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Farlie-Gumbel-Morgenstern Bivariate Exponentaited Weibull Distribution with Applications

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Abstract

The bivariate Exponentaited Weibull distribution is an important lifetime distribution in survival analysis. In this paper, Farlie-Gumbel-Morgenstern (FGM) copula and Exponentaited Weibull marginal distribution are used for creating bivariate distribution which is called FGM bivariate Exponentaited Weibull (FGMBEW) distribution. FGMBEW distribution is used for describing bivariate data that have weak correlation between variables in lifetime data. It is agood alternative to bivariate several lifetime distributions for modeling real-valued data in application. Some properties of the FGMBEW distribution are obtained such as product moment, moment generating function, reliability function and hazard function. Estimation method for parameters estimation is discussed for FGMBEW distribution namely maximum likelihood estimation. A simulations study is conducted to evaluate the performance of the estimators. Also, a real data set is introduced, analyzed to investigate the model and useful results are obtained for illustrative purposes.

Keywords Exponentaited Weibull distribution . FGM copula . Maximum likelihood estimation.

1 Introduction

The Exponentaited Weibull distribution has been attained more attention in the literature and has inherent flexibility. Mudholkar and Srivastava (1993) introduced the probability density function and the cumulative density function of univariate Exponentaited Weibull distribution respectively as

$$F(x;\alpha,\beta,\lambda)=\left(1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right)^{\lambda};\ x>0,\qquad \alpha,\beta,\lambda>0, \tag{1}$$

and

$$f(x;\alpha,\beta,\lambda)=\frac{\alpha\lambda}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\left(1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right)^{\lambda-1};\ x>0,\qquad \alpha,\beta,\lambda>0, \tag{2}$$

where β and α , λ are the scale and shape parameters respectively. Suppose that there exist two related failure time variables X_1 and X_2. To describe them, various bivariate models or distributions have been proposed in the literature and this is especially the case when X_1 and X_2 represent the times of the two components of a system in a reliability study. The references for this include Block (1977), Nair and Nair (1988), Balakrishnan and Basu (1995), Sahu and Dey (2000), Iyer et al. (2002). Galiani (2003) concluded that bivariate Weibull are specifically oriented towards applications in economics, finance and risk management. Flores (2009) used Weibull marginal to construct bivariate Weibull distributions. And others such as Hanagal and Ahmadi (2009), Kundu and Gupta (2009), Regoli (2009), Diawara and Carpenter (2010), and Xie et al. (2011). Recent researches have been made for the bivariate Weibull distribution. Kundu and Gupta (2013) introduced the Marshall-Olkin bivariate Weibull distribution. Almetwally et al. (2020) introduced FGM Bivariate Weibull Distribution and others.

A copula is a convenient approach for description of a multivariate distribution. Nelsen (2006) introduced Copulas as following; copula is function that join multivariate distribution functions with uniform [0, 1] margins. A copula is a convenient approach to describe amultivariate distribution with dependence structure. The n-dimensional copula (C) exists for all $x_1, x_2, \dots, x_n$, $F(x_1, x_2, \dots, x_n) = C\big(F_1(x_1), F_2(x_2), \dots, F_n(x_n)\big)$, if F is continuous, then C is uniquely defined.

Sklar (1973) states that, considered the two random variables X_1 *and* X_2, with distribution functions $F_1(x_1)$ *and* $F_2(x_2)$ the following cdf and pdf for copula are given respectively as

$$F(x_1, x_2) = C\big(F_1(x_1), F_2(x_2)\big), \tag{3}$$

and

$$f(x_1, x_2) = f_1(x_1) f_2(x_2) c\big(F_1(x_1), F_2(x_2)\big). \tag{4}$$

Farlie-Gumbel-Morgenstern (FGM) is one of the most popular parametric families of copulas, the family was discussed by Gumbel (1960). The joint cdf and joint pdf for FGM copula given as following respectively

$$C(x_1, x_2) = F_1(x_1)F_2(x_2)\Big(1 + \theta\big(1 - F_1(x_1)\big)\big(1 - F_2(x_2)\big)\Big), \tag{5}$$

and

$$c(x_1, x_2) = \Big(1 + \theta\big(1 - 2F_1(x_1)\big)\big(1 - 2F_2(x_2)\big)\Big), \quad -1 < \theta < 1 \quad . \tag{6}$$

Fredricks and Nelsen (2007) introduced the formula for Spearman's and Kendall's correlation coefficient in case of FGM copula as follows

$$\rho_{sperman} = \left(12 \iint uv(1+\theta(1-u)(1-v)dudv)\right) - 3 = \frac{\theta}{3},$$

$$\rho_{kendall} = 1 - 4 \iint \frac{\partial C}{\partial u} C(u,v) \frac{\partial C}{\partial v} C(u,v) dudv = \frac{2}{9}\theta,$$

where

$$\frac{\partial C}{\partial u} C(u,v) = v + \theta v - \theta v^2 - 2\theta uv + 2\theta uv^2,$$

$$\frac{\partial C}{\partial v} C(u,v) = u + \theta u - \theta u^2 - 2\theta uv + 2\theta u^2 v,$$

such that $\frac{-1}{3} \le \rho_{sperman} \le \frac{1}{3}$, $\frac{-1}{9} \le \rho_{kendall} \le \frac{1}{9}$, the correlation coefficient measures
the strength and direction of a linear relationship between two variables, where $(-1 < \theta < 1)$.

In this article, we study the bivariate extension of the Exponentaited Weibull distribution based on FGM copula function (FGMBEW) and discuss its statistical properties. FGMBEW distribution is used for describing bivariate data that have weak correlation between variables in lifetime data. It is a good alternative to bivariate several lifetime distributions for modeling non-negative real-valued data in application.

The objective of this article is to study the properties of the FGMBEW distribution, and to estimate the parameters of the model. The attractive feature of the marginal function of FGMBEW distribution is the same as the basic distribution (Ex Weibull). Other features of the FGMBEW distribution: it contains closed forms for its cdf, product moment, moment generation function and hazard rate function. The final motivation of the article is to develop a guideline for introducing the best estimation method for the FGMBEW distribution, which we think would be of deep interest to statisticians. A simulation study is conducted to the MLE estimation method. Also, a real data set is introduced and analyzed to investigate the model. The uniqueness of this study comes from the fact that we introduce a comprehensive description of mathematical and statistical properties of FGMBEW distribution with the hope that they will attract wider applications in medicine, economics, life testing and other areas of research.

The rest of this paper is organized as follows: FGM bivariate Exponentaited Weibull distribution is obtain in section 2. Some statistical properties of FGMBEW distribution in section 3. Parameter estimation method for the FGMBEW distribution based on copula in section 4. In section 5, asymptotic confidence intervals are discussed. Application of real data are discussed in section 6. In section 7 the potentiality of the new model is illustrated by simulation study. Finally, Conclusion of some remarks for FGMBEW model are addressed in section 8.

2 FGM Bivariate Exponentaited Weibull Distribution

Let $X_1 \sim EW(\alpha_1, \beta_1, \lambda_1)$ *and* $X_2 \sim EW(\alpha_2, \beta_2, \lambda_2)$, then according to Sklar theorem the joint pdf of bivariate Exponentaited Weibull distribution for any copula is defined as follows

$$\mathbf{f}(\mathbf{x}_1, \mathbf{x}_2) = \frac{\alpha_1\lambda_1}{\beta_1}\left(\frac{x_1}{\beta_1}\right)^{\alpha_1-1} e^{-\left(\frac{x_1}{\beta_1}\right)^{\alpha_1}}\left(1 - e^{-\left(\frac{x_1}{\beta_1}\right)^{\alpha_1}}\right)^{\lambda_1-1}\frac{\alpha_2\lambda_2}{\beta_2}\left(\frac{x_2}{\beta_2}\right)^{\alpha_2-1} e^{-\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}\left(1 - e^{-\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}\right)^{\lambda_2-1} \mathbf{c}\left(\left(1 - e^{-\left(\frac{x_1}{\beta_1}\right)^{\alpha_1}}\right)^{\lambda_1}, \left(1 - e^{-\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}\right)^{\lambda_2}\right), \quad (7)$$

and the joint cdf of FGMBEW for any copula is defined as follows

$$\mathbf{F}(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{C}\left(\left(1 - e^{-\left(\frac{x_1}{\beta_1}\right)^{\alpha_1}}\right)^{\lambda_1}, \left(1 - e^{-\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}\right)^{\lambda_2}\right) \quad (\mathbf{8})$$

