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An effective comparison with Least Square method for solving fractional gas dynamic equations.

A. Hassan[1,*], A. A. M. Arafa[2,3], S. Z. Rida[4], M. A. Dagher[1], H. M. El Sherbiny[5]

[1]Department of Science and Mathematical Engineering, Faculty of Petroleum and Mining Engineering, Suez University, Suez, Egypt

[2]Department of Mathematics, College of Science and Arts, Qassim University, Al Mithnab, Saudi Arabia

[3]Department of Mathematics, Faculty of Science, Port Science University, Port Said, Egypt

[4]Department of Mathematics, Faculty of Science, South Valley University, Qena, Egypt

[5]Department of Mathematics and Computer Science, Faculty of Science, Suez University, Suez, Egypt

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ABSTRACT

Nonlinear time-fractional partial differential equations, especially nonlinear time-fractional Gas dynamic equations, can be resolved by applying the Optimal Homotopy Asymptotic Method (OHAM) and the Least Square Residual Power Series Method (LSRPSM). A Fractional-order derivative that has numerical values in the closed interval [0, 1] is being employed in the Caputo meaning. These approaches are compared based on their computing complexity, convergence rate, and approximation error. The present study demonstrates that when these techniques are assigned to nonlinear differential equations of fractional order, they exhibit differing convergence rates and approximation errors. Using the Matlab software, perform numerical computations and graphics for fractional gas differential equations. The results of this comparison are compared to the exact solution to demonstrate how much more efficient and precise our methods are at solving nonlinear differential equations. In comparison to (OHAM), the results demonstrate the validity and efficiency of the series solution utilizing (LSRPSM), showing the importance of these methods in the study of fractional differential equations.

1. Introduction

Partial differential equations with nonlinear solutions are commonly used to describe a wide range of phenomena in a wide range of disciplines [1]. Fractional calculus utilized the use of the mathematical concept of non-integral ordered differentiation and integration. The development of fractional calculus begins with classical calculus. However, fractional calculus is receiving greater attention nowadays because of its extensive applications in several technical areas [2]. Nonlinear fractional partial differential equations have recently become prevalent in several branches of applied mathematics, physics, and engineering, including mathematical biology, fluid mechanics, viscoelasticity, aerodynamics, and electrodynamics [11], [12], [13], [14], [15], and [16].

Therefore, solving nonlinear fractional partial differential equations is crucial. For the reason numerically solving fractional PDEs (FPDEs) is important in various fields, many important researchers have made contributions to this topic, and various influential numerical techniques have been proposed [3].

Gas dynamics is a subfield of fluid dynamics that studies the motion of gases and how it impacts physical systems. It is based on the principles of fluid mechanics and thermodynamics. Numerous numerical and analytical techniques were employed in literature to resolve fractional gas dynamic equations. The Homotopy analysis technique, NIM, DTM, FNDM, reduced differential transform, and Elzaki transform method were applied to determine the optimal solution for the gas dynamics equations [4–10]. Regardless matter how big or tiny a physical element is, the Optimal Homotopy Asymptotic Method (OHAM) offers a simple method to ensure solution series convergence. Marinca et al. developed the approach initially [17-19]. This study also uses the terms least-squares methodology and residual power series technique to refer to the least-squares residual power series method (LSRPSM). Compared to the classical residual power series method, the new methodology can generate an estimate that is

* Corresponding author at Suez University

E-mail addresses: a.kamal@suezuni.edu.eg (A. Hassan)

more accurate while using fewer expansion terms [20]. In this work, a nonlinear partial fractional differential is solved using the techniques outlined above, and their results are compared using a nonlinear fractional Gas-Dynamic equation.

Following is the structure of the current paper: For the fractional calculus and Wronskian theory, several fundamental concepts and mathematical fundamentals are discussed in section 2. Both the Optimal Homotopy Asymptotic and the least square residual power series technique's fundamentals are covered in Section 3 of this article. The suggested approaches are demonstrated using the same example (Gas-Dynamic Equation) in Section 4. Section 5 ends with a conclusion.

2. Definitions and properties

The concept of partial fractional Wronskian and some definitions of fractional calculus theory are introduced, and will be used extensively throughout this article.

Definition 2.1. The Riemann-Liouville fractional integral operator of order α is defined as

$$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \quad \alpha > 0, x > 0 \tag{1}$$

Where,

$$J^0 f(x) = f(x). \tag{2}$$

The Riemann-Liouville fractional integral operator has the following properties:

$$J^{\alpha} J^{\beta} f(x) = J^{\alpha+\beta} f(x), \tag{3}$$

$$J^{\alpha} J^{\beta} f(x) = J^{\beta} J^{\alpha} f(x), \tag{4}$$

$$J^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}. \quad \alpha > 0, \beta > -1, t > 0 \tag{5}$$

Definition 2.2. A continuous function $f(x)$ has the following The Riemann-Liouville fractional integral operator fractional derivatives:

$$D_*^{\alpha} f(x) = D^m\left(J^{m-\alpha} f(x)\right), \tag{6}$$

Where $m-1 < \alpha \leq m, m \in N$.

Definition 2.3. The Caputo fractional derivative:

For a function $f(x)$, the Caputo fractional integral is defined as follows:

$$D^{\alpha} f(x) = J^{m-\alpha}\left(D^m f(x)\right) = \frac{1}{\Gamma(m+\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^m(t)dt, \tag{7}$$

Where $m-1 < \alpha \leq m, m \in N, x > 0$. The Caputo fractional derivative has the following properties:

1) If $-1 < \alpha \leq m, m \in N, \mu \geq -1$ it holds

$$D^{\alpha} J^{\alpha} f(x) = f(x),$$

2) $J^{\alpha} D^{\alpha} f(x) = f(x) + \sum_{k=0}^{m-1} f^k(0^+) \frac{x^k}{k!}. \ x > 0$

3) $D^{\alpha}(k) = 0$

$$D^{\alpha}\left(\xi f(t) + \omega g(t)\right) = \xi D^{\alpha}\left(f(t)\right) + \omega D^{\alpha}\left(g(t)\right),$$

Where ω, ξ, and k are real constants.

