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Similarity Transforming Techniques of Partial Differential Equations and its Applications: A review

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ABSTRACT
The current paper is a review of some transformation techniques of partial differential equations (PDEs) using similarity techniques which have the ability to reduce the number of independent variables to be a single variable resulting in a simpler ordinary differential equation (ODEs). These techniques have the benefit to obtain some completely new solutions. The paper introduces a summary and literature review for method of characteristic lines, Bäcklund transformations, Cole-Hopf Transformation, Lie infinitesimals, Non-classical symmetry method, Potential Symmetry, direct methods and characteristic function method.

Keywords: *Partial differential equations, Transformation Methods, Similarity solutions*

1. Introduction

This article may be broken down into two primary sections. In the first section, we will look at terminology as well as a review of highly nonlinear issues that are common in engineering applications. Some examples of these problems include:
i. Non-Newtonian flow for viscous, pseudo plastic, and volatile fluids
ii- Problems with boundary layers in accordance with Newton's law
iii- The issue of the development of waves and the interaction of waves This particular sort of issue manifests itself in physical systems, such as plasmas, elastic strings, or the lattice vibration of a crystal when the temperature is kept low.
Methods that involve simplifying a complex set of nonlinear equations or determining a starting point for a solution are discussed in the second section of this work. These methods are divided into two categories:
i- methods that reduce the complexity of the system of nonlinear equations, and (ii) methods that evaluate starting points for solutions.
ii- Techniques by which the partial differential equation may be transformed into an ordinary one.
A review of prior work done in the topic is offered across all of the parts.

2. Classification of partial differential equations according to nonlinearity

2.1. Heat or fluid losses at the boundaries

A non-Newtonian shape may be seen for the heat or fluid loss that occurs at the boundary in certain circumstances. For example:

$$K\frac{\partial T}{\partial x} = -\varepsilon\left(T^n - T_\infty^n\right) \tag{1}$$

where K represents thermal conductivity, ε is the surface emissivity coefficient which indicates the radiation of heat from a 'grey body' according the Stefan-Boltzmann Law, compared with the radiation of heat from an ideal 'black body', T represents temperature, T_∞ represents ambient temperature, and n represents power index. You may get further information from here (R.Wiley et al, 1982)

2.2. Nonlinearity of the dependent variables

In some instances, it was discovered that the dependent variables followed a power law. For instance, the thermal conductivity may not be constant in some circumstances and may instead take the shape of a power law;

$$K = T^s \tag{2}$$

In this equation, K represents the thermal conductivity, T represents the temperature, and s represents the power law index. Inside a boundary layer, the same nonlinearity may be seen in the concentration of a chemical compound denoted by the letter C. The following is the mathematical description of this phenomenon:

$$\frac{\partial u}{\partial x}+\frac{\partial \mathrm{v}}{\partial y}=0 \tag{3}$$

$$u\frac{\partial u}{\partial x}+\mathrm{v}\frac{\partial u}{\partial y}-g\beta(c-c_\infty)-\nu\frac{\partial^2 u}{\partial y^2}=0 \tag{4}$$

$$u\frac{\partial c}{\partial x}+\mathrm{v}\frac{\partial c}{\partial y}+k(c-c_\infty)^n-D\frac{\partial^2 c}{\partial y^2}=0 \tag{5}$$

where u and v are the velocities, c and c_∞ are the concentrations of chemical within and beyond the boundary layer, respectively, n is the power law index, and k, g, D, and v are flow parameters defined in the notation index.

2.3. Non linearity in the derivative of the dependent variable

In some case the derivative of velocity perpendicular to a plate has a power index n. This case is described by the equations;

$$\frac{\partial u}{\partial x}+\frac{\partial \mathrm{v}}{\partial y}=0 \tag{6}$$

$$u\frac{\partial u}{\partial x}+\mathrm{v}\frac{\partial u}{\partial y}=\mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{x}}+\frac{k}{\rho}\left[\left(\frac{\partial u}{\partial y}\right)^n\right]_y \tag{7}$$

where k is the coefficient of consistency, ρ is the fluid density and n the power law index. This problem was investigated by numerous researchers such as (Anderson et al, 1979), (Gorla et al, 1995) and (Kumari et al, 1997) through the application of deductive transformation methods and finite difference schemes.

2.4. Non-Newtonian flow under forced convective heat

In some case the non linear flow is heated. In this case an energy equation is added to the two previous (6) and (7) equations;

$$\frac{\partial u}{\partial x}+\frac{\partial \mathrm{v}}{\partial y}=0, \tag{8}$$

$$u\frac{\partial u}{\partial x}+\mathrm{v}\frac{\partial u}{\partial y}=g\beta(t-t_\infty)+\frac{k}{\rho}\left[\left(\frac{\partial u}{\partial y}\right)^n\right]_y \tag{9}$$

$$u\frac{\partial \theta}{\partial x}+\mathrm{v}\frac{\partial \theta}{\partial y}=\frac{k}{\rho C_p}\frac{\partial^2\theta}{\partial y^2} \tag{10}$$

Several investigators looked into this matter to see what they could find. (Kassem et al, 2008) and (Abd-el Malek et al, 2005) conducted research on the influence of fluid density on the laminar velocity and thermal distribution of non-Newtonian fluids, respectively.