Then the pdf of a FGMBEW distribution can be given as

$$f(x_1,x_2)=\frac{\alpha_1\lambda_1}{\beta_1}\left(\frac{x_1}{\beta_1}\right)^{\alpha_1-1}e^{-\left(\frac{x_1}{\beta_1}\right)^{\alpha_1}}\left(1-e^{-\left(\frac{x_1}{\beta_1}\right)^{\alpha_1}}\right)^{\lambda_1-1}\times\frac{\alpha_2\lambda_2}{\beta_2}\left(\frac{x_2}{\beta_2}\right)^{\alpha_2-1}e^{-\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}\left(1-e^{-\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}\right)^{\lambda_2-1}\times\left[1+\theta\left(1-2\left(1-e^{-\left(\frac{x_1}{\beta_1}\right)^{\alpha_1}}\right)^{\lambda_1}\right)\left(1-2\left(1-e^{-\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}\right)^{\lambda_2}\right)\right] \quad (9)$$

and the cdf of a FGMBEW distribution can be expressed as

$$F(x_1,x_2)=\left(1-e^{-\left(\frac{x_1}{\beta_1}\right)^{\alpha_1}}\right)^{\lambda_1}\left(1-e^{-\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}\right)^{\lambda_2}\times\left[1+\theta\left(1-\left(1-e^{-\left(\frac{x_1}{\beta_1}\right)^{\alpha_1}}\right)^{\lambda_1}\right)\left(1-\left(1-e^{-\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}\right)^{\lambda_2}\right)\right]. \quad (10).$$

We can show the flexability of new distribution by figures with various value of the parameters especially the copula parameter in figure (1), the curve would have very light tails. It is not unusual. This does not necessarily mean that FGMBEW distribution differs significantly from a bivariate normal distribution. Figure (1) show the plot 3-dimension for the pdf and cdf of FGMBEW distribution with diferent value of $(\alpha_1,\beta_1,\lambda_1,\alpha_2,\beta_2,\lambda_2)$ and θ.

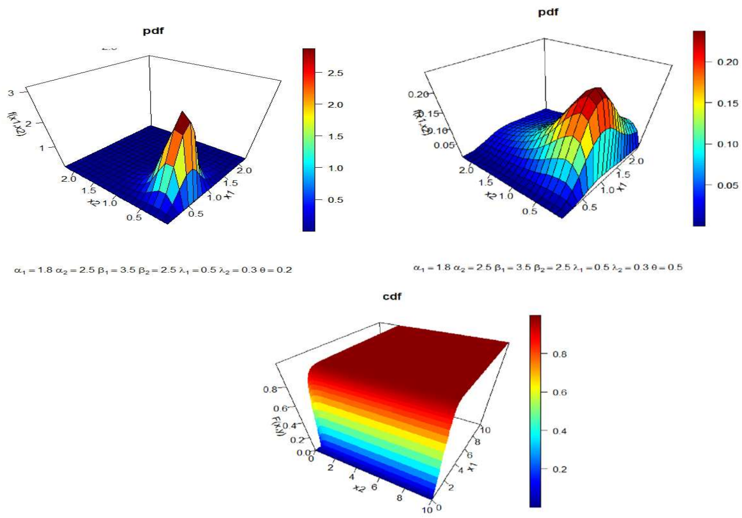

Figure (1) The pdf and cdf of FGMBEW distribution with various value of the parameters

3 Properties of FGMBEW Distribution

In this section, we give some important statistical properties of the FGMBEW distribution such as marginal distributions, product moments, moment generating function, conditional distribution, generating random variables, reliability function. Establishing algebraic expressions to determine some statistical properties of the FGMBEW distribution can be more efficient than computing them directly by numerical simulation.

3.1 The Marginal and Conditional Distributions

The marginal density functions for X_1 and X_2 can be shown respectively as,

$$f(x_1;\alpha_1,\beta_1,\lambda_1)=\frac{\alpha_1\lambda_1}{\beta_1}\left(\frac{x_1}{\beta_1}\right)^{\alpha_1-1}e^{-\left(\frac{x_1}{\beta_1}\right)^{\alpha_1}}\left(1-e^{-\left(\frac{x_1}{\beta_1}\right)^{\alpha_1}}\right)^{\lambda_1-1},x_1>0,$$

$$\alpha_1,\beta_1,\lambda_1>0, \qquad (11)$$

and

$$\mathbf{f(x_2;\alpha_2,\beta_2,\lambda_2)=\frac{\alpha_2\lambda_2}{\beta_2}\left(\frac{x_2}{\beta_2}\right)^{\alpha_2-1}e^{-\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}\left(1-e^{-\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}\right)^{\lambda_2-1},x_2>0,\alpha_2,\beta_2,\lambda_2>0,} \qquad (\mathbf{12})$$

which are Ex Weibull distributed, where the marginal distribution of X_1 and X_2 can be calculated directly by

$$f(x_i)=\int_{allx_j} f(x_1,x_2)\,dx_j\ ;\ i,j=1,2\ ,i\neq j.$$

The conditional probability distribution of X_2 given X_1 is given as

$$f(x_2\mid x_1)=\frac{f(x_1,x_2)}{f(x_1)}=v[1+\theta(1-2U)(1-2V)] \qquad (13)$$

$$= \frac{\alpha_2\lambda_2}{\beta_2}\left(\frac{x_2}{\beta_2}\right)^{\alpha_2-1} e^{-\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}\left(1-e^{-\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}\right)^{\lambda_2-1} \times\left[1+\theta\left(1-2\left(1-e^{-\left(\frac{x_1}{\beta_1}\right)^{\alpha_1}}\right)^{\lambda_1}\right)\left(1-2\left(1-e^{-\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}\right)^{\lambda_2}\right)\right], \quad (14)$$

where $f(x_1)$ is the above marginal density functions for X_1.

The conditional probability distribution of X_1 *given* X_2 is given as

$$f(x_1 \mid x_2) = \frac{f(x_1,x_2)}{f(x_2)} = u[1+\theta(1-2U)(1-2V)] \quad (15)$$

$$= \frac{\alpha_1\lambda_1}{\beta_1}\left(\frac{x_1}{\beta_1}\right)^{\alpha_1-1} e^{-\left(\frac{x_1}{\beta_1}\right)^{\alpha_1}}\left(1-e^{-\left(\frac{x_1}{\beta_1}\right)^{\alpha_2}}\right)^{\lambda_1-1} \times\left[1+\theta\left(1-2\left(1-e^{-\left(\frac{x_1}{\beta_1}\right)^{\alpha_1}}\right)^{\lambda_1}\right)\left(1-2\left(1-e^{-\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}\right)^{\lambda_2}\right)\right], \quad (16)$$

where $f(x_2)$ is the above marginal density functions for X_2.

The conditional cdf is

$$F(x_2 \mid x_1) = \int_0^{x_2} f(x_2 \mid x_1)\,dx_2 = V(1+\theta-\theta V-2\theta U+2\theta UV) \quad (17)$$

$$= \left(1-e^{-\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}\right)^{\lambda_2}\left[1+\theta-\theta\left(1-e^{-\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}\right)^{\lambda_2}-2\theta\left(1-e^{-\left(\frac{x_1}{\beta_1}\right)^{\alpha_1}}\right)^{\lambda_1}+2\theta\left(1-e^{-\left(\frac{x_1}{\beta_1}\right)^{\alpha_1}}\right)^{\lambda_1}\left(1-e^{-\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}\right)^{\lambda_2}\right], \quad (18)$$

where $v = f_2(x_2),\ u = f_1(x_1),\ U = F_1(x_1),\ V = F_2(x_2)$.