Definition 2.4. The fractional partial Wronskian (see [21]).

Let $\varphi_1, \varphi_2, \ldots, \varphi_n$ represent how many functions of the variables x and t are defined on the domain Ω

The fractional partial Wronskian of $\varphi_1, \varphi_2, \ldots, \varphi_n$ follows :

$$\omega^{\alpha}[\varphi_0,\varphi_1,\dots,\varphi_n] = \begin{vmatrix} \varphi_0 & \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ D^{\alpha}(\varphi_1) & D^{\alpha}(\varphi_2) & D^{\alpha}(\varphi_3) & \dots & D^{\alpha}(\varphi_n) \\ D^{2\alpha}(\varphi_1) & D^{2\alpha}(\varphi_2) & D^{2\alpha}(\varphi_3) & \cdots & D^{2\alpha}(\varphi_n) \\ \vdots & \vdots & \vdots & & \vdots \\ D^{(n-1)\alpha}(\varphi_1) & D^{(n-1)\alpha}(\varphi_n) & D^{(n-1)\alpha}(\varphi_n) & \cdots & D^{(n-1)\alpha}(\varphi_n) \end{vmatrix} \neq 0, \quad (8)$$

Where $0 < \alpha \leq 1,\ D^{\alpha}(\varphi_i) = \left(\frac{\partial}{\partial x} + \frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)(\varphi_i), where\ i = 1,2,3,\dots,n,$

$D^{n\alpha} = D^{\alpha} D^{\alpha} \dots D^{\alpha}(\ n - times)$, $\varphi_1(x,t), \varphi_2(x,t), \dots, \varphi_n(x,t)$ if all n functions have a fractional partial Wronskian, then all n functions are regarded as linearly independent. In the domain $\Omega = [a,b] \times [a,b]$, $\varphi_1(x,t), \varphi_2(x,t), \dots, \varphi_n(x,t)$ is nonzero, at least at one point.

3. Overview of two numerical approaches

With the least square residual power series method (LSRPSM) and the optimal homotopy asymptotic approach (OHAM), we discuss two effective strategies for solving nonlinear Caputo time-fractional partial differential equations in this part.

3.1. The OHAM Methodology (Optimal Homotopy Asymptotic Method)

The subsequent steps provided the OHAM approach for nonlinear partial fractional differential equations:

Assume the subsequent partial differential equation:

$$D\big(u(x,t) + g(x,t)\big) = 0\ ,\ x\epsilon\Gamma \quad ,\ t \geq 0 \tag{9}$$

$$B\left(u, \frac{\delta u}{\delta t}\right) = 0, \tag{10}$$

Where D indicates the differential operator, which could be an integer or a fractional order. x and t Denote a variable that is independent, Unknown function $u(x,t)$, boundary operator B, Γ is the boundary of the domain Ω, and known expression $g(x,t)$ are all present in Equation (9)

The differential operator D can now be divided into the terms of L and N differential operators, providing:

$$L(u(x,t) + N(u(x,t) + g(x,t)) = 0.\ x\epsilon\Gamma. \tag{11}$$

Below, L refers to the simpler linear differential operator, which could represent the linear and uncomplicated portion of the Eq. (9) that can be solved using any auxiliary analytical method, whereas N the operator denotes the differential operator, that's is a non-linear and complicated portion of the Eq. (9).

We begin by creating the homotopy as:

$$(1-q)L(\xi(x,t;q)) = H(q)(L(\xi(x,t;q) + N\big(\xi(x,t;q)\big), \tag{12}$$

$$B\left(\xi(x,t;q), \frac{\delta\xi(x,t;q)}{\delta t}\right) = 0, \tag{13}$$

An unidentified function is represented $\xi(x,t;q)$, where $q \in [0,1]$ forms an embedding parameter, H(q) defines a nonzero function for $q \neq 0$ and H(0)=0,

Obviously, it holds when $q = 0$ and $q = 1$:

$\xi(x,t;0) = u_0(x,t), \qquad \xi(x,t;1) = u(x,t).$

As a result, once q increases from $0\ to\ 1$, the solution $\xi(x,t)$changes from $u_0(x,t)$ to $u(x,t)$, which ensures a quick convergence to the exact solutions.

The auxiliary function $H(q)$ gives us a simple way for controlling and governing the convergence while increasing the precision of the results and effectiveness of the procedure.

The auxiliary function $H(q)$ is selected in the following approach:

$$H(q) = q\,c_1 + q^2 c_2 + q^3 c_3 + \cdots \quad , \tag{14}$$

Since the auxiliary convergence control parameters $c_i\ , i = 1,2,3, \dots$ are present.

In Taylor's series relating to q, expand $\xi(x,t;q,c)$ to get approximations of the following solutions:

$$\xi(x,t;q,c_i) = u_0(x,t) + \sum_{k=1}^{\infty} u_k(x,t;c_i)\, q^k. \tag{15}$$

One of the main contributors to the convergence of series (15) has been determined to be the auxiliary convergence-control parameters.