2.4. Non linearity of the dependent variable subjected to first order derivative

The dependent variable in the case of a magneto-hydrodynamic heat and mass transfer is non-linear, and the Newtonian laminar flow is regulated by the equations (EL-Kabeir et al, 2008)

$$\frac{\partial u}{\partial x}+\frac{\partial \mathrm{v}}{dy}=0, \tag{11}$$

$$\frac{\partial u^n}{\partial y}+K\frac{\partial u^{2n}}{\partial y}+2g\frac{\sigma K B^2}{\mu}\frac{\partial u}{\partial y}=\frac{gK_B}{\nu}\left(B_T\frac{\partial T}{\partial y}+B_C\frac{\partial C}{\partial y}\right) \tag{12}$$

where u and v represent the velocity components along the x- and y-directions, respectively; n represents the power law index or the viscosity index; σ represents the electrical conductivity of the fluid; B represents the magnetic induction; K and ν represent the inertial coefficient and viscosity of the fluid, respectively; K_B represents the permeability of the porous medium; T and C represent the temperature and concentration, respectively; and σ represents the fluid electrical conductivity. While B_T and B_C represent the coefficients of heat expansion and concentration expansion, respectively, g denotes the acceleration due to gravity.

3. Boundary Layer Problems

Schowalter, in 1978, was the one who first presented the idea of a boundary layer. A layer of fluid that is found in the immediate proximity of a bounding surface is what is referred to as it. The planetary boundary layer is the layer of air near the ground that is impacted by the movement of diurnal heat, moisture, or momentum to or from the surface. This layer is located in the atmosphere of the earth. The portion of the flow that is closest to the surface of an aircraft wing is referred to as the boundary layer. The boundary layer effect takes place in the field area, which is the location where all of the changes in the flow pattern take place. There are many distinct varieties of laminar boundary layers, each of which may be roughly categorized based on the structure of the layer itself as well as the conditions that led to its formation. One example of a Stokes boundary layer is the thin shear layer that forms on an oscillating body. On the other hand, a Blasius boundary layer is the well-known similarity solution for the stable boundary layer that is connected to a flat plate that is held in an approaching unidirectional flow. When a fluid is rotated, the Coriolis effect, rather than the convective inertia that would normally balance viscous forces, may do it instead. This results in the production of an Ekman layer. In the process of heat transport, thermal boundary layers also occur. It is possible for many kinds of boundary layers to live near a surface at the same time. The equations that determine the behavior of a boundary layer in two dimensions are provided by

$$\frac{\partial u}{\partial x}+\frac{\partial \mathrm{v}}{\partial y}=0 \tag{13}$$

$$u\frac{\partial u}{\partial x}+\mathrm{v}\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \tag{14}$$

$$u\frac{\partial \mathrm{v}}{\partial x}+\mathrm{v}\frac{\partial \mathrm{v}}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 \mathrm{v}}{\partial x^2}+\frac{\partial^2 \mathrm{v}}{\partial y^2}\right) \tag{15}$$

where u and v are the velocity components in the x,y directions, ρ is the density, p is the pressure and ν is the kinematic viscosity of the fluid at a point.

4. Methods of transformations

4.1 Method of characteristic lines

This approach is applicable to partial differential equations that are quasi-linear. Characteristic lines serve as an envelope for family solutions and are represented by the symbol s. The parameter s stands for the scaled distance that is travelled along a characteristic line. After this, the partial differential equation is converted into a system of ordinary differential equations by use of a differentiation of the dependent variable with respect to s. This set explains how the answer varies along every particular attribute by describing the changes.
Suppose that we need to solve the following quasi linear partial differential equation

$$\frac{\partial u}{\partial x}+x^2\frac{\partial u}{\partial y}=-yu$$

$$u(0,y)=f(y) \tag{16}$$

by chain rule we differentiate u(x,y) with respect to the variable *s*;

$$\frac{du}{ds}=\left(\frac{\partial x}{\partial s}\right)u_x+\left(\frac{\partial y}{\partial s}\right)u_y \tag{17}$$

Comparing (16) with (17), we get

$$\frac{\partial x}{\partial s}=1 \tag{18}$$

$$\frac{\partial y}{\partial s}=x^2 \tag{19}$$

$$\frac{du}{ds}=-yu \tag{20}$$

The initial condition can be rewritten as

$$\begin{aligned} x(s=0)&=0\\ y(s=0)&=t_1\\ u(s=0)&=f(t_1)\end{aligned} \tag{21}$$

The solution of (18) using (21) is given by

$$x(s) = s \tag{22}$$

Therefore (19) can be rewritten as

$$\frac{\partial y}{\partial s} = s^2 \tag{23}$$

From (21) we obtain

$$y(s,t_1) = \frac{s^3}{3} + t_1 \tag{24}$$

And finally, the solution of (20) using (21) and (24) is:

$$u(s,t_1) = f(t_1)\ \exp(-\frac{s^4}{12} - st_1) \tag{25}$$

Then

$$s = x,\ t_1 = y - \frac{x^3}{3} \tag{26}$$

Finally, by substituting in (25) the solution of (16) has the form

$$u(x,y) = f(y - \frac{x^3}{3})\ \exp(\frac{x^4}{4} - xy) \tag{27}$$

4.2 Bäcklund Transformations

If a Backlund transformation can be found, the solution to a nonlinear partial differential equation can be used to obtain either a different solution to the same partial differential equation or a solution to a different nonlinear partial differential equation. This is only possible if the Backlund transformation can be found. Take, for instance, the sine-Gordon equation as an illustration.

$$\frac{\partial^2 u}{\partial x \partial t} = \sin u \tag{28}$$

this equation was studied by (Darboux, 1984) in connection with the pseudo spherical surfaces. An auto Bäcklund transformation is given by the pair of partial differential equations.