3.2 Moment Generating and Product Moment Functions

Let (X_1, X_2) denote abivariate random variable with the probability density function of FGMBEW. Then, the moment generating function of (X_1, X_2) is given by

$$M_{(x_1,x_2)}(t_1,t_2) = E(e^{t_1x_1}e^{t_2x_2})$$
$$= \int_0^\infty \int_0^\infty e^{t_1x_1}e^{t_2x_2}\ f(x_1,x_2)dx_1dx_2 \qquad (19)$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\lambda_1-1}\left(\frac{(t_1)^n\lambda_1\binom{\lambda_1-1}{k}(-1)^k}{n!\beta_1(k+1)^{\frac{n}{\alpha_1}+1}}\Gamma\left(\frac{n}{\alpha_1}+1\right)\right)\times$$
$$\sum_{m=0}^{\infty}\sum_{k=0}^{\lambda_2-1}\left(\frac{(t_2)^m\lambda_2\binom{\lambda_2-1}{k}(-1)^k}{m!\beta_2(k+1)^{\frac{m}{\alpha_2}+1}}\Gamma\left(\frac{m}{\alpha_2}+1\right)\right)\times\Big[1+\theta-$$
$$2\theta\sum_{k=0}^{\lambda_2}\binom{\lambda_2}{k}-2\theta\sum_{k=0}^{\lambda_1}\binom{\lambda_1}{k}+4\theta\sum_{k=0}^{\lambda_1}\sum_{k=0}^{\lambda_2}\binom{\lambda_1}{k}\binom{\lambda_2}{k}\Big].$$

If the random variable (X_1, X_2) is distributed as FGMBEW, then its r^{th} and s^{th} moments around zero can be expressed as follows

$$\acute{\mu}_{rs} = E(X_1^rX_2^s)$$
$$= \int_0^\infty \int_0^\infty x_1^rx_2^s\ f(x_1,x_2)dx_1dx_2 \qquad (20)$$

$$= \sum_{k=0}^{\lambda_1-1}\left(\frac{\lambda_1\binom{\lambda_1-1}{k}(-1)^k}{\beta_1{}^r(k+1)^{\frac{r}{\alpha_1}+1}}\Gamma\left(\frac{r}{\alpha_1}+1\right)\right)\times\sum_{k=0}^{\lambda_2-1}\left(\frac{\lambda_2\binom{\lambda_2-1}{k}(-1)^k}{\beta_2{}^s(k+1)^{\frac{s}{\alpha_2}+1}}\Gamma\left(\frac{s}{\alpha_2}+1\right)\right)\Big[1+$$
$$\theta-2\theta\sum_{k=0}^{\lambda_2}\binom{\lambda_2}{k}-2\theta\sum_{k=0}^{\lambda_1}\binom{\lambda_1}{k}+4\theta\sum_{k=0}^{\lambda_1}\sum_{k=0}^{\lambda_2}\binom{\lambda_1}{k}\binom{\lambda_2}{k}\Big].$$

Mardia (1970) defined measures of multivariate and bivariate skewness (SK) and kurtosis (KU), and we used this measures to introduced table (1) respectively as

$$SK = (1-\rho^2)^{-3}[\gamma_{30}^2 + \gamma_{03}^2 + 3(1+2\rho^2)(\gamma_{12+}^2 \gamma_{21}^2) - 2\rho^3\gamma_{30}\gamma_{03} + 6\rho\{(\gamma_{30}(\rho\gamma_{21}-\gamma_{12}) + \gamma_{03}(\rho\gamma_{21}-\gamma_{12}) - (2+\rho^2)\gamma_{21}\gamma_{12})\}], \quad (21)$$

$$KU = \frac{\gamma_{40} + \gamma_{04} + 2\gamma_{22} + 4\rho(\rho\gamma_{22} - \gamma_{13} - \gamma_{31})}{(1-\rho^2)^2} \quad (22)$$

Table (1) Covariance, skewness, and kurtosis of FGMBEW distribution

θ	Cov	P	*SK*	*KU*
1	0.7207	0.3003	1.3523	29.5719
0.8	0.5545	0.2340	1.3644	21.8483
0.6	0.3884	0.1728	1.3664	20.4443
0.4	0.2232	0.1225	1.3640	19.4227
0.2	0.1552	0.0512	1.3622	14.4085
0	0.0000	0.0000	1.3609	9.4224
-0.2	-0.1552	-0.0512	1.3625	4.5549
-0.4	-0.2232	-0.1225	1.3673	-0.6369
-0.6	-0.3884	-0.1728	1.3752	-6.2269
-0.8	-0.5545	-0.2340	1.3848	-12.727
-1	-0.7207	-0.3003	1.4027	-20.457

Where $(\alpha_1 = 0.6, \beta_1 = 0.5, \lambda_1 = 0.4, \alpha_2 = 0.5, \beta_2 = 0.7, \lambda_2 = 0.3)$ the strength and direction of a linear relationship between two variables, where $(-1 < \theta < 1)$.

3.4 Reliability Function

Osmetti and Chiodini (2011) discussed that the reliability function is more convenient to express a joint survival function as a copula of its marginal survival functions, where X_1 and X_2 be random variable with survival functions $\bar{F}_{(x_1)}$ and $\bar{F}_{(x_2)}$ as following.
The reliability function of the marginal distributions is defined as

$$R(x_j;\alpha_j,\beta_j,\lambda_j) = 1 - F(x_j;\alpha_j,\beta_j,\lambda_j) = 1 - \left(1 - e^{-\left(\frac{x_j}{\beta_j}\right)^{\alpha_j}}\right)^{\lambda_1} \quad (23)$$

The expression of the joint survival function for copula is given as follow

$$R(x_1,x_2) = C\big(R(x_1),R(x_2)\big) = 1 - F(x_1) - F(x_2) + F(x_1,x_2)$$

$$= R(x_1) + R(x_2) - 1 + C\big(F(x_1),F(x_2)\big).$$

Then the reliability function of FGMBEW distribution is

$$R(x_1,x_2) = 1 - \left(1 - e^{-\left(\frac{x_1}{\beta_1}\right)^{\alpha_1}}\right)^{\lambda_1} - \left(1 - e^{-\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}\right)^{\lambda_2} + \left(1 - e^{-\left(\frac{x_1}{\beta_1}\right)^{\alpha_1}}\right)^{\lambda_1}\left(1 - e^{-\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}\right)^{\lambda_2} \times \left[1 + \theta\left(1 - \left(1 - e^{-\left(\frac{x_1}{\beta_1}\right)^{\alpha_1}}\right)^{\lambda_1}\right)\left(1 - \left(1 - e^{-\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}\right)^{\lambda_2}\right)\right]. \quad (24)$$

The first one uses the bivariate failure rate function defined in Basu (1971) by

$$h(x_1,x_2)=\frac{f(x_1,x_2)}{R(x_1,x_2)}$$

for all (x_1,x_2) such that $R(x_1,x_2)>0$. Puri and Rubin (1974) characterized a mixture of exponential distributions by $h(x_1,x_2)=c\ for\ x_1>0$ and $x_2>0$. However, in general, h does not necessarily determine R. This fact was noted by Yang and Nachlas (2001) and Finkelstien and Esaulova (2005). The second option is to use the hazard gradient defined in Johnson and Kotz (1975) by

$$h(x_1,x_2)=\big(h_1(x_1,x_2),h_2(x_1,x_2)\big)$$

where

$$h_i(x_1,x_2)=-\frac{\partial}{\partial x_i}\ln R(x_1,x_2)$$

for $i=1,2$ (x_1,x_2) such that $R(x_1,x_2)>0$.

