Value (+/-) | unit | Devisor | Sensitivity coefficient | |
|---|---|---|---|---|---|
| A | 0.2 | mm | √3 | 0.011 | 0.001 |
| Zo | 0.5 | mm | √3 | 0.0030 | 0.001 |
| d | 0.3 | mm | 1 | 1.5735 | 0.472 |
| m | 0.02 | | √3 | 2.152 | 0.0248 |

Expanded uncertainty =0.945 Mpa mm-1/2

The calculated SIF by diameter D= 23.3±0.945 Mpa mm$^{-1/2}$

From the two uncertainty budgets above it was shown that the most effective sources of error are the magnification ratio, and the thickness of the specimen, they have the maximum sensitivity coefficient. The calculated SIF by area reduced uncertainty from 1.73 to 0.945 Mpa mm$^{-1/2}$ it means that the method of calculating SIF by area is more accurate and precise.

## 5. Conclusion:

By conducting a comprehensive review of error analysis in caustic curve calculations, this research aims to provide researchers, scientists, and engineers with a better understanding of the challenges and strategies involved in quantifying errors in this critical field. The insights gained from this study can help improve the accuracy and reliability of caustic curve calculations. The error analysis and uncertainty budget of the new accurate method for calculating SIF by measuring the area have been thoroughly investigated. The error analysis and uncertainty budget of the new accurate method for calculating SIF by measuring the area have been thoroughly investigated. The error analysis and uncertainty budget of the new accurate method for calculating SIF by measuring the area have been thoroughly From the error analysis, it was found that the most effective source of error is the magnification ratio, followed by the thickness of the specimen. From the error analysis, it was found that the most effective source of error is the magnification ratio, followed by the thickness of the specimen.The calculated SIF by area reduced uncertainty from 1.73 to 0.945 Mpa mm-1/2 i.g approximately 50%. It means that the method of calculating SIF by area is more accurate and precise.

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