$$\begin{aligned} \mathrm{v}_x &= u_x + 2\lambda \sin\left(\frac{\mathrm{v}+u}{2}\right) \\ \mathrm{v}_t &= -u_t + \frac{2}{\lambda}\sin\left(\frac{\mathrm{v}-u}{2}\right) \end{aligned} \tag{29}$$

where λ is a constant. i) Starting in (29) with the solution $u_0 = 0$

Then, (29) becomes

$$\begin{aligned} \mathrm{v}_x &= 2\lambda \sin\frac{\mathrm{v}}{2} \\ \mathrm{v}_t &= \frac{2}{\lambda}\sin\frac{\mathrm{v}}{2} \end{aligned} \tag{30}$$

ii) This system of equation is solved giving a new solution of the sine-Gordon equation v= u_1

$$\mathrm{v} = C[t + \lambda^2 x], \tag{31}$$

where C is a constant of integration. Steps (i) and (ii) are repeated twice, giving u[2] and u[3]. According to (Bianchi, 1922), the knowledge of a third solution enables one to generate an infinite number of solutions through the relation;

$$\tan\left(\frac{u[3]-u[0]}{4}\right) = \frac{\lambda + C}{C - \lambda}\ \tan\left(\frac{u[1]-u[2]}{4}\right) \tag{32}$$

this relation is known as the *theorem of permutability* or nonlinear superposition and was illustrated by (Bianchi, 1922). More examples and details for this method may be found in (Zwillinger, 1997).

4.3 Cole-Hopf Transformation

Both Hopf (1950) and Cole (1951) separately developed a transformation that simplifies the solution to a quasi-nonlinear partial differential equation by transforming it into a linear form. After that, the solution to the initial problem is found by using the Cole Hopf transformation formula. Consider, for the sake of illustration, the equation developed by Burgers.

$$u_t + uu_x = \sigma u_{xx} \tag{33}$$

where σ is a constant and *u* is transformed to another variable *w* using Cole Hopf transformation

$$u = -2\sigma \frac{w_x}{w} \tag{34}$$

from (34) into (33) we get

$$w_t = \sigma w_{xx} \tag{35}$$

which is a linear diffusion equation whose solution w(x,t) is easily obtained. Then through Cole Hopf transformation, the function u(x,t) which is the solution of Burger equation, is derived.

4.4 Hodograph Transformation

This type of transformation is similar to the Cole Hopf transformation, aims to linearize the partial differential equations as precisely as possible, which ultimately results in a new formulation of the equations themselves. The constant two-dimensional gas dynamic equations provide an excellent illustration of this strategy.

$$\begin{aligned} &v_y + uv_x + bvu_x = 0 \\ &u_y + uu_x + \frac{1}{b} vv_x = 0 \end{aligned} \tag{36}$$

where b is a constant. The concerned transformation can be Applied to (36) by inverting $u(x, y)$ and $\mathrm{v}(x, y)$ to find

$$\begin{aligned} x_u &= -\frac{\mathrm{v}_y}{J}, \qquad x_v = \frac{u_y}{J}, \\ y_u &= \frac{\mathrm{v}_x}{J}\ , \qquad y_v = -\frac{u_x}{J} \end{aligned} \tag{37}$$

Where J is the Jacobian of the transformation, $J = u_y v_x - v_y u_x$. From (37) in (36) leads to

$$\begin{aligned} &x_u - uy_u + b\mathrm{v}y_v = 0 \\ &x_v - uy_v + \frac{1}{b}\mathrm{v}y_u = 0 \end{aligned} \tag{38}$$

As equations (38) are quasi-linear, they are solved using the method of characteristics.

4.5 Lie Infinitesimals

From the year 1870, Lie has discovered that it is feasible to subordinate all prior theories of integration of ordinary differential equations to a general approach. This discovery was made possible by the fact that Lie was the first person to do so. In this manner, it became able to deduce the earlier theories from a common source, and at the same time, to build a larger point of view for the general theory of differential equations. Both of these accomplishments were made possible as a result of this method.

Take into consideration the generic instance of a non-linear system of differential equations with an arbitrary number q of unknown functions that rely on p separate independent variables.

$$\Delta^i(x, u_{(k)}) = 0, \quad i = 1,2,...,m \tag{39}$$

where m is the number of differential equations describing the system and the term $u_{(k)}$ is the *k*th derivative of *u* with respect to *x*. Given a family of coordinates x and dependent variable u. Consider a one parameter transformation

$$\bar{x} = \Xi(x, u; \alpha) \tag{40}$$

$$\bar{u} = \Theta(x, u; \alpha) \tag{41}$$

where α is the transformation parameter. We assume that Ξ and Θ are differentiable a sufficient number of time with respect to α. If ε is an infinitesimally small value of α, expanding the variables $\bar{x}, \bar{u}$ around ε generates.

$$\bar{x} = \Xi(x,u;0) + \varepsilon \frac{\partial \Xi}{\partial \alpha}(x,u;\alpha)\bigg|_{\alpha=0} + \frac{\varepsilon^2}{2!}\frac{\partial^2 \Xi}{\partial \alpha^2}(x,u;\alpha)\bigg|_{\alpha=0} + \dots \tag{42}$$

$$\bar{u} = \Theta(x,u;0) + \varepsilon \frac{\partial \Theta}{\partial \alpha}(x,u;\alpha)\Big|_{\alpha=0} + \frac{\varepsilon^2}{2!}\frac{\partial^2 \Theta}{\partial \alpha^2}(x,u;\alpha)\Big|_{\alpha=0} + \dots \quad (43)$$

These two equations reduce to

$$\bar{x}_i = x_i + \varepsilon \xi_i(x,u) + O(\varepsilon^2), \quad i = 1,2,\dots,p \quad (44)$$

$$\bar{u}^a = u^a + \varepsilon \theta_a(x,u) + O(\varepsilon^2), \quad a = 1,2,\dots,q \quad (45)$$

where ξ_i and θ_a are the independent and dependent variables infinitesimals transformations described by

$$\xi_i(x,u) = \frac{\partial \Xi}{\partial \alpha}(x,u;\alpha)\Big|_{\alpha=0} \quad (46)$$

$$\theta_a(x,u) = \frac{\partial \Theta}{\partial \alpha}(x,u;\alpha)\Big|_{\alpha=0} \quad (47)$$