Then the hazard rate function of FGMBEW distribution is

$$h(x_1,x_2)$$
$$=\frac{f_1(x_1)f_2(x_2)\left[1+\theta\left(1-2\left(1-e^{-\left(\frac{x_1}{\beta_1}\right)^{\alpha_1}}\right)^{\lambda_1}\right)\left(1-2\left(1-e^{-\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}\right)^{\lambda_2}\right)\right]}{1-\left(1-e^{-\left(\frac{x_1}{\beta_1}\right)^{\alpha_1}}\right)^{\lambda_1}-\left(1-e^{-\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}\right)^{\lambda_2}+\left(1-e^{-\left(\frac{x_1}{\beta_1}\right)^{\alpha_1}}\right)^{\lambda_1}\left(1-e^{-\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}\right)^{\lambda_2}\times\left[1+\theta\left(1-\left(1-e^{-\left(\frac{x_1}{\beta_1}\right)^{\alpha_1}}\right)^{\lambda_1}\right)\left(1-\left(1-e^{-\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}\right)^{\lambda_2}\right)\right]} \quad (25)$$

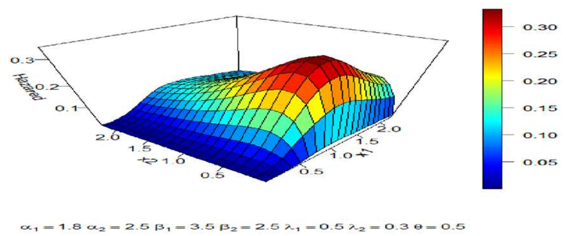

Figure (2) show the plot 3-dimension of hazared function of FGMBEW distribution with different value of $(\alpha_1,\beta_1,\lambda_1,\alpha_2,\beta_2,\lambda_2)\ and\ \theta$

4 Maximum Likelihood Estimation (MLE)

Elaal and Jarwan (2017), discussed the maximum likelihood estimator to estimate all model parameters jointly, it is a one-step parametric method. Therefore, the log-likelihood is given as

$$lnl = \sum_{i=1}^{n}\left[ln\left(f_1(x_{1i})f_2(x_{2i})c(F_1(x_{1i},\delta_1)F_2(x_{2i},\delta_2);\theta)\right)\right] \qquad (26)$$

The parameter estimates are obtained by maximizing the log-likelihood function with expect to each parameter separately.

The likelihood function of a FGMBEW distribution is defined as

$$L(x_1,x_2\ |\Theta) = \prod_{i=1}^{n} f(x_1,x_2) = \left(\frac{\alpha_1\alpha_2\lambda_1\lambda_2}{\beta_1\beta_2}\right)^n \prod_{i=1}^{n}\left(\left(\frac{x_1}{\beta_1}\right)^{\alpha_1-1}\left(\frac{x_2}{\beta_2}\right)^{\alpha_2-1}\right)\left(e^{-\left(\frac{x_1}{\beta_1}\right)^{\alpha_1}} e^{-\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}\right)$$
$$\times \prod_{i=1}^{n}\left(1-e^{-\left(\frac{x_1}{\beta_1}\right)^{\alpha_1}}\right)^{\lambda_1-1}\left(1-e^{-\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}\right)^{\lambda_2-1}$$
$$\times \prod_{i=1}^{n}\left[1+\theta\left(1-2\left(1-e^{-\left(\frac{x_1}{\beta_1}\right)^{\alpha_1}}\right)^{\lambda_1}\right)\left(1-2\left(1-e^{-\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}\right)^{\lambda_2}\right)\right],$$

$$where\ \Theta = (\alpha_1,\beta_1,\lambda_1,\alpha_2,\beta_2,\lambda_2,\theta).$$

Then let

$$a(x_j;\alpha_j,\beta_j.\lambda_j) = \left(1-2\left(1-e^{-\left(\frac{x_j}{\beta_j}\right)^{\alpha_j}}\right)^{\lambda_j}\right);j = 1,2,$$

then the likelihood function can be written as

$$L(x_1, x_2 \mid \Theta) = \left(\frac{\alpha_1\alpha_2\lambda_1\lambda_2}{\beta_1\beta_2}\right)^n \prod_{i=1}^{n}\left(\left(\frac{x_1}{\beta_1}\right)^{\alpha_1-1}\left(\frac{x_2}{\beta_2}\right)^{\alpha_2-1}\right)\left(e^{-\left(\frac{x_1}{\beta_1}\right)^{\alpha_1}} e^{-\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}\right)$$
$$\times \prod_{i=1}^{n}\left(1-e^{-\left(\frac{x_1}{\beta_1}\right)^{\alpha_1}}\right)^{\lambda_1-1}\left(1-e^{-\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}\right)^{\lambda_2-1}$$
$$\times \prod_{i=1}^{n}\left[1+\theta\big(a(x_1;\alpha_1,\beta_1.\lambda_1)\big)\big(a(x_2;\alpha_2,\beta_2.\lambda_2)\big)\right] \qquad (27)$$

and the log-likelihood function of a FGMBEW can be written as

$$l(x_1, x_2 \mid \Theta) = lnL = n(ln\alpha_1 + ln\lambda_1 - ln\beta_1) + n(ln\alpha_2 + ln\lambda_2 - ln\beta_2) + (\alpha_1 - 1)\sum_{i=1}^{n} ln\left(\frac{x_1}{\beta_1}\right) + (\alpha_2 - 1)\sum_{i=1}^{n} ln\left(\frac{x_2}{\beta_2}\right) - \sum_{i=1}^{n}\left(\frac{x_1}{\beta_1}\right)^{\alpha_1} - \sum_{i=1}^{n}\left(\frac{x_2}{\beta_2}\right)^{\alpha_2} + (\lambda_1 - 1)\sum_{i=1}^{n} ln\left(1-e^{-\left(\frac{x_1}{\beta_1}\right)^{\alpha_1}}\right) + (\lambda_2 - 1)\sum_{i=1}^{n} ln\left(1-e^{-\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}\right) + \sum_{i=1}^{n} ln\left[1 + \theta\big(a(x_1;\alpha_1,\beta_1.\lambda_1)\big)\big(a(x_2;\alpha_2,\beta_2.\lambda_2)\big)\right].$$

The estimates of all parameters are obtained by diferentiating the log-likelihood function with respect to each parameter separately, as following:

$$\frac{\partial l(x_1, x_2 \mid \Theta)}{\partial \alpha_1} = \frac{n}{\alpha_1} + \sum_{i=1}^{n} ln\left(\frac{x_1}{\beta_1}\right) - \sum_{i=1}^{n}\left(\frac{x_1}{\beta_1}\right)^{\alpha_1} ln\left(\frac{x_1}{\beta_1}\right) + (\lambda_1 - 1)\sum_{i=1}^{n}\frac{e^{-\left(\frac{x_1}{\beta_1}\right)^{\alpha_1}}\left(\frac{x_1}{\beta_1}\right)^{\alpha_1} ln\left(\frac{x_1}{\beta_1}\right)}{\left(1-e^{-\left(\frac{x_1}{\beta_1}\right)^{\alpha_1}}\right)}$$
$$+\sum_{i=1}^{n}\frac{-2\theta\lambda_1\big(a(x_2;\alpha_2,\beta_2.\lambda_2)\big)e^{-\left(\frac{x_1}{\beta_1}\right)^{\alpha_1}}\left(\frac{x_1}{\beta_1}\right)^{\alpha_1} ln\left(\frac{x_1}{\beta_1}\right)\left(1-e^{-\left(\frac{x_1}{\beta_1}\right)^{\alpha_1}}\right)^{\lambda_1-1}}{\left[1+\theta\big(a(x_1;\alpha_1,\beta_1,\lambda_1)\big)\big(a(x_2;\alpha_2,\beta_2.\lambda_2)\big)\right]},$$

$$\frac{\partial l(x_1,x_2 \mid \Theta)}{\partial \alpha_2} = \frac{n}{\alpha_2} + \sum_{i=1}^{n} ln\left(\frac{x_2}{\beta_2}\right) - \sum_{i=1}^{n}\left(\frac{x_2}{\beta_2}\right)^{\alpha_2} ln\left(\frac{x_2}{\beta_2}\right) + (\lambda_2 - 1)\sum_{i=1}^{n}\frac{e^{-\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}\left(\frac{x_2}{\beta_2}\right)^{\alpha_2} ln\left(\frac{x_2}{\beta_2}\right)}{\left(1-e^{-\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}\right)} + \sum_{i=1}^{n}\frac{-2\theta\lambda_2\big(a(x_1;\alpha_1,\beta_1.\lambda_1)\big)e^{-\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}\left(\frac{x_2}{\beta_2}\right)^{\alpha_2} ln\left(\frac{x_2}{\beta_2}\right)\left(1-e^{-\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}\right)^{\lambda_2-1}}{\left[1+\theta\big(a(x_1;\alpha_1,\beta_1,\lambda_1)\big)\big(a(x_2;\alpha_2,\beta_2.\lambda_2)\big)\right]},$$