When we replace $\xi(x,t;q,c_i)$, and $H(q)$ with identical coefficients of the same powers of q in equations (11), we produce series problems and zeroth-order problems, as follows:

$$L\big(u_0(x,t)\big) = 0 \ , \qquad B\left(u_0(x,t), \frac{\partial u_0(x,t)}{\partial t}\right) = 0, \tag{16}$$

$$L\big(u_1(x,t)\big) = c_1\, N_0\big(u_0(x,t)\big), \qquad B\left(u_1(x,t), \frac{\partial u_1(x,t)}{\partial t}\right) = 0, \tag{17}$$

$$L\big(u_2(x,t)\big) = c_2\, N_0\big(u_0(x,t)\big) + c_1\, N_1\big(u_0(x,t), u_1(x,t)\big) + (1 + c_1)\, L\big(u_1(x,t)\big),$$

$$B\left(u_2(x,t), \frac{\partial u_2(x,t)}{\partial t}\right) = 0 \tag{18}$$

The analytical solution's general controlling k^{th} - order problem is written as $u_k(x,t)$, and it has the following form

$$L\big(u_k(x,t)\big) - L\big(u_{k-1}(x,t)\big) = c_k\, N_0\big(u_0(x,t)\big) + \sum_i^{k-1} c_i\big[L\big(u_{k-i}(x,t)\big) + N_{k-i}\big(u_0(x,t), u_1(x,t), \ldots, u_{k-1}(x,t)\big)\big],$$

$$k = 2,3, \ldots, \tag{19}$$

$$B\left(u_k(x,t), \frac{\partial u_k(x,t)}{\partial t}\right) = 0, \tag{20}$$

The coefficient of q^i in the nonlinear operator is N_i , $i > 0$.

$$N\,\big(u(x,t)\big) = N_0\big(u_0(x,t)\big) + q\, N_1\big(u_0(x,t), u_1(x,t)\big) + q^2 N_2\big(u_0(x,t), u_1(x,t), u_2(x,t)\big) + \cdots \tag{21}$$

Series (15) is said to converge at q = 1 if it satisfies:

$$\tilde{u}(x,t,c_i) = u_0(x,t) + \sum_{k=1}^{\infty} u_k\,(x,t;c_i).\, i = 1,2,3, \ldots \tag{22}$$

When Eq. (15) is substituted into Eq. (11), the residual is as described as:

$$R(x,t;c_i) = L\big(\tilde{u}(x,t,c_i)\big) + N\big(\tilde{u}(x,t,c_i)\big).\, i = 1,2,3, \ldots \tag{23}$$

As $R(x,t;c_i) = 0$ then $\tilde{u}(x,t,c_i)$ just so seems to be the same solution.

There are numerous techniques that can be employed to determine the auxiliary convergence-control parameters c_1,c_2,c_3, . . . (Ritz, Least Square, Collocation, and Galerkin's Method). In the brief overview that follows, the least square method is used to obtain the optimal values for the auxiliary convergence-control parameters.

$$J(c_i) = \int_0^t \int_\Omega R^2(x,t,c_i)\, dx\, dt, \tag{24}$$

R refers for the residual. The following conditions provide an optimal opportunity to identify the unknown constants $c_i (i = 1,2,3, \ldots, m)$):

$$\frac{\delta J}{\delta c_1} = \frac{\delta J}{\delta c_2} = \cdots = \frac{\delta J}{\delta c_m} = 0. \tag{25}$$

The nonlinear algebraic system certainly can be solved rapidly while m is small, but as m increases. The solution becomes more complex.

3.2. The Least Square Residual Power Series (LSRPS) Methodology

An approach based on the least-squares method and the classical (RPS) method has been proposed for time-fractional differential equations.

3.2.1. Residual Power Series (RPS) Method.

The general fractional differential equation:

$$l^\alpha(u(x,t)) + N\big(u(x,t)\big) = 0, \qquad t > 0, \qquad 0 < \alpha \le 1 \tag{26}$$

Where N is the nonlinear operator and l is the linear operator.

Applying the classical (RPS) approach [23], an algorithm could have been described as follows:

$$u(x,t) = \sum_{n=0}^{\infty} f_n(\mathrm{x}) \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}, \; 0 < \alpha \le 1, \; x \in I, \; 0 \le t < R \tag{27}$$

To provide an accurate approximation for (26), we present The kth series of $u(x,t)$:

$$u_k(x,t) = \sum_{n=0}^{\infty} f_n(\mathrm{x}) \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}, \; 0 < \alpha \le 1, \; x \in I, \; 0 \le t < R \tag{28}$$

The 0-th RPS approximate solution of $u(x,t)$ is:

$$u_0(x,t) = u(x,0) = f(x) \quad (29)$$

Equation (26) can be written by:

$$u_k(x,t) = f(x) + \sum_{n=1}^{\infty} f_n(\mathrm{x}) \frac{t^{n\alpha}}{\Gamma(1+\mathrm{n}\alpha)}, 0 < \alpha \le 1, \ x \in I, \ 0 \le t < R, \quad k = 1,2,3,\ldots \quad (30)$$

For Equation (26) we express the residual function as follows:

$$Res_u(x,t) = l^\alpha(u_k(x,t)) + N(u_k(x,t)), \quad t > 0, \quad 0 < \alpha \le 1 \quad (31)$$

The k^th residual function $Res_{u,k}$ define as follow:

$$Res_{u,k}(x,t) = l^\alpha(u_k(x,t)) + N(u_k(x,t)), \quad t > 0, \quad 0 < \alpha \le 1 \quad (32)$$

We investigate the solution of the following to obtain $f_n(\mathrm{x}), n \in N^*$:

$$D_t^{(n-1)\alpha} Res_{u,k}(x,0) = 0, k \in N^* \quad (33)$$

Where $N^* = \{1,2,3,\ldots,n\}$.

