Estimation of Rényi entropy of linear failure rate distribution based on Generalized Type- II Hybrid Censored Samples

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Abstract

Entropy is an essential term in statistical mechanics that was originally defined in the second law of thermodynamics. In this paper, we estimate the Rényi entropy measure of a linear failure rate distribution when data are generalized type-II hybrid censored. The estimations with the maximum likelihood are obtained. With regard to the linex loss function, the Bayes estimates are suggested. The Bayes estimates for closed-form formulations are unavailable. The methods of significance sampling and Tierney and Kadane's approximation are thus used. To demonstrate the suggested approaches, two real datasets based on a generalized type-II hybrid censored scheme have also been analyzed for illustrative purposes. Simulation studies to evaluate the performance of the estimates with various sample sizes are described. Additionally, many criteria are suggested for contrasting various sample plans, such as relative mean squared error and relative bias for various censored samples. The Bayes estimators of entropy are superior to the maximum likelihood.

Keywords: Bayes estimation, generalized type-II hybrid censored, linear failure rate distribution, Tierney and Kadane approximation.

1. Introduction

The two-parameter linear failure rate distribution (LFR) has been used quite successfully to analyze lifetime data. This distribution is also known as the linear exponential distribution, having exponential and Rayleigh distributions as special cases. It is a very well-known distribution for modeling lifetime data in reliability and medical studies. It is also used to model phenomena with increasing failure rate. The cumulative distribution function (CDF) and probability density function (pdf) of LFR distribution are given by

$$F(z) = 1 - e^{-\gamma z - \frac{\theta}{2} z^2} \quad z > 0, \gamma > 0, \theta > 0 \;, \qquad (1)$$

and

$$f(z) = (\gamma + \theta z)\, e^{-\gamma z - \frac{\theta}{2} z^2},\ z > 0, \gamma > 0, \theta > 0\ , \qquad (2)$$

respectively, where γ is scale parameter and θ is shape parameter. Note that if $\theta = 0$ and $\gamma \neq 0$, then the LFR distribution is refers to as exponential distribution with parameter γ and if $\gamma = 0$ and $\theta \neq 0$ then we can obtain the Rayleigh distribution with parameter θ.

In the recent past, many researchers have taken a keen interest in the measurement of uncertainty associated with a probability distribution. Of particular interest in probability and statistics is the notion of entropy. One of the first and most popular ways to estimate entropy is with Shannon's entropy. This measurement has shown to be successful in the research of communication networks. Let Z be a random variable with cumulative distribution function (cdf) $F(z)$, and probability density function (pdf) $f(z)$, then the Shannon's entropy H_z of the random variable Z is defined as:

$$H_Z = H(f) = -E[\ln f(z)] = \int_{-\infty}^{\infty} f(z) \log f(z) dz,$$

One of Shannon's measure's biggest drawbacks is that it could be negative for specific probability distributions, rendering it worthless as a measure of uncertainty. In order to create a new generalized entropy, Rényi researched the ideas of uncertainty and randomness. Rényi entropy is characterized by:

$$H_R(\alpha) = \frac{1}{1-\alpha} \log\left[\int_{-\infty}^{\infty} f^{\alpha}(z) dz\right],$$

where, $\alpha > 0$ and $\alpha \neq 1$

Many authors worked on the estimation entropy for different life distributions. Kayal (2015) studied a generalized residual entropy of record values and weighted distributions. Cho et al. in (2015) considered the estimation of the entropy of Weibull distribution based on the generalized progressively censored sample. Mahdy & Samir (2017) introduced the differential entropy and $\beta -$ entropy for Nakagami$-\mu$ distributions and their associated distributions were gained. Lee (2017) introduced the classical and Bayes estimation of the entropy of the inverse weibull distribution under under generalized progressive hybrid censored data. Lee (2020) suggested using the extended type I hybrid censoring approach to estimate the entropy of a Weibull distribution. He proposed symmetric and asymmetric loss function-based Bayes estimators for the Weibull distribution's entropy.