The infinitesimal generator associated with (44) and (45) is given by the vector field:

$$\vec{\mathrm{v}} \equiv \sum_{i=1}^{p} \xi_i(x,u)\frac{\partial}{\partial x_i} + \sum_{a=1}^{q} \theta_a(x,u)\frac{\partial}{\partial u_a} \quad (48)$$

and the prolongations of this field contain the derivatives of u_a with respect to the independent variables.

$$\mathrm{Pr}\left(\vec{\mathrm{v}}\right) = \vec{\mathrm{v}} + \sum_{a=1}^{q}\sum_{k}^{J} \theta_a^k\left(x, u_{(k)}\right)\frac{\partial}{\partial u_{a(k)}} \quad (49)$$

The invariance condition regulating similarity transformations is written as

$$\frac{\partial x_i}{\xi_i} = \frac{\partial u_j}{\theta_j}, \quad i = 1,2\dots\dots p, \quad j = 1,2,\dots\dots q \quad (50)$$

The method starts with applying the chain rule to the variables that have been converted in the equations (40) and (41), and then substitute these changed variables for the derivatives in the system of equations (39). From this, we have a system of ordinary differential equations whose solution leads to the infinitesimal transformation of the system variable by setting equal to zero the coefficients of the k^{th} derivative of $\bar{u}$ not present in (39). After that, we make advantage of the invariance requirement (50), which allows us to define both the similarity variable $\eta(x_i)$ and the similarity functions. Detailed examples can be found in (R. Seshadri et al, 1985, P.L. Sachdev, 2000, Baumann, 2000).

4.5.1 Types of Infinitesimal Transformation

Three types are known of infinitesimal transformation

i- Translational transformation

a- Translation in x ; $\mathbf{X} = \partial_x \quad x_\varepsilon = x + \varepsilon, \quad y_\varepsilon = y$ (51)

b- Translation in y; $\mathbf{X} = \partial_y \quad x_\varepsilon = x, \quad y_\varepsilon = y + \varepsilon$ (52)

ii- Amplitude transformation

$$\mathbf{X} = x\partial_x + y\partial_y, \qquad x_\varepsilon = e^{\varepsilon}x, \; y_\varepsilon = e^{\varepsilon}y \quad (53)$$

iii- Rotation in the (*x; y*) plane

$$\mathbf{X} = -y\partial_x + x\partial_y, \qquad x_\varepsilon = x\cos\varepsilon - y\sin\varepsilon, \qquad y_\varepsilon = x\sin\varepsilon - y\cos\varepsilon \quad (54)$$

4.6 Non-classical Symmetry Method

The classical method developed by Lies has been expanded upon in the non-classical symmetry method. This method was compiled by Bluman and Cole in 1974 in connection with the analysis of the heat equation. Lie's classical procedure was enhanced by the addition of a surface invariance condition. This constraint can result in solutions that are distinct from those obtained by applying Lies' infinitesimal method in certain circumstances. In point of fact, the accomplishments of the non-classical method paved the way for additional generalizations of the application of symmetries to the production of completely new exact solutions.

Consider a system of differential equation of the form

$$\Delta(x,u_{(k)})=0 \tag{55}$$

where $x=(x_1,x_2,...,x_n)$ and $u=(u^1,u^2,...,u^q)$ are respectively the independent and dependent variables. The system variables are linearly transformed through an infinitesimal incrimination of ξ^i and ϕ^α (independent and dependent variables). These increments are determined by the use of an extended differential operator already defined in the classical method

$$\mathrm{v}=\sum_{i=1}^{n}\xi^i(x,u)\frac{\partial}{\partial x_i}+\sum_{\alpha=1}^{q}\phi^\alpha(x,u)\frac{\partial}{\partial u^\alpha} \tag{56}$$

and the invariant surface condition

$$Q^\alpha(x,u^{(1)})=\phi^\alpha(x,u)-\sum_{i=0}^{n}\xi^i(x,u)\frac{\partial u^\alpha}{\partial x_i}=0, \qquad \alpha=1,...,q \tag{57}$$

The remaining stages in this approach are very much like the phases in the conventional infinitesimal method that were discussed in the previous section.

4.7 Potential Symmetry

Local symmetries are referred to as infinitesimal and non-classical symmetries. This is because the coordinates are directly engaged in the local transformations, which gives rise to the name "local." On the other hand, potential symmetry is also known as non-local symmetry. This is due to the fact that the resulting symmetry does not depend solely on the variables in the initial equation, but rather on the variables in a system of equations derived from the initial problem and whose solutions are solutions of the initial equation. In other words, the potential symmetry does not depend on the variables in the initial equation. The new system is analyzed using similarity approaches, which, in the vast majority of instances, result in the creation of new similarity variables. After this, several instances from the most current research will be shown.