The standard residual power series strategy will result in k-th-order approximation solutions in the present case.

$$u_k = \varphi_0 + \varphi_1 + \varphi_2 + \cdots \varphi_k, \quad (34)$$

where,

$$\begin{aligned} \varphi_0 &= f_0(\mathrm{x}), \\ \varphi_1 &= f_1(\mathrm{x}) \frac{t^\alpha}{\Gamma(1+\alpha)}, \\ \varphi_2 &= f_2(\mathrm{x}) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}, \\ &\vdots \\ \varphi_k &= f_k(\mathrm{x}) \frac{t^{k\alpha}}{\Gamma(1+\mathrm{k}\alpha)}. \end{aligned} \quad (35)$$

3.2.2. Least-Squares Residual Power Series Method (LSRPS)

The (LSRPS) method's methodology is covered in this section, along with some expressions that we think are important.

The remainder $\widetilde{Res}$ for Equation (26) is:

$$\widetilde{Res}(x,t,\tilde{u}) = l^\alpha(\tilde{u}(x,t)) + N(\tilde{u}(x,t)), \ t > 0, \ 0 < \alpha \le 1 \quad (36)$$

regarding $I(\tilde{u}) = 0$ and $\tilde{u}$ representing the approximate solution of the equation.

Remark:

$$\lim_{i \to \infty} \widetilde{Res}(x,t,s^{i\alpha}(x,t)) = 0, \quad (37)$$

In this case, $\{s^{i\alpha}(x,t)\}_{\ i \in N^*}$ converge to the solution of equation (26).

The $\tilde{u}$ is the ε –approximate (RPS) method solution of equation (26) on domain Ω if:

$$\left|\widetilde{Res}(x,t,\tilde{u})\right| < \varepsilon, \quad (38)$$

and $I(u) = 0$ is also satisfied by $\tilde{u}$.

Assuming $\tilde{u}$ is the weak approximation (RPS) method solution of equation (26), then we refer to it as:

$$\iint \widetilde{Res}^2\,(x,t,\tilde{u})\;dx\,dt \leq \;\varepsilon. \tag{39}$$

where $I(u) = 0$ is also satisfied by $\tilde{u}$.

For the least-squares (RPS) approach, we suggest the procedures below.

1st step:

We implement the classical residual power series approach. The form of $u_k(x,t)$ could be represented as:

$$u_k(x,t) = \sum_{n=0}^{\infty} f_n(\mathrm{x}) \frac{t^{n\alpha}}{\Gamma(1+\mathrm{n}\alpha)},\;\; 0 < \alpha \leq 1,\;\; x \in I,\;\; 0 \leq t < R \tag{40}$$

and the kth residual function $Res_{u,k}$ as follows:

$$Res_{u,k}(x,t) = l^{\alpha}\big(u_k(x,t)\big) + N\big(u_k(x,t)\big).\; t > 0,\; 0 < \alpha \leq 1 \tag{41}$$

After that, we explore for $f_n(\mathrm{x})$ solutions by:

$$D_t^{(n-1)\alpha} Res_{u,k}(x,0) = 0, k \in N^* \tag{42}$$

Where $N^* = \{1,2,3,\ldots,n\}$.

In this case, the classical (RPS) technique yields kth-order approximation solutions with:

$$u_k = \varphi_0 + \varphi_1 + \varphi_2 + \cdots \varphi_k\,, \tag{43}$$

Where $\varphi_0, \varphi_1, \varphi_2$ can be computed by equation (35)

2nd Step:

The linearly independent functions can be validated using:

$$\omega^{\alpha}[\varphi_0, \varphi_1, \ldots, \varphi_n] = \begin{vmatrix} \varphi_0 & \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ D^{\alpha}(\varphi_1) & D^{\alpha}(\varphi_2) & D^{\alpha}(\varphi_3) & \cdots & D^{\alpha}(\varphi_n) \\ D^{2\alpha}(\varphi_1) & D^{2\alpha}(\varphi_2) & D^{2\alpha}(\varphi_3) & \cdots & D^{2\alpha}(\varphi_n) \\ \vdots & \vdots & \vdots & & \vdots \\ D^{(n-1)\alpha}(\varphi_1) & D^{(n-1)\alpha}(\varphi_n) & D^{(n-1)\alpha}(\varphi_n) & \cdots & D^{(n-1)\alpha}(\varphi_n) \end{vmatrix} \neq 0, \tag{44}$$

Where $D^{\alpha}(\varphi_i) = \left(\frac{\partial}{\partial x} + \frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)(\varphi_i)$.

Let $s_k = \{\varphi_0, \varphi_1, \ldots, \varphi_n\}$ be a set defined on R, where i is any integer between 0 and $1, 2, \ldots$

Remark:

If we are unable to identify the point where $\omega^{\alpha}[\varphi_1, \varphi_2, \ldots, \varphi_n]$ is not equal to 0, the set s_k is linearly dependent. As a result, we should use the classic residual power series strategy in this case.

3rd Step:

We assume the analytical solution for Equation (26)

$$\tilde{u}_k = \sum_{n=0}^{k} c_k^n\, \varphi_r, \tag{45}$$

then we get:

$$\widetilde{Res}(x,t,c_k^n) = \widetilde{Res}(x,t,\tilde{u}_k). \tag{46}$$

Here, we calculate some constants c_k^n.

4th Step:

We associate with the following functional:

$$\int \int_{\Omega} \widetilde{Res}^2\,(x,t,\tilde{u})\;\; dx\; dt\;\; = min\, J. \tag{47}$$

4. Numerical applications

We will examine the numerical example below that applies both techniques to the same equation in order to evaluate the positive features and precision of the LSRPSM and OHAM for the resolution of nonlinear Caputo time-fractional Gas Dynamic equations.