$$\frac{\partial l(x_1,x_2 \mid \Theta)}{\partial \beta_1} = \frac{-n\alpha_1}{\beta_1} + \frac{\alpha_1}{\beta_1}\sum_{i=1}^{n}\left(\frac{x_1}{\beta_1}\right)^{\alpha_1} - (\lambda_1 - 1)\frac{\alpha_1}{\beta_1}\sum_{i=1}^{n}\frac{e^{-\left(\frac{x_1}{\beta_1}\right)^{\alpha_1}}\left(\frac{x_1}{\beta_1}\right)^{\alpha_1}}{\left(1-e^{-\left(\frac{x_1}{\beta_1}\right)^{\alpha_1}}\right)} + \sum_{i=1}^{n}\frac{2\theta\lambda_1\big(a(x_2;\alpha_2,\beta_2.\lambda_2)\big)e^{-\left(\frac{x_1}{\beta_1}\right)^{\alpha_1}}\left(\frac{x_1}{\beta_1}\right)^{\alpha_1}\left(1-e^{-\left(\frac{x_1}{\beta_1}\right)^{\alpha_1}}\right)^{\lambda_1-1}}{\left[1+\theta\big(a(x_1;\alpha_1,\beta_1,\lambda_1)\big)\big(a(x_2;\alpha_2,\beta_2.\lambda_2)\big)\right]},$$

$$\frac{\partial l(x_1, x_2 \mid \Theta)}{\partial \beta_2}$$
$$= \frac{-n\alpha_2}{\beta_2} + \frac{\alpha_2}{\beta_2}\sum_{i=1}^{n}\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}$$
$$- (\lambda_2 - 1)\frac{\alpha_2}{\beta_2}\sum_{i=1}^{n}\frac{e^{-\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}{\left(1 - e^{-\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}\right)}$$
$$+ \sum_{i=1}^{n}\frac{2\theta\lambda_2\big(a(x_1;\alpha_1,\beta_1.\lambda_1)\big)e^{-\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}\left(1 - e^{-\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}\right)^{\lambda_2 - 1}}{\left[1 + \theta\big(a(x_1;\alpha_1,\beta_1,\lambda_1)\big)\big(a(x_2;\alpha_2,\beta_2.\lambda_2)\big)\right]},$$

$$\frac{\partial l(x_1, x_2 \mid \Theta)}{\partial \lambda_1}$$
$$= \frac{n}{\lambda_1} + \sum_{i=1}^{n} ln\left(1 - e^{-\left(\frac{x_1}{\beta_1}\right)^{\alpha_1}}\right)$$
$$+ \sum_{i=1}^{n}\frac{-2\theta\big(a(x_2;\alpha_2,\beta_2.\lambda_2)\big)\left(1 - e^{-\left(\frac{x_1}{\beta_1}\right)^{\alpha_1}}\right)^{\lambda_1} ln\left(1 - e^{-\left(\frac{x_1}{\beta_1}\right)^{\alpha_1}}\right)}{\left[1 + \theta\big(a(x_1;\alpha_1,\beta_1,\lambda_1)\big)\big(a(x_2;\alpha_2,\beta_2.\lambda_2)\big)\right]},$$

$$\frac{\partial l(x_1, x_2 \mid \Theta)}{\partial \lambda_2} = \frac{n}{\lambda_2} + \sum_{i=1}^{n} ln\left(1 - e^{-\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}\right)$$
$$+ \sum_{i=1}^{n}\frac{-2\theta\lambda_2\big(a(x_1;\alpha_1,\beta_1.\lambda_1)\big)\left(1 - e^{-\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}\right)^{\lambda_2} ln\left(1 - e^{-\left(\frac{x_2}{\beta_2}\right)^{\alpha_2}}\right)}{\left[1 + \theta\big(a(x_1;\alpha_1,\beta_1,\lambda_1)\big)\big(a(x_2;\alpha_2,\beta_2.\lambda_2)\big)\right]},$$

and

$$\frac{\partial l(x_1, x_2 \mid \Theta)}{\partial \theta} = \sum_{i=1}^{n}\frac{\big(a(x_1;\alpha_1,\beta_1,\lambda_1)\big)\big(a(x_2;\alpha_2,\beta_2,\lambda_2)\big)}{\left[1 + \theta\big(a(x_1;\alpha_1,\beta_1,\lambda_1)\big)\big(a(x_2;\alpha_2,\beta_2,\lambda_2)\big)\right]}.$$

Where

$$\boldsymbol{a}(\boldsymbol{x_j};\boldsymbol{\alpha_j},\boldsymbol{\beta_j}.\boldsymbol{\lambda_j})=\left(\mathbf{1}-\mathbf{2}\left(\mathbf{1}-\boldsymbol{e}^{-\left(\frac{x_j}{\beta_j}\right)^{\alpha_j}}\right)^{\lambda_j}\right);\boldsymbol{j}=\mathbf{1},\mathbf{2}.$$

The MLE $\hat{\delta}=\left(\widehat{\alpha_1},\widehat{\beta_1},\widehat{\lambda_1},\widehat{\alpha_2},\widehat{\beta_2},\widehat{\lambda_2},\hat{\theta}\right)$ can be obtained by solving simultaneously the likelihood equations

$$\frac{\partial l}{\partial \theta}\,|_{\theta=\hat{\theta}}=0,\frac{\partial l}{\partial \alpha_j}\,|_{\alpha=\hat{\alpha}}=0,\frac{\partial l}{\partial \beta_j}\,|_{\beta=\hat{\beta}}=0,\frac{\partial l}{\partial \lambda_j}\,|_{\lambda=\hat{\lambda}}=0, j=1,2.$$

But the equations has to be performed numerically using a nonlinear optimization algorithm.

5 Asymptotic Confidence Intervals

We propose important method to construct confidence intervals (CI) for the parameters of FGMBEW distribution, which called asymptotic confidence interval. The most common method to set confidence bounds for the parameters is to use the asymptotic normal distribution of the MLE. In relation to the asymptotic variance-covariance matrix of the MLEs of the parameters, Fisher information matrix I(Θ), where it is composed of the negative second derivatives of the natural logarithm of the likelihood function evaluated at

$\hat{\Theta}=\left(\widehat{\alpha_1},\widehat{\beta_1},\widehat{\lambda_1},\widehat{\alpha_2},\widehat{\beta_2},\widehat{\lambda_2},\hat{\theta}\right)$. Assuming the regularity condition are satisfied. The MLEs of the parameters based on the log-likelihood functions for MLE, we have the second derivatives.

Therefore the asymptotic variance-covariance matrix of the parameter vector Θ can be written as follows

$$V.COV(\Theta)=-I(\Theta)^{-1}=-\begin{bmatrix} I_{11} & I_{12} & I_{13} & I_{14} & I_{15} & I_{16} & I_{17}\\ I_{21} & I_{22} & I_{23} & I_{24} & I_{25} & I_{26} & I_{27}\\ I_{31} & I_{32} & I_{33} & I_{34} & I_{35} & I_{36} & I_{37}\\ I_{41} & I_{42} & I_{43} & I_{44} & I_{45} & I_{46} & I_{47}\\ I_{51} & I_{52} & I_{53} & I_{54} & I_{55} & I_{56} & I_{57}\\ I_{61} & I_{62} & I_{63} & I_{64} & I_{65} & I_{66} & I_{67}\\ I_{71} & I_{72} & I_{73} & I_{74} & I_{75} & I_{76} & I_{77} \end{bmatrix}^{-1}$$

Assuming the regularity condition are satisfied. An approximate 95% two side confidence intervals for parameter $\Theta=$

$(\alpha_1, \beta_1, \lambda_1, \alpha_2, \beta_2, \lambda_2, \theta)$ can be constructed based on the asymptotic normality conditions of the MLE as $(\widehat{\alpha_1}, \widehat{\beta_1}, \widehat{\lambda_1}, \widehat{\alpha_2}, \widehat{\beta_2}, \widehat{\lambda_2}, \hat{\theta})$

$$\widehat{\alpha_1} \pm Z_{\frac{\gamma}{2}}\sqrt{V_{11}}\,, \quad \widehat{\beta_1} \pm Z_{\frac{\gamma}{2}}\sqrt{V_{22}}\,, \quad \widehat{\lambda_1} \pm Z_{\frac{\gamma}{2}}\sqrt{V_{33}}\,,$$

$$\widehat{\alpha_2} \pm Z_{\frac{\gamma}{2}}\sqrt{V_{44}}\,, \quad \widehat{\beta_2} \pm Z_{\frac{\gamma}{2}}\sqrt{V_{55}}\,, \quad \widehat{\lambda_2} \pm Z_{\frac{\gamma}{2}}\sqrt{V_{66}}\,, \quad \hat{\theta} \pm Z_{\frac{\gamma}{2}}\sqrt{V_{77}}$$

where $Z_{\frac{\gamma}{2}}$ is the percentile of the standard normal distribution with right tail probability $\frac{\gamma}{2}$.