Consider a Burgers' equation

$$u_t+uu_x-u_{xx}=0 \tag{58}$$

This equation is rewritten in a conservation form as

$$u_t+\left(\frac{u^2}{2}-u_x\right)_x=0 \tag{59}$$

A potential variable or stream function is introduced as

$$\begin{aligned}\mathrm{v}_x&=u\\ \mathrm{v}_t&=u_x-\frac{u^2}{2}\end{aligned} \tag{60}$$

and (59) reduces to

$$\mathrm{v}_t+\frac{\mathrm{v}_x^2}{2}-\mathrm{v}_{xx}=0 \tag{61}$$

Because of the variables $u(x,t),\mathrm{v}(x,t)$ satisfy the auxiliary system (60), they also satisfy Burgers equation (59) and the integrated form (61). The problem is then solved using the non-classical approach where variables t, x, u and v infinitesimal transformations are described by:

$$\begin{aligned}\bar{t}&=t+\varepsilon\tau\\ \bar{x}&=x+\varepsilon\chi\\ \bar{u}&=u+\varepsilon U\\ \bar{\mathrm{v}}&=\mathrm{v}+\varepsilon \mathrm{V}\end{aligned} \tag{62}$$

the initial problem is augmented with the surface condition

$$U\equiv\tau\frac{\partial}{\partial t}+\chi\frac{\partial}{\partial x} \tag{63}$$

and is related to the auxiliary problem through the condition

$$\left(\frac{\partial\tau}{\partial \mathrm{v}}\right)^2+\left(\frac{\partial\chi}{\partial \mathrm{v}}\right)^2+\left(\frac{\partial U}{\partial v}\right)^2\neq 0 \tag{64}$$

The optimal linear increments τ, χ and V are then derived through the infinitesimal generator

$$O=\tau\frac{\partial}{\partial t}+\chi\frac{\partial}{\partial x}+V\frac{\partial}{\partial v} \tag{65}$$

then from the nonlocal symmetries of (61), Burgers' new symmetries are derived

4.8 Direct Method

This technique is based on the assumption of the solution in the form:

$$u(x,t) = g(t)\, f(\eta) \tag{66}$$

and the reduction of the PDE into ODE through chain rule with respect to the similarity variable $\eta(x,t)$. This technique developed by Abbott and Kline (1965) was generalized by Clarkson and Kruskal (1989) through an assumption in the form $u(x,t)=\alpha(x,t)+\beta(x,t)\,f(\eta)$

Consider the heat equation

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t} \tag{67}$$

Assume

$$u(x,t) = g(t)\, f(\eta) \tag{68}$$

where

$$g(t) = g_0\, t^{\lambda}, \quad \eta = x\, t^m \tag{69}$$

thus

$$u(x,t) = g_0\, t^{\lambda} f(x\, t^m) \tag{70}$$

Using chain rule for all derivatives in (68) it transforms to;

$$f''(\eta)=\frac{g'(t)}{g(t)t^{2m}}\,f(\eta)+f'(\eta)m\frac{\eta}{t^{1+2m}} \tag{71}$$

For this equation to be an ordinary differential equation the coefficients of $f(\eta)$, $f'(\eta)$, $f''(\eta)$ must be constant or function of η. The last coefficient in (71) is constant if $m=-1/2$

The function g(t) is assumed to be in the form

$$g(t) = g_0\, t^{\lambda} \tag{72}$$

so, (67) reduces to

$$f''(\eta)+\frac{1}{2}\eta f'(\eta)-\lambda f(\eta)=0 \tag{73}$$

And

$$\eta=\frac{x}{\sqrt{t}} \tag{74}$$

The solution of (73) yields a solution similar to Green function.

4.9 The characteristic function method

In (Na and Hansen's,1971), non-classical infinitesimal technique, the inclusion of the characteristic function allows the ensuing mathematical description to be expressed in terms of a single variable. This method is known as the characteristic function method. This method, which is based on the invariance of the partial differential equation and the surface condition, gives an overdetermined system of differential equations, which, in turn, results in a transformation vector with dimensions that are greater than those obtained using the classical method, the non-classical method, the potential method, or the group method. This technique involves describing the infinitesimals of the group in terms of a single function denoted by the letter W. This function is referred to as the characteristic function. Let's look at the heat equation as an example to help demonstrate this point.

$$\frac{\partial^2 u}{\partial y^2}=\frac{\partial u}{\partial t} \tag{75}$$

Using the following notations

$$p_1=\frac{\partial u}{\partial t},\, p_2=\frac{\partial u}{\partial y},\, p_{11}=\frac{\partial^2 u}{\partial t^2}$$
$$p_{22}=\frac{\partial^2 u}{\partial y^2}\ \ and\ \ p_{12}=\frac{\partial^2 u}{\partial t\partial y} \tag{76}$$

The heat equation takes the form

$$p_1-p_{22}=0 \tag{77}$$

and define the infinitesimal group of the transformations as follows

$$\begin{aligned}
\bar{u} &= u + \varepsilon\zeta(y,t,u) + \mathrm{O}(\varepsilon^2) \\
\bar{y} &= y + \varepsilon\eta(y,t,u) + \mathrm{O}(\varepsilon^2) \\
\bar{t} &= t + \varepsilon\xi(y,t,u) + \mathrm{O}(\varepsilon^2) \\
\bar{p}_1 &= p_1 + \varepsilon\pi_1(y,t,u,p_1,p_2) + \mathrm{O}(\varepsilon^2) \\
\bar{p}_2 &= p_2 + \varepsilon\pi_2(y,t,u,p_1,p_2) + \mathrm{O}(\varepsilon^2) \\
\bar{p}_{22} &= p_{22} + \varepsilon\pi_{22}(y,t,u,p_1,p_2,p_{11},p_{12},p_{22}) + \mathrm{O}(\varepsilon^2)
\end{aligned} \tag{78}$$

To generate the characteristic function W

$$u(\bar{t},\bar{y}) = \bar{u}(t,y,\varepsilon) \tag{79}$$

$$\begin{aligned}
u(t+\varepsilon\xi, y+\varepsilon\eta) &= u(t,y) + \varepsilon\zeta(t,y,u) + \mathrm{O}(\varepsilon^2) \\
&= u(t,y) + \varepsilon\xi\frac{\partial u}{\partial t} + \ldots + \varepsilon\eta\frac{\partial u}{\partial y} + \mathrm{O}(\varepsilon^2)
\end{aligned} \tag{80}$$