Example 4.1. Nonlinear Gas Dynamical Equation Solved by OHAM Using Caputo Time Fractional Derivative:

$$D_t^{\alpha} u + \frac{1}{2}(u^2)_x - u(1-u)u_{xx} = 0 \,,\ 0 < \alpha \le 1,\ t > 0 \tag{48}$$

From the initial condition

$$u(x,0) = e^{-x}. \tag{49}$$

The equation (48) becomes the standard gas dynamics equation of order one if we choose α =1, which has

$$u(x,t) = e^{t-x}. \tag{50}$$

The OHAM concept described in Section 3 will be followed by following steps:

$$L(\phi(x,t;q) = D_t^{\alpha} u,$$

$$N(\phi(x,t;q) = -\left(\frac{1}{2}(u^2)_x - u(1-u)u_{xx}\right).$$

Using the initial condition:

$$\phi(x,0;q) = e^{-x}. \tag{51}$$

Collecting identical powers of q by setting each coefficient of q to zero creates the zeroth order problem-solving.

$$\frac{\delta^{\alpha} u_0}{\delta t^{\alpha}} = 0. \tag{52}$$

Starting with an initial estimate of $u_0(x,t) = (x,0) = e^{-x}$, we obtain the first order problem as follows:

$$\frac{\delta^{\alpha} u_1}{\delta t^{\alpha}} = -c_1\,(e^{-x}\ ). \tag{53}$$

When we apply the J^{α}operator, referred to as the inverse operator of the D^{α} operator in (53) we get:

$$u_1 = \frac{-c_1 t^{\alpha}}{\Gamma(\alpha+1)} e^{-x}. \tag{54}$$

We compare the coefficient of q^2 and apply the J^{α}operator to the two sides of $\frac{\delta^{\alpha} u_2}{\delta t^{\alpha}}$ to find the value of the second-order problem:

$$u_2 = (c_1)^2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} (e^{-x}) - \frac{t^{\alpha}}{\Gamma(\alpha+1)} (e^{-x})((c_1)^2 + c_1 + c_2) \tag{55}$$

The third order problem is:

$$u_3 = (c_1)^3 \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} (-e^{-x}) + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} (e^{-x})(2(c_1)^2 + 2c_1c_2) - e^{-x} \frac{t^{\alpha}}{\Gamma(\alpha+1)} (2(c_1)^2 + 2c_1c_2 + c_1 + c_2 + c_3 + (c_1)^3) \tag{56}$$

By using the initial condition (49), Eq. (54), and Eq. (55), (56) we find out third-order approximate solution of Eq. (48)

$$u_{OHAM}(x,t) = e^{-x} + \frac{-c_1 t^{\alpha}}{\Gamma(\alpha+1)} e^{-x} + (c_1)^2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} (e^{-x}) - \frac{t^{\alpha}}{\Gamma(\alpha+1)} (e^{-x})((c_1)^2 + c_1 + c_2) + (c_1)^3 \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} (-e^{-x}) + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} (e^{-x})(2(c_1)^2 + 2c_1c_2) - e^{-x} \frac{t^{\alpha}}{\Gamma(\alpha+1)} (2(c_1)^2 + 2c_1c_2 + c_1 + c_2 + c_3 + (c_1)^3) \tag{57}$$

We find the values of constants c_1, c_2, c_3 for different values α of using the least squares approach described above. Subsequently when α =1, we determine the following unknown coefficients:

$$\boldsymbol{c_1 = 0.215278960, c_2 = 1.070093466, c_3 = -4.4172810492} \tag{58}$$

Table 1. The results of the OHAM method's third order solution (56) at multiple points of x, t and Alpha=1

x	t	OHAM at Alpha=1	Exact at Alpha=1	Absolute Error(OHAM) at Alpha=1
0.1	0.1	0.9999792	1	2.07553118E-5
0.2	0.1	0.90481863	0.90483741	1.87801827E-5
0.3	0.1	0.81871376	0.81873075	1.69930120E-5
0.4	0.1	0.74080284	0.74081822	1. 5375913E-5
0.5	0.1	0.67030613	0.67032004	1.3912701E-5