6 Application to Real Data Sets

We study the parameter estimation of the appropriate distribution of data, where the correlation between the two variables (bivariate data) is low. And through this access to a fit model specialized in the study of weak relations and the extent of their impact and effectiveness.

A comparison has been done between FGM Bivariate Weibull (FGMBW), which was discussed by Almetwally et al. (2020), FGM Bivariate Generalized Exponential (FGMBGE), which was discussed by Abd Elaal and Jarwan (2017), FGM bivariate Gamma (FGMBG), which was discussed by Kotz et al. (2004).

6.1 Economic Data

The economic data set, which is reproduced in table (2) consists of 31 yearly time series observations [1980-2010] on response variable: exports of goods and services (X_1) and GDP growth (X_2), which was discussed by Almetwally et al. (2020), The main reasons for selecting the economic data for the present study may due to the fact that, economic is an important sector for many developed and developing countries. Thus, the government is interested in increasing GDP growth and Exports of goods and services.
To show the usefulness of the proposed bivariate estimators obtained from section 2 to section 4 with real situations, we considered here the real economic data to estimate parameters of FGMBEW distribution for the

GDP growth and exports of goods and services. The data is relevant to the FGMBEW distribution, since the correlation between data is weak $\left(\frac{-1}{3}:\frac{1}{3}\right)$, see table (3) the correlation coefficient and test of correlation for data of economics. Goodness of fit test of FGM copula is obtain in table (4). We obtained the proposed estimators for economic data in table (5) the estimates parameters of FGMBEW distributions. Table (6) show the Corresponding Stander Erorr and L.CI for FGMBEW distribution using Economics Data and

table (7) show the variance - covariance matrix of FGMBEW distribution.

Table (2): Economics Data

Years	x_1	x_2	Years	x_1	x_2
1980	30.51	10.01	1996	20.75	4.99
1981	33.37	3.76	1997	18.84	5.49
1982	27.03	9.91	1998	16.21	4.04
1983	25.48	7.40	1999	15.05	6.11
1984	22.35	6.09	2000	16.20	5.37
1985	19.91	6.60	2001	17.48	3.54
1986	15.73	2.65	2002	18.32	2.37
1987	12.56	2.52	2003	21.80	3.19
1988	17.32	7.93	2004	28.23	4.09
1989	17.89	4.97	2005	30.34	4.48
1990	20.05	5.70	2006	29.95	6.85
1991	27.82	1.08	2007	30.25	7.09
1992	28.40	4.43	2008	33.04	7.16
1992	25.84	2.90	2009	24.96	4.67
1994	22.57	3.97	2010	21.35	5.15
1995	22.55	4.64			

X_1: Exports of goods and services (EGS) (% of GDP) 1980:2010, X_2: GDP growth (% per year) 1980:2010.

Source: The employed of economics data are collected by World Bank National Accounts data and OECD National Accounts data

Table (3): The Correlation Coffiecient and Test of Correlation for Data of Economics

	Corr	P- value
Pearson's	0.2705	0.1411
Kendall's	0.1397	0.2791

Table (4): Goodness of Fit test of FGM Copula for Economics Data

	Statistic	$\hat{\theta}$	p-value
Anderson Darling-type	0.5263	0.6271	0.1794

Table (5): The Estimates Parameters of Bivariate Distributions for Economics Data

	FGMBEW	FGMBW	FGMBG	FGMBGE	BMOW
$\widehat{\alpha_1}$	1.9390	4.5225	15.7109	50.9233	2.2795
$\widehat{\beta_1}$	14.971	25.3230	1.4633	0.19577	24.4983
$\widehat{\lambda_1}$	5.8760	-	-	-	-
$\widehat{\alpha_2}$	1.7371	2.6953	5.6643	7.4094	2.3545
$\widehat{\beta_2}$	4.1712	5.8395	0.9113	0.5080	3.6319
$\widehat{\lambda_2}$	2.2701	-	-	-	-
$\hat{\theta}$	0.5950	0.6712	0.6049	0.6338	-
LL	-146.938	-162.810	-162.850	-164.210	-167.070
AIC	307.877	335.617	335.703	338.418	342.133
BIC	317.915	342.789	342.869	345.589	347.680
CAIC	303.007	338.017	338.103	340.818	344.533

Table (6): The Estimates, the Corresponding Stander Erorr and L.CI for FGMBEW distribution using Economics Data

Par	Estimates	SE	L.CI	CI
$\widehat{\alpha_1}$	1.939	0.160	0.628	[1.625 , 2.254]
$\widehat{\beta_1}$	14.971	1.062	4.163	[12.890 , 17.053]
$\widehat{\lambda_1}$	5.876	0.007	0.031	[5.861 , 5.891]
$\widehat{\alpha_2}$	1.737	0.272	1.068	[1.203 , 2.271]
$\widehat{\beta_2}$	4.171	0.229	0.897	[3.723 , 4.620]
$\widehat{\lambda_2}$	2.270	0.472	1.850	[1.345 , 3.195]
$\hat{\theta}$	0.595	0.501	1.964	[-0.387 , 1.577]

Table (7): The variance - covariance matrix of FGMBEW distribution by MLE for Economics Data

	$\widehat{\alpha_1}$	$\widehat{\beta_1}$	$\widehat{\lambda_1}$	$\widehat{\alpha_2}$	$\widehat{\beta_2}$	$\widehat{\lambda_2}$	$\hat{\theta}$
$\widehat{\alpha_1}$	0.026	-0.013	0.083	0.0017	0.011	-0.0056	-0.011
$\widehat{\beta_1}$	-0.013	-0.074	0.119	-0.0014	0.083	0.0095	0.0017
$\widehat{\lambda_1}$	0.083	0.119	1.128	-0.032	-0.083	0.079	-0.029
$\widehat{\alpha_2}$	0.0017	-0.0014	-0.032	0.052	-0.0086	-0.056	0.0051
$\widehat{\beta_2}$	0.011	0.083	-0.083	-0.0086	-0.0061	0.0023	0.0025
$\widehat{\lambda_2}$	-0.0056	0.0095	0.079	-0.056	0.0023	0.223	-0.0020
$\hat{\theta}$	-0.011	0.0017	-0.029	0.0051	0.0025	-0.0020	0.251

Conclusion from tables, in table (4) it is observed that, the economic data is fit for FGM model, and in table (5) it is observed that, the FGMBEW model provides a better fit than the other tested models (FGMBW FGMBG FGMBGE), because it has the smallest value of LL, AIC, BIC, CAIC and HQIC. Table (6) shown the Estimates, the Corresponding Stander Erorr and L.CI for FGMBEW distribution using Economics Data and table (7) shown the variance covariance matrix of FGMBEW distribution by MLE for Economics Data. The FGMBEW distribution is a good alternative to bivariate several lifetime distributions for modeling non negative real-valued data in application.