Hence we get

$$\xi\frac{\partial u}{\partial t} + \eta\frac{\partial u}{\partial y} - \zeta = 0 \tag{81}$$

We can now define

$$W = \xi\, p_1 + \eta\, p_2 - \zeta \quad ; p_1 = \frac{\partial u}{\partial t},\ p_2 = \frac{\partial u}{\partial y} \tag{82}$$

Following (Seshadri and Na, 1985), we finally obtain the characteristic function

$$\begin{aligned}
W(t,y,u,p_1,p_2) = W_{111}(t,y) &+ \left(2c_2t + c_3 + c_1y + c_2y^2\right)u \\
&+ \left(2c_1t + c_4 + (4c_2t + c_5)y\right)p_2 + \left(4c_2t^2 + 2c_5t + c_6\right)p_1
\end{aligned} \tag{83}$$

whose components are the infinitesimal changes in y, t and u and W_{111} satisfies the heat equation;

$$\frac{\partial W_{111}}{\partial t} - \frac{\partial^2 W_{111}}{\partial y^2} = 0 \tag{84}$$

The symmetry transformations are then obtained from

$$\frac{dt}{\xi} = \frac{dy}{\eta} = \frac{du}{\zeta} \tag{85}$$

where

$$\begin{aligned}
\xi &= 4c_2t^2 + 2c_5t + c_6 \\
\eta &= 2c_1t + c_4 + (4c_2t + c_5)y \\
\zeta &= -W_{111}(t,y) - (2c_2t + c_3 + c_1y + c_2y^2)u
\end{aligned} \tag{86}$$

the details of reduction of the heat equation to an ordinary differential equation for different values of the coefficients c_1 to c_6 is to be found.

5. Literature review

By reducing the number of independent variables, one may convert partial differential equations to ordinary differential equations. This transformation is achieved by using the method of ordinary differential equation. This reduction is controlled by invariance principles that are embedded within the laws of nature. Some examples of these principles include the presence of a conservation law for momentum and energy in fluids and mechanical systems, as well as the realization that a phase change in a wave that is transporting charged particles must necessarily maintain the electric charges in their original state.

In older times, finding answers to similarity problems required using direct physical and dimensional considerations. The explosion and implosion difficulties presented by (Taylor et al,1950) and (Sedov, 1959) are the two most well-known instances of this phenomenon (1959). The work of (Ovsyannikov,1982), in which he presented an algorithmic approach to the determination of similarity solutions by employing both finite and infinitesimal transformation, resulted in significant shifts in this technique, which had previously been relatively unchanged. The non-classical symmetry approach was created by (Bluman and Cole, 1974), and it is distinguished by the fact that the invariance surface condition is related with the symmetry requirements. (Vinogradov,1989) came up with a new technique called the Potential method in 1989. He did this by transforming the original PDE by making use of a stream function, and he provided a criterion that must be fulfilled by nonlocal symmetries. This method is considered to be very recent. Clarkson and Kruskal came up with the Direct approach in (Clarkson et al, 1989). This approach does not entail the use of infinitesimals and is quite straightforward and easy to put into practice. The non-classical infinitesimal approach that (Na and

Hansen,1971) developed is called the characteristic function method. This method is based on the idea that the introduction of the characteristic function may cause the following mathematical description to be recast in terms of a single variable. The next paragraphs will elaborate on these procedures.

Under the influence of a magnetic field, (Bhardwaj et al, 1999) investigated the classes of motion that are cylindrically symmetrically identical in a non-equilibrium gas dynamic flow. The profiles of flow and field variables were acquired using numerical calculations, and the findings obtained reveal that the magnetic field has an impact on the flow pattern. Flow and field variables were obtained. By a Lie-symmetry reduction of three-dimensional Navier Stokes equations (Grassi et al, 2000) were able to categorize the generated vortices and deduce novel precise solutions, including the Burgers vortex and shear-layer solutions. It was considered what the possible physical significance of such solutions may be. Lie infinitesimal transformation approach and announce a new similarity variable by (Vaganan et al, 2004). With the use of Lie symmetries, (Gandarias et al, 2005) were able to provide a solution for a highly nonlinear wave equation known as the Calogero–Degasperis–Fokas (CDF) problem. Under the context of this equation, the wave is said to be moving at a speed that varies depending on the value of some arbitrary function. The solutions that were produced display a diverse range of qualitative patterns and characteristics. Some of these problems may be solved using solitons, which are solutions that are localized on a curve but decay exponentially as they move away from that curve. (Ivanova et al, 2006) identified Lie infinitesimals transformations for a nonlinear diffusion equation with a source term. Using Lie infinitesimal approach, (Boutros et al., 2006) evaluated unsteady flows in a semi-infinite conduit that was being exposed to injection or suction via a porous wall whose radius changed throughout the course of the experiment. After that, the governing equations were simplified to a nonlinear ordinary differential equation of the fourth order, and an exact solution was found using the perturbation technique. After that, the analytical findings are contrasted with the numerical solution for a variety of Reynolds numbers and wall expansion ratios. It was discovered that the axial velocity is greatest at the centre of the structure, and it is lowest along the wall. This relationship is inversely proportional to the expansion ratio of the wall. Using the use of infinitesimal Lie point transformations, (Shahzad et al, 2007) were able to deduce the analytical solution to a time-independent two-dimensional micropolar fluid. Via the use of an extended order generator, the system of partial differential equations that were originally used to describe the flow were transformed into ordinary differential equations, which were then solved analytically. Thus, an interpretation of the translational parameter m, which is given as u(x,y) = x-my on the x and y components of the velocity u. A symmetry analysis was carried out by (Arrigo et al., 2007) on a (2+1)-dimensional linear diffusion equation that had a source term. The purpose of this research was to identify source words that are capable of admitting nontrivial symmetries. The traditional Lie approach is used in this investigation, which results in the identification of many nonlinear source term configurations. Then, after using the non-classical technique (the Lie method combined with the surface invariant condition), it was discovered that the classical symmetries could be restored. This study demonstrates that there are no non-classical symmetries for problems of this kind; they do not exist. The symmetry