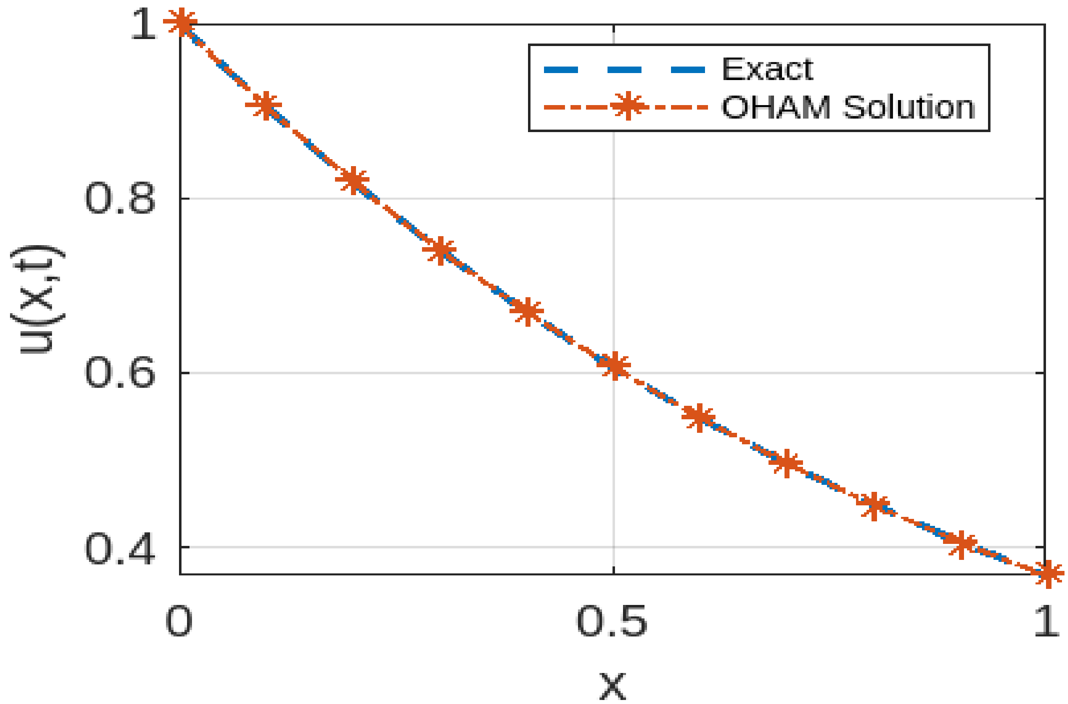

Figure 1. The approximate solution obtained using OHAM for equation(48) is compared to the exact solution for equation(50) when Alpha = 1 and t = 0.

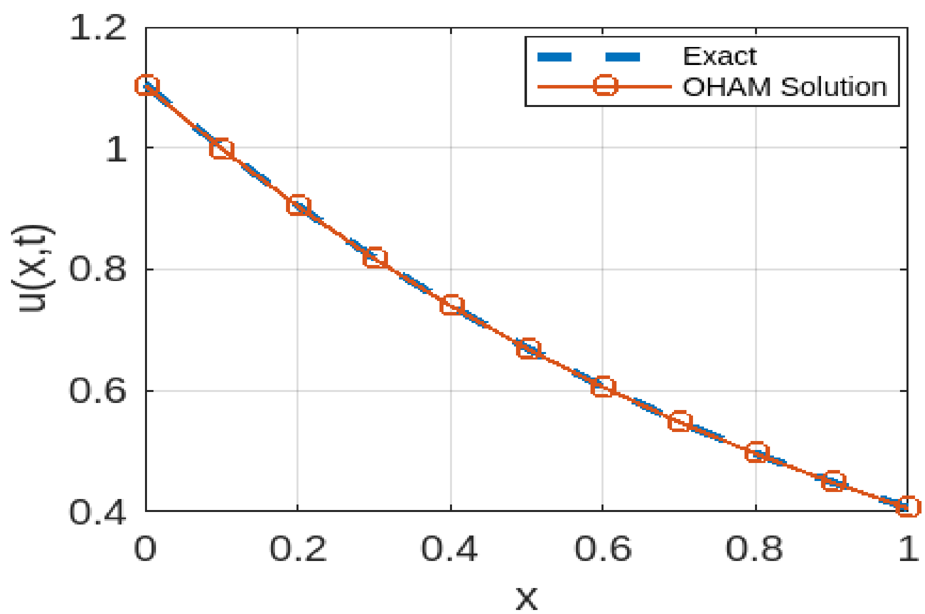

Figure 2. When Alpha = 1 and t = 0.1, the approximate solution via OHAM for equation(48) is compared to the exact solution for equation(50).

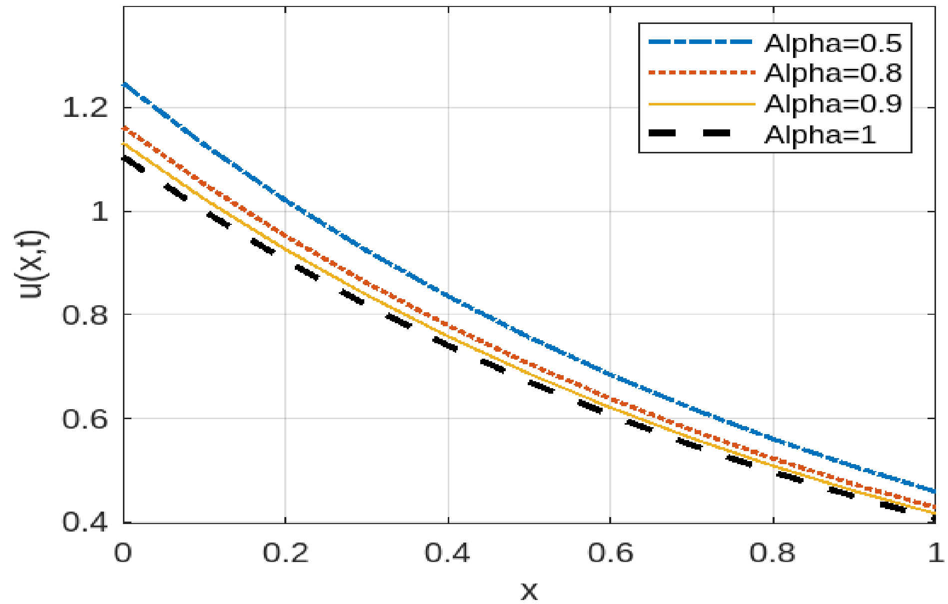

Figure 3. The behavior of the exact and approximate solutions provided by OHAM for equation(48) at alpha = 0.5,0.8,0.9,1 when t = 0.1.

Second: we will use the LSRPSM for solving the same equations for the nonlinear gas dynamic problem which take the form:

$$D_t^{\alpha} u + \frac{1}{2}(u^2)_x - u(1-u)u_{xx} = 0\ ,\ 0 < \alpha \leq 1,\ t > 0 \tag{59}$$

$u(x,0) = e^{-x}$ is the initial condition.