5.2 Medical Data

The data for 30 patients set from Mc Gilchrist and Aisbett in (1991). Let (X_1) refers to first recurrence time and (X_2) to second recurrence time in table (8). Abd Elaal and Jarwan (2017), discussed the estimation of the parameters of bivariate generalized exponential distribution for this data. The correlation coefficient and test of correlation for medical data are obtained in table (9), Goodness of fit test of FGM copula for medical data is obtain in table (10).We obtained the proposed estimators for medical Data in table (11) the estimates parameters of FGMBEW distributions. Table (12) show the Corresponding Stander Erorr and L.CI for FGMBEW distribution using Medical Data. Table (13) show the variance - covariance matrix of FGMBEW distribution.

Table (8): Medical Data

No.	x_1	x_2	No.	x_1	x_2
1	8	16	16	17	4
2	23	13	17	185	117
3	22	28	18	292	114
4	447	318	19	22	159
5	30	12	20	15	108
6	24	245	21	152	362
7	7	9	22	402	24
8	511	30	23	13	66
9	53	196	24	39	46
10	15	154	25	12	40
11	7	333	26	113	201
12	141	8	27	132	156
13	96	38	28	34	30
14	149	70	29	2	25
15	536	25	30	130	26

X_1: refers to the first recurrence time, X_2: refers to the second recurrence time.

Table (9): The Correlation Coefficient and Test of Correlation for Medical Data

	Corr	P-value
Pearson's	0.13342	0.4902
Kendall's	0.00495	0.4851

Table (10): Goodness of Fit test of FGM Copula for Medical Data

	Statistic	$\hat{\theta}$	P-value
Anderson Darling-type	0.29031	0.46704	0.3936

Table (11): The Estimates Parameters of Bivariate Distributions for Medical Data

	FGMBEW	FGMBW	FGMBG	FGMBGE
$\widehat{\alpha_1}$	0.26813	0.75106	0.67780	0.66607
$\widehat{\beta_1}$	0.83764	100.119	175.526	0.00631
$\widehat{\lambda_1}$	6.47635	-	-	-
$\widehat{\alpha_2}$	0.37933	0.92435	0.92321	0.92584
$\widehat{\beta_2}$	3.62927	98.2466	107.753	0.00958
$\widehat{\lambda_2}$	7.55139	-	-	-
$\hat{\theta}$	0.27716	0.34801	0.37959	0.3780
LL	-327.111	-338.907	-339.492	-339.545
AIC	668.222	687.814	688.984	689.090
BIC	678.031	694.826	695.986	696.106
CAIC	663.131	690.314	691.484	691.590
HQIC	671.266	688.061	689.221	689.870

Table (12): The Estimates, the Corresponding Stander Erorr and L.CI for FGMBEW distribution using Medical Data

Par	Estimates	SE	L.CI	CI
$\widehat{\alpha_1}$	0.359	0.029	0.115	[0.302 , 0.417]
$\widehat{\beta_1}$	7.649	0.700	2.746	[6.276 , 9.021]
$\widehat{\lambda_1}$	4.858	0.337	1.322	[4.197 , 5.519]
$\widehat{\alpha_2}$	0.421	1.847	7.238	[-3.198 , 4.040]
$\widehat{\beta_2}$	9.132	0.035	0.138	[9.063 , 9.201]
$\widehat{\lambda_2}$	5.435	0.989	3.876	[3.497 , 7.372]
$\hat{\theta}$	0.226	0.493	1.934	[-0.741 , 1.193]

Table (13) : The variance - covariance matrix of FGMBEW distribution by MLE for Medical Data

	$\widehat{\alpha_1}$	$\widehat{\beta_1}$	$\widehat{\lambda_1}$	$\widehat{\alpha_2}$	$\widehat{\beta_2}$	$\widehat{\lambda_2}$	$\hat{\theta}$
$\widehat{\alpha_1}$	0.0086	-0.0290	0.0180	0.0020	0.0290	-0.0070	0.0051
$\widehat{\beta_1}$	-0.0290	-3.410	1.120	-0.0011	3.631	0.0043	-0.0010
$\widehat{\lambda_1}$	0.0180	1.120	0.491	0.00109	-1.103	0.0380	-0.0200
$\widehat{\alpha_2}$	0.0020	-0.0011	0.00109	0.00123	-0.0076	-0.0091	0.0049
$\widehat{\beta_2}$	0.0290	3.631	-1.103	-0.0076	0.114	0.0024	-0.0028
$\widehat{\lambda_2}$	-0.0070	0.0043	0.0380	-0.0091	0.0024	0.977	-0.014
$\hat{\theta}$	0.0051	-0.0010	-0.0200	0.0049	-0.0028	0.014	0.243

Conclusion from table (9), it is observed that, the FGMBEW model provides a better than the other tested models (FGMBW, FGMBG, FGMBGE), because it has the smallest value of LL, AIC, BIC, CAIC and HQIC. The FGMBEW distribution is a good alternative to bivariate several lifetime distributions for modeling non negative real-valued data in application.

6 Simulation Study

In this section; a simulation study is done for estimation method based on copula which is MLE. For estimating FGMBEW distribution parameters by Mathcad language.

Simulation Algorithm: the simulation experiments were carried out based on the following data generated form Exponentiated Weibull Distributions, where X_1, X_2 are distributed as Exponentiated Weibull with α_i, λ_i shape parameters and β_i scale parameter, $i = 1,2$, the values of the parameters $(\alpha_1, \beta_1, \lambda_1, \alpha_2, \beta_2, \lambda_2)$ and θ is chosen as the following cases for the random variables generating:

Case 1: $(\alpha_1 = 0.6, \beta_1 = 0.5, \lambda_1 = 0.4, \alpha_2 = 0.5, \beta_2 = 0.7, \lambda_2 = 0.3\ and\ \theta = 0.2)$

Case 2: $(\alpha_1 = 0.6, \beta_1 = 0.5, \lambda_1 = 0.4, \alpha_2 = 0.5, \beta_2 = 0.7, \lambda_2 = 0.3\ and\ \theta = 0.5)$,

for different sample size (n = 50, 100, 160, 200). All computations are obtained based on the Mathcad language. The simulation method is performed by calculate in the Bias and MSE as following cases for the random variables generating:

$$Bias = \hat{\Theta} - \Theta, \qquad MSE = E(\hat{\Theta} - \Theta) = Var(\hat{\Theta}) + \left(Bias(\hat{\Theta})\right)^2,$$
$$where\ \hat{\Theta} = \left(\widehat{\alpha_1}, \widehat{\beta_1}, \widehat{\lambda_1}, \widehat{\alpha_2}, \widehat{\beta_2}, \widehat{\lambda_2}, \hat{\theta}\right)$$

We restricted the number of repeated-samples to 1000.

On the basis of the results summarized in tables (10,11), some conclusions can be drawn which are stated as follows: It is observed that as sample size increases and fixed vector value of Θ, the Bias and MSE of the estimates decreases in the considered method. Also when the sample size increases and fixed vector value of Θ in each cases, Total MSE of the estimates decreases in the considered method. In large sample size all of them are nearly equivalent, where the difference is less and there are no significant differences in Bias and MSE values for MLE method