Vectors of wave equation in the form $u_{tt} = u_{xx} + u_x \cot x + \frac{1}{\sin^2 x} u_{xx}$ were investigated using MathLie program in Mathematica Package by (Azad et al, 2007) obtaining some exact solutions for subgroups of transformation. Benney equation was studied by (Ozer et al, 2007), which occurs when a wave with a long wavelength travels through an ideal fluid that is incompressible and has a limited depth. In this study, the Benney equation is presented in a format that is just two-dimensional. In order to convert the Benney equation to an ordinary differential equation, the independent variables of the issue, t, x, and v, are subjected to a transformation under a Lie group with a single parameter. Using the use of Dirac's delta type distribution function, a connection between shallow water and the Benney one-dimensional equation may be constructed. Three-dimensional Euler equations was solved by (Raja Sekhar et al, 200) that describe the behavior of gases. After deriving the symmetry generators of these partial differential equations, the system is then reduced to a set of ordinary differential equations. In some situations, the problems can be solved to an accurate degree. In the presence of a uniform magnetic field and heat generation/absorption effects, (EL-Kabeir et al, 2008) examined heat transfer within an unsteady, three-dimensional boundary layer flow for a viscous, incompressible, and electrically conducting fluid over an inclined permeable surface embedded in a porous medium. Both the macroscopic viscosity term and the microscopic permeability of the porous media were accounted for in the extended flow model. Both the infinitesimal generators and the Lie algebra extension are derived. Boundary conditions' constraints on the generators are considered. Using a subset of the transformation vector group, we were able to minimize the number of independent variables from n to n-1. After this, the resultant equations are numerically solved using the perturbation technique at a number of different times. For different values of Prandtl number, Hartmann number, Darcy number, heat generation/absorption coefficient, and surface mass-transfer coefficient, the velocity, temperature, and pressure profiles, surface shear stresses, and wall-heat transfer rate are analyzed. Based on Lie theory, a group method was created and used in many researches. The natural convection from a vertical plate was investigated in steady state by (Rashed et al, 2008) and in unsteady state (Rashed et al, 2009). Hidden vectors were created to get more solutions (Rashed et al, 2014). BLMP equation was investigated in (3+1) dimensions (Mabrouk et al, 2017). More applications were executed using Lie theorems (Kassem et al, 2019), (Mabrouk et al, 2019) and (Rashed et al, 2019).

Non classical solution to the heat and Burgers equation was investigated by (Mansfield et al, 1999). The heat equation was solved by obtaining a superposition of solutions to equations of lower order. Burgers equation has a specific family of generalized symmetries that have been derived. (Romero et al, 2000) derived the non-classical symmetries of a one-dimensional reaction diffusion equation of the kind where and are arbitrary functions. In the first example, we discovered conventional answers, while in the second, we found novel solutions for $n = 1$. Asymptotically periodic plane waves arise as similarity solutions of systems, one of the kinds of precise solutions produced. A group of solutions displaying a blow-up mechanism was also discovered. Non-classical symmetry reduction of a dissipation-modified KdV equation by (Bruzón et al, 2003) yielded numerous novel solutions that cannot be obtained using just classical Lie symmetries. The dissipation-modified Korteweg-de Vries equation was transformed into ordinary equations with the Painlevé property, where the moveable singularities of all solutions are poles, thanks to the authors' successful pursuit of nonclassical symmetries. (Nucci et al, 2003) devised an approach that is simpler to implement than the standard way to locate non-classical symmetries, and used this to derive non-classical symmetries as special solutions of particular families of second- and third-order evolution equations. The author has also shown that an unlimited number of equations accepting nonclassical symmetries may be retrieved using the novel approach. New methods for determining the equations of nonclassical symmetries were developed by (Nicoleta Bîlă et al., 2004). This method included determining the value of the non-classical symmetry operator by use of equations whose coefficients depended on the classical symmetry. This means that all programs for symbolic derivation that were written for the latter method can be used directly for deriving non-classical symmetries. The algorithm, GENDEFNC, was written as a MAPLE routine, and the DESOLV package was utilized. Exemplifications are provided. Two evolutionary partial differential equations, the Burgers and KdV equations, have their non-classical symmetries derived by (Arrigo et al, 2004). The authors uncovered a collection of non-classical symmetries and demonstrated that determining equations may be built by only enforcing compatibility between the initial equation and a related first-order quasi-linear PDE. Symmetry analysis of the two-dimensional diffusion equation with a source term was performed by (Arrigo et al, 2007). The authors made advantage of classical and nonclassical symmetries. Part one of this study uses the classical approach to categorise source words that admit a nontrivial symmetry. The paper's second section employed the non-classical approach to prove that the resultant symmetries could be achieved using just classical techniques. Non-classical symmetry solutions to a reaction-diffusion equation were investigated by (Bradshaw-Hajek et al, 2007). Many biological situations, including predator-prey systems, population genetics issues, and the modelling of calcium waves on the surface of amphibian eggs, are described by this equation, which also describes physical phenomena like the transmission of impulses along a nerve axon, heating by microwave radiation, chemical reactions, the theory of superconductivity, and so on. In order to rigorously accept non-classical symmetries for the cubic source term, the authors constructed mathematical representations for the spatial dependency. The use of one of these answers in diploid population genetics led to the derivation of many new precise solutions. Nonlinear fin equations were categorized into groups by (Vaneeva et al, 2008). The authors discovered new groups of equivalence transformations and groups of conditional equivalence transformations. The categorization and future applications are made easier with these findings. For an inhomogeneous nonlinear diffusion equation, (Gandarias et al., 2008) performed non classical symmetry reductions. Classes of non-classical symmetries of the equation and associated system were related by the authors. The authors were able to enhance the total number of solutions by using the found symmetries. Several of these solutions displayed unusual behavior that could not have been achieved using just classical symmetries.