With exact solution $u(x,t) = e^{t-x}$ (60)

To introduce the Gas-Dynamic equation, we can employ the well-known (RPS) method. This method offers a solution for the equation, as indicated by reference [22].

$$u(x,t) = f(x) + f_1(x)\frac{t^{\alpha}}{\Gamma(1+\alpha)} + f_2(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + f_3(x)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \cdots, \tag{61}$$

Where

$$f(x) = e^{-x}.$$
$$f_1(x) = e^{-x}$$
$$f_2(x) = e^{-x}$$
$$f_3(x) = e^{-x} \tag{62}$$

The linearly independent functions could be validated by using:

$$\omega^{\alpha}[\varphi_0, \varphi_1, \ldots, \varphi_n] = \begin{vmatrix} e^{-x}. & \left(e^{-x}\frac{t^{\alpha}}{\Gamma(1+\alpha)}\right) & \left(e^{-x}\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right) & e^{-x}\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \\ D^{\alpha}(e^{-x}.) & D^{\alpha}\left(e^{-x}\frac{t^{\alpha}}{\Gamma(1+\alpha)}\right) & D^{\alpha}\left(e^{-x}\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right) & D^{\alpha}\left(e^{-x}\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\right) \\ D^{2\alpha}(e^{-x}) & D^{2\alpha}\left(e^{-x}\frac{t^{\alpha}}{\Gamma(1+\alpha)}\right) & D^{2\alpha}\left(e^{-x}\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right) & D^{2\alpha}\left(e^{-x}\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\right) \\ \vdots & \vdots & \vdots & \vdots \\ D^{(n-1)\alpha}(e^{-x}) & D^{(n-1)\alpha}\left(e^{-x}\frac{t^{\alpha}}{\Gamma(1+\alpha)}\right) & D^{(n-1)\alpha}\left(e^{-x}\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right) & D^{(n-1)\alpha}\left(e^{-x}\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\right) \end{vmatrix} \neq 0 \tag{63}$$

$\alpha = 1, t = 0.5, x = 0, c = 1, \omega^1[\varphi_0, \varphi_1, \varphi_2] \neq 0$ Hence, the functions $\varphi_0, \varphi_1, \varphi_2$ are linearly independent define as:

$\varphi_0 = f(x),$

$\varphi_1 = f_1(x)\dfrac{t^{\alpha}}{\Gamma(1+\alpha)},$

$$\varphi_2 = f_2(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)},$$
$$\varphi_3 = f_3(x)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)},$$

Consequently, we can obtain an approximation that can be formulated as follows:

$$\tilde{u} = c_0(e^{-x}) + c_1(e^{-x})\frac{t^{\alpha}}{\Gamma(1+\alpha)} + c_2((e^{-x})\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + c_3((e^{-x})\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \quad (64)$$

The residual function can be obtained by:

$$\widetilde{Res}(x,t,\tilde{u}) = D_t^{\alpha}\tilde{u} + \frac{1}{2}(\tilde{u}^2)_x - \tilde{u}(1-\tilde{u})\tilde{u}_{xx} = 0\,. \quad (65)$$

With the initial condition:

$$\tilde{u}_0 = c_0(e^{-x}). \quad (66)$$

By using $\tilde{u}_0$ put $c_0 = 1$ then $\tilde{u}$ can be written by:

$$\tilde{u} = (e^{-x} + c_1(e^{-x})\frac{t^{\alpha}}{\Gamma(1+\alpha)} + c_2(e^{-x})\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + c_3((e^{-x})\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \quad (67)$$

By substituting ($\tilde{u}$) into ($\widetilde{Res}(x,t,\tilde{u})$)., we can obtain $\widetilde{Res}$ As a result, the functional J can be expressed as:

$$\int\int_{\Omega} \widetilde{Res}^2\,(x,t,\tilde{u})\,dx\,dt = J(c_1,c_2,c_3). \quad (68)$$

We determine the functional J. As a result, we have two algebraic equations:

$$\frac{\partial J}{\partial c_1} = \frac{\partial J}{\partial c_2} = \frac{\partial J}{\partial c_3} = 0. \quad (69)$$

And following that, we calculate the unknown coefficients of (67) when $\alpha = 1$:

$$c_1 = 1.000008699704,\ c_2 = 0.9989611594288\ ,\ c_3{=}1.01514550095504 \quad (70)$$

The following formula can be used to demonstrate the absolute error between the exact and approximate solutions using the proposed approach:

$$Error{=}|\tilde{u}_i(x,t) - u(x,t)|. \quad (71)$$

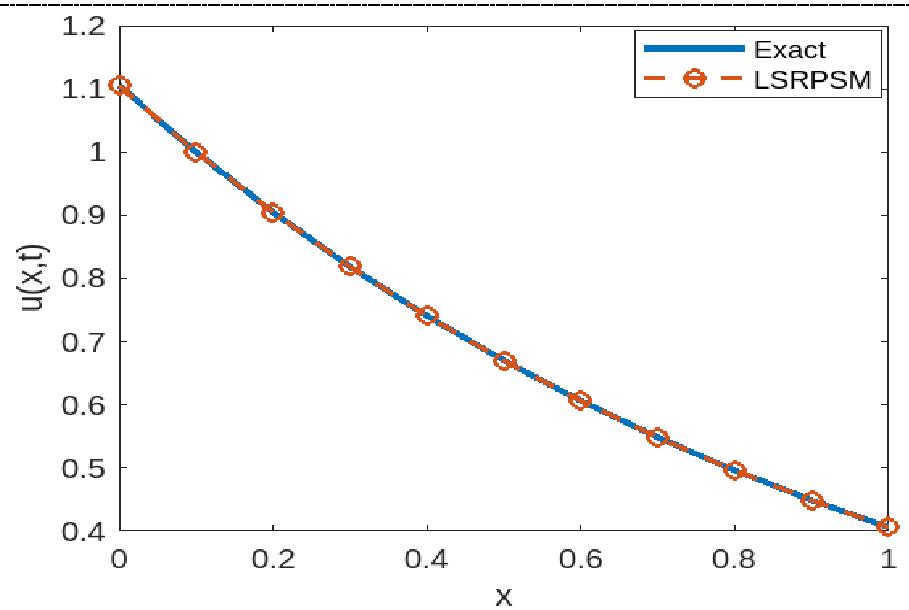

Figure 4. demonstrates the contrast between the Exact Solutions for the Gas-Dynamic Equation (60) at t= 0.1 and the Method (LSRPSM).