Table (14): Estimation of the Parameters of FGMBEW Distribution: Case 1

n	par	Mean	Bias	MSE	SE	L.CI
50	$\widehat{\alpha_1}$= 0.6	0.720	0.120	0.022	0.085	0.333
	$\widehat{\beta_1}$= 0.5	0.570	0.070	0.024	0.112	0.439
	$\widehat{\lambda_1}$= 0.4	0.697	0.297	0.095	0.007	0.005
	$\widehat{\alpha_2}$= 0.5	1.221	0.721	0.538	0.010	0.040
	$\widehat{\beta_2}$ = 0.7	0.062	-0.638	0.407	0.143	0.562
	$\widehat{\lambda_2}$ = 0.3	0.548	0.248	0.066	0.075	0.293
	$\widehat{\theta}$= 0.2	0.198	-0.009	0.017	0.377	1.478
				T=0.685		
100	$\widehat{\alpha_1}$= 0.6	0.732	0.132	0.020	0.067	0.264
	$\widehat{\beta_1}$= 0.5	0.546	0.046	0.086	0.084	0.330
	$\widehat{\lambda_1}$= 0.4	0.685	0.285	0.089	0.001	0.003
	$\widehat{\alpha_2}$= 0.5	1.211	0.711	0.533	0.006	0.027
	$\widehat{\beta_2}$ = 0.7	0.065	-0.635	0.404	0.101	0.395
	$\widehat{\lambda_2}$ = 0.3	0.539	0.239	0.063	0.054	0.210
	$\widehat{\theta}$= 0.2	0.204	0.003	0.015	0.272	1.064
				T=0.679		
160	$\widehat{\alpha_1}$= 0.6	0.347	-0.253	0.017	0.059	0.230
	$\widehat{\beta_1}$= 0.5	0.213	-0.287	0.026	0.071	0.276
	$\widehat{\lambda_1}$= 0.4	0.231	-0.169	0.118	0.001	0.003
	$\widehat{\alpha_2}$= 0.5	0.401	-0.099	0.017	0.003	0.016
	$\widehat{\beta_2}$ = 0.7	0.356	-0.344	0.030	0.073	0.287
	$\widehat{\lambda_2}$ = 0.3	0.221	-0.079	0.007	0.043	0.167
	$\widehat{\theta}$= 0.2	0.082	-0.118	0.018	0.216	0.848
				T=0.165		

200	$\widehat{\alpha_1}= 0.6$	**0.380**	**-0.220**	**0.012**	**0.049**	**0.130**
	$\widehat{\beta_1}= 0.5$	**0.214**	**-0.286**	**0.025**	**0.061**	**0.165**
	$\widehat{\lambda_1}= 0.4$	**0.253**	**-0.147**	**0.023**	**0.011**	**0.0003**
	$\widehat{\alpha_2}= 0.5$	**0.453**	**-0.047**	**0.012**	**0.003**	**0.014**
	$\widehat{\beta_2} = 0.7$	**0.356**	**-0.344**	**0.018**	**0.051**	**0.276**
	$\widehat{\lambda_2} = 0.3$	**0.246**	**-0.054**	**0.005**	**0.032**	**0.155**
	$\hat{\theta}= 0.2$	**0.081**	**-0.119**	**0.015**	**0.206**	**0.606**
				T=0.044		

Table (15): Estimation of the Parameters of FGMBEW Distribution: Case 2

n	Par	Mean	Bias	MSE	SE	L.CI
50	$\widehat{\alpha_1}= 0.6$	0.676	0.076	0.019	0.082	0.323
	$\widehat{\beta_1}= 0.5$	0.561	0.061	0.058	0.106	0.414
	$\widehat{\lambda_1}= 0.4$	0.659	0.259	0.088	0.099	0.037
	$\widehat{\alpha_2}= 0.5$	1.230	0.730	0.713	0.056	0.022
	$\widehat{\beta_2} = 0.7$	0.020	-0.680	0.462	0.151	0.592
	$\widehat{\lambda_2} = 0.3$	0.567	0.267	0.126	0.078	0.305
	$\hat{\theta}= 0.5$	0.423	-0.077	0.054	0.731	2.865
				T=0.867		
100	$\widehat{\alpha_1}= 0.6$	0.670	0.070	0.013	0.063	0.248
	$\widehat{\beta_1}= 0.5$	0.628	0.128	0.050	0.078	0.414
	$\widehat{\lambda_1}= 0.4$	0.683	0.283	0.094	0.040	0.015
	$\widehat{\alpha_2}=0.5$	1.110	0.610	0.429	0.033	0.013
	$\widehat{\beta_2} = 0.7$	0.020	-0.680	0.460	0.112	0.582
	$\widehat{\lambda_2} = 0.3$	0.500	0.200	0.058	0.058	0.205
	$\hat{\theta}= 0.5$	0.474	-0.026	0.023	0.538	2.110
				T=0.641		

160	$\widehat{\alpha_1}$= 0.6	0.282	-0.318	0.003	0.055	0.214
	$\widehat{\beta_1}$= 0.5	0.204	-0.296	0.091	0.068	0.267
	$\widehat{\lambda_1}$= 0.4	0.187	-0.213	0.046	0.026	0.013
	$\widehat{\alpha_2}$=0.5	0.317	-0.183	0.038	0.013	0.050
	$\widehat{\beta_2}$ = 0.7	0.351	-0.349	0.122	0.094	0.370
	$\widehat{\lambda_2}$ = 0.3	0.181	-0.119	0.015	0.048	0.199
	$\hat{\theta}$= 0.5	0.203	-0.297	0.092	0.402	1.575
				T=0.188		
200	$\widehat{\alpha_1}$= 0.6	0.319	-0.281	0.001	0.022	0.211
	$\widehat{\beta_1}$= 0.5	0.221	-0.279	0.081	0.160	0.125
	$\widehat{\lambda_1}$= 0.4	0.211	-0.189	0.036	0.0034	0.013
	$\widehat{\alpha_2}$=0.5	0.336	-0.164	0.033	0.0016	0.006
	$\widehat{\beta_2}$ = 0.7	0.351	-0.349	0.122	0.069	0.272
	$\widehat{\lambda_2}$ = 0.3	0.189	-0.111	0.013	0.034	0.189
	$\hat{\theta}$= 0.5	0.220	-0.280	0.081	0.288	0.872
				T=0.174		

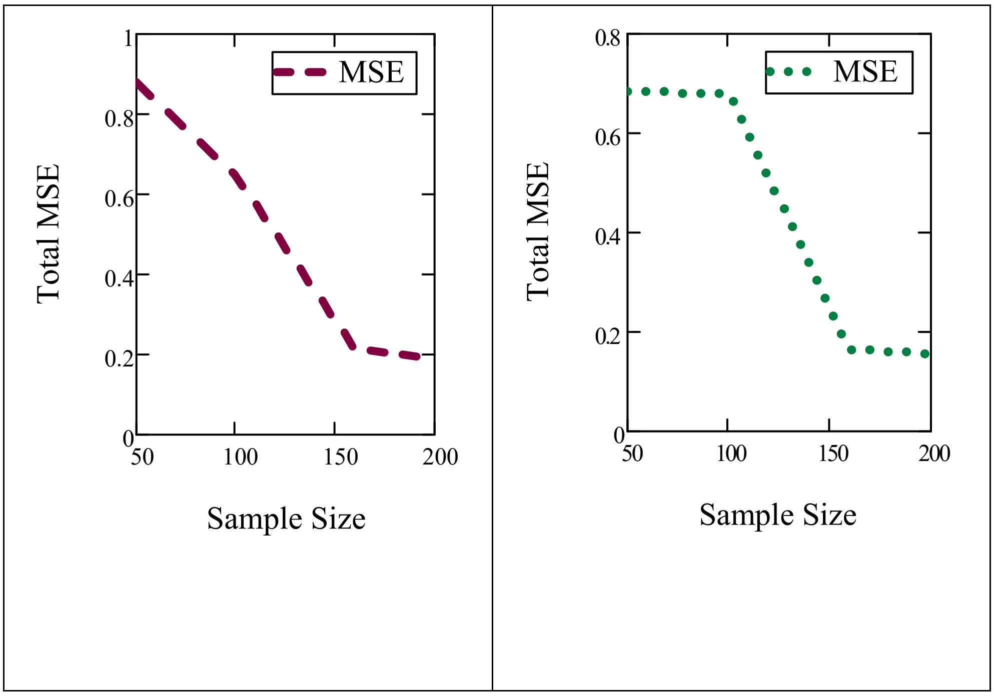

Figure (3) show the plot of MSE of FGMBEW distribution parameters with different sample size(50,100,160,200)

7 Conclusion

In this paper, we have proposed a FGMBEW distribution based on FGM copula function. Moreover, we have the reliability functions for FGMBEW distribution; therefore, it can be used quite effectively in life testing data. Additionally, the new FGMBEW model can be used as an alternative to any bivariate Weibull distribution; it might work better, where the marginal function of FGMBEW distribution has the same basic distribution and has closed forms for product moment. The MLE estimation method of the FGMBEW distribution is concluded. Hence, we can argue that MLE are the best performing estimators for FGMBEW distribution.

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