Using a non-classical potential symmetry approach, (Gandarias et al, 2000) solved the Cahn-Hilliard diffusion equation in 2000. Many physical chemistry issues may be understood by using this equation as a model. A fresh set of solutions were found. The authors categorised the discovered possible (nonlocal) symmetries. Non-classical symmetry reductions are also compared to those produced in earlier work. Possible symmetries for differential equations arising from group reductions of various diffusion equations were derived by (Gandarias et al, 2001). Even though ordinary differential equations don't accept classical symmetries, the authors demonstrated that the order of the equations might be decreased using these prospective symmetries. It was also discovered that equations that allowed for point symmetry would have even more reductions than those using the symmetries themselves. Potential symmetries and invariant solutions for the extended one-dimensional Fokker Planck equation were derived by (Khater et al, 2002). A magneto hydrodynamic system of equations was decoupled by (Khater et al., 2004) into two equations, each of which describes the flow of plasma via a magnetic field. After that equation, we used the Fokker–Planck equation, then we used the potential symmetry approach. The effort resulted in new precise answers. The authors (Davison et al, 2004) study applied the ideas of potential symmetries to primarily nonlinear systems that had a small parameter perturbed. The potential (or auxiliary) form of the perturbed system needs knowledge of an estimated conservation rule of the system. Perturbed wave, Burgers and perturbed nonlinear diffusion equation were presenting examples for the approach. The authors came to the conclusion that even though the unperturbed equations do not have potential symmetries, the corresponding perturbed equation may have approximate potential symmetries depending on the kind of perturbation that is used. Significant interest in soil science and mathematical biology has led (Sophocleous et al., 2006) to investigate a system of diffusion equations, which they have linearized and potentially symmetrical. Diffusion system solutions may be obtained by several types of linearization and hodograph-like mappings. By defining a potential auxiliary function and applying full classical and also a non-classical symmetry transformation to a second order nonlinear degenerate parabolic equation related to non-Newtonian ice sheet dynamics in the isothermal case, (Daz et al., 2006) derived similarity solutions for a free boundary problem arising in glaciology. In this work, we first obtain a general result linking the ice sheet thickness function to the solution of the nonlinear equation, then we present a specific example of a similarity solution to a problem with Cauchy boundary conditions, and finally we derive several qualitative properties of the free moving boundary when the accumulation-ablation function has physically plausible characteristics. New potential symmetries were developed by (Gandarias, 2008) for the Fokker-Planck equation, the inhomogeneous diffusion issue, and the quasi-linear equation representing the sources and temperature evolution in a porous medium. Instead of using the natural potential system or a generic integral variable, a generalized potential system was used to find these symmetries.

In 2000, (Simon Hood, 2000) investigated a parabolic and nonlinear diffusion problem and found an exact and analytical solution. The author used a modified form of the direct method to overcome the difficulty of having an arbitrary function as a coefficient of the dependent variable. To solve it analytically, we first transform it into an ordinary differential equation. The symbolic computing system was used by (Xuelin Yong, 2006) based on the Clarkson and Kruskal direct approach to study integrability for a nonlinear system of partial differential equations. Several kinds of reductions in similarity were found.

To solve Burger's problem, (Kassem, 2003) used this approach. A greater transformation vector was achieved with this technique than using the classical approach, the non-classical approach, the potential approach, or the group approach. Analyses were performed on three instances, and the findings were compared to other research in the area. An equation for nonlinear diffusion was discussed by the same author. Well-known physical issues were seen when alternative values of the storativity S(u) and diffusion K(u) coefficients were examined using the three discovered similarity variables. Previous research in the subject was used to evaluate the accuracy of the analytical or numerical answers. The classes of solutions to the first-order nonlinear hydrodynamic equations of a perfect fluid for a range of Coriolis parameters are determined and explored by (Abd-el-Malek et al, 2005). The study demonstrated the efficiency of this approach in getting invariant solutions for the system of nonlinear partial differential equations defining an ideal fluid in a two-dimensional geometry, which would be challenging by some other approaches. Because of the Coriolis factors, conventional approaches to the problem's analysis have a number of fundamental hurdles to overcome. These same authors in 2006 used the characteristic function technique to provide an answer to the problem of free-surface, first-order, nonlinear sheared flows subject to gravity. Several solutions were determined for various scenarios in terms of a hyper-geometric function, typical profiles of the free surface, and the horizontal and vertical velocities. Time has a negative correlation with the free surface's profile and the velocity components. Parabolic change with "x" distance was also seen in the free surface.

6. Conclusion

The similarity methods are powerful tools that could be used to transform partial differential equations or system of partial differential equations to simpler form of ordinary differential equations. This can help to facilitate solving the original equations. In many cases, these reductions may help obtaining new exact solutions that are completely different to the common solutions. The applications are wide to cover evolution equations, fluid dynamics, natural convection, compressible gases, diffusion and heat equations and more.

Disclosure

The author reports no conflicts of interest in this work.

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