Table 2. presents the absolute errors between the approximate and exact solutions obtained using the (LSRPS) approach.

x	t	LSRPSM at Alpha=1	Exact at Alpha=1	Absolute Error(LSRPSM) at Alpha=1
0.1	0.1	0.999999999999948	1	5.19689956877948e-14
0.2	0.1	0.904837418035913	0.90483741803596	4.70234918760661e-14
0.3	0.1	0.818730753077939	0.818730753077982	4.25486149761746e-14
0.4	0.1	0.740818220681679	0.740818220681718	3.8499578916048e-14
0.5	0.1	0.670320046035604	0.670320046035639	3.48358595818685e-14

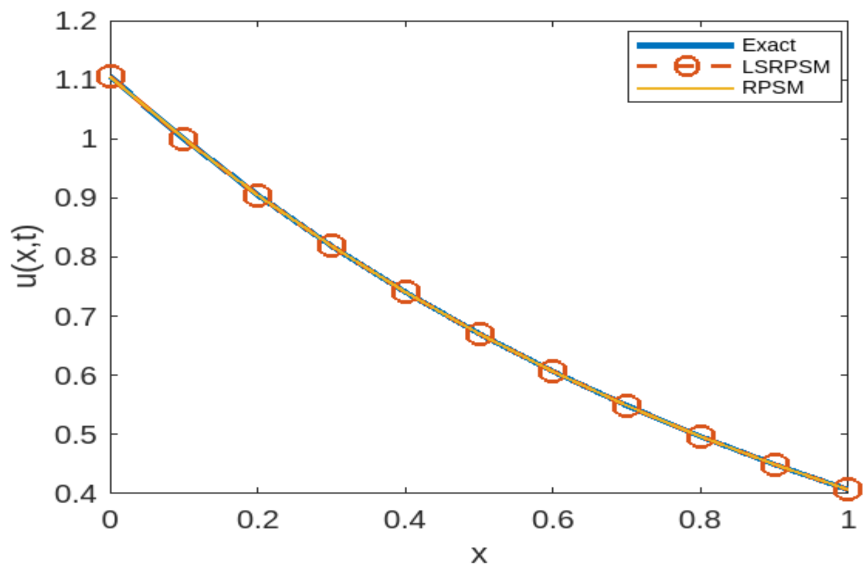

Figure 5. compares three techniques The Exact Solutions for the Gas-Dynamic Equation(60), the Method (RPSM), and the Method (LSRPSM) at t= 0.1

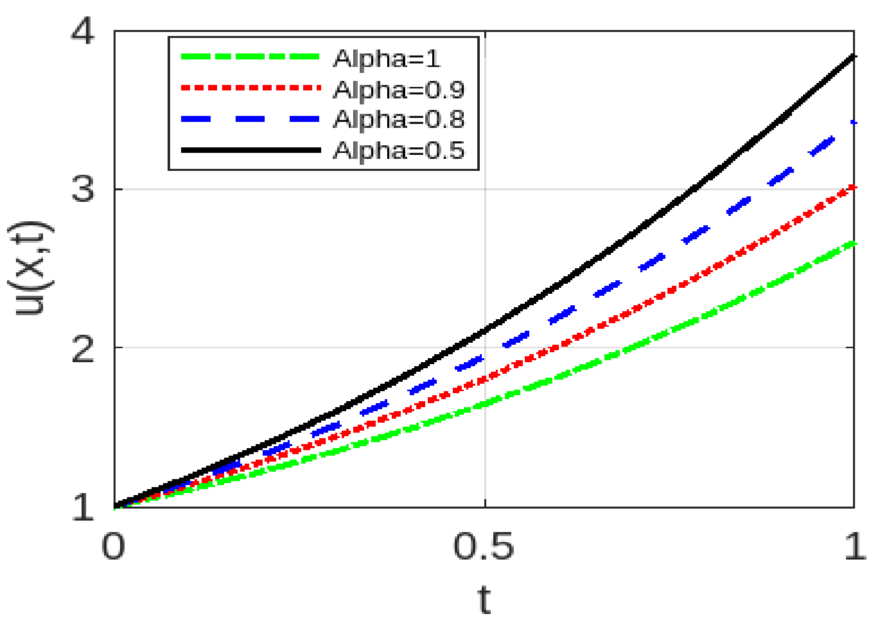

Figure 6. A description of the exact and approximate solutions provided through LSRPSM for various alpha values= 0.5, 0.8, 0.9,1 when x = 0.

Conclusion

Time-fractional Gas-Dynamic equations based on Caputo fractional derivation were approximated using the least Square Residual Power Series approach and the Optimal Homotopy Asymptotic method. At the same value of t and with the same number of term approximations, the acquired solutions in the Least Square Residual Power Series Method (LSRPSM) see (Table 2) and fig(4to6) converge to the exact solutions more rapidly than the Optimal Homotopy Asymptotic Method (OHAM) in table(1)and fig from 1to 3. This implies that the approximate solutions in (LSRPSM) technique are significantly closer to the exact solutions than the solutions in (OHAM) approach. According to the numerical results, the current approaches are simple, efficient, and provide extremely high precision for getting approximate solutions to various nonlinear fractional physical differential equations.

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