Mahmoud et al. (2021, a) introduced estimating the entropy of a Lomax distribution under Generalized Type-I hybrid censoring. In the generalized type-II hybrid censored sample, Ahmad (2021) proposed the Bayes estimator for the residual entropy of the Inverse Weibull distribution. Mahmoud et al. (2021, b) introduced the estimation of the entropy and residual entropy of a two parameter Lomax distribution under a generalized Type-II hybrid censoring scheme. Al-Babtain et al. (2021) discussed the maximum likelihood and Bayesian methods of estimation for dynamic cumulative residual Rényi entropy of Lomax distribution. Jose and Abdul Sathar (2022) introduced Rényi entropy as a function of the ordered random variable named k-records instead of the unordered random variable to derive some new characterizations of probability distributions. Shrahili et al. (2022) discussed estimation problem of certain entropy measures such as Rényi entropy, A-entropies, Arimoto entropy, Havrda and Charvat entropy and Tsallis entropy for log-logistic distribution under the progressive type II censoring (PT2C) scheme.

In a lifetime experiment, the researcher will probably stop the experiment before all of the items fail. This is due to the fact that the last failure's waiting period is unknown or that the study's subject matter can be costly. Due to these causes, the experiment is stopped before the last failure, and the data samples that result from this are known as censored samples. There are several censoring schemes. We say that we have a "type I censoring scheme" if the experiment is over at a fixed, predetermined time T. If the experiment is stopped at the r^{th} failure, it is said to have a "type II censorship system." A mixture of Type-I and Type-II censoring schemes is a hybrid censoring scheme. Type I hybrid censoring scheme is used when the experiment ends when either the pre-determined censoring period T or the pre-fixed number of failures (r) have occurred (Type-I HCS). We write $T = min\, X_{r:n}$, T to represent the time at which the experiment ends. A type II hybrid censoring strategy is used when the experiment ends after either the last of a set of pre-fixed failure numbers has failed or a set censoring time T is achieved (Type-II HCS). We write $T = max\, X_{r:n}$, T as the time at which the experiment ends. The prefixed time T is likely to occur in type I hybrid censoring before there are enough failure times to draw conclusions. On the other hand, type II hybrid censoring may take long time before we see the necessary number of failures. Chandrasekar et al. (2004) presented generalized type I and type II hybrid censoring methods to overcome these drawbacks.

This paper's goal is to introduce both the classical and Bayesian estimation of Rényi entropy of LFR distribution under (G-Type-II HCS) scheme. But as it seems out, the entropy's MLE cannot be derived in closed form. Therefore, we must simultaneously solve two nonlinear equations. Furthermore, using flexible priors, we obtain the Bayes estimation of the entropy. For the entropy of LLF, the Bayes estimators are obtained. We construct the Bayes estimates by discarding the Tierney and Kadane approximate method because the Bayes estimators cannot be produced in closed form. The remainder of this essay is structured as follows. The entropy of LFR distribution is estimated using both classical and Bayesian methods in section 2 using the (G-Type-II HCS) scheme. Two real datasets have been analyzed in Section 3. In Section 4, the description of different estimators that are compared by performing the Monte Carlo simulation is introduced.

2. Entropy Estimation

Consider the LFR distribution with pdf (2), survival function

$$S(z) = e^{-\gamma z - \frac{\theta}{2} z^2}, \quad z > 0. \tag{3}$$

Thus, the Rényi entropy function associated with the LFR distribution (2) can be obtained as:

$$H_R(\alpha) = \frac{1}{1-\alpha} \log \left[\int_0^\infty (\gamma + \theta z)^\alpha e^{(-\gamma z - \frac{\theta}{2} z^2)\alpha} dz \right].$$

After some calculations the Rényi entropy function for the linear failure rate model is

$$H_R(\alpha) = \frac{1}{1-\alpha} \left[\log \left(\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \gamma^{\alpha-j} \binom{\alpha}{j} \frac{\alpha^j \theta^{i-2}}{j!} \Gamma(\mathrm{i}+\mathrm{j}+1) \right) \right]. \quad (٤)$$

2.1 Maximum likelihood estimation

Consider a life-testing experiment in which n identical units are tested at time 0 on a life-test. The equivalent lifetimes from a distribution with cdf $F(x)$ and pdf $f(x)$ are denoted by $Z_1, Z_2, Z_3, \ldots Z_n$. As an example, a G-Type-II HCS is characterized as; fix time points T_1 and $T_2 \in (0, \infty)$ and an integer $r \in \{1,2,3,\ldots,n\}$ where $T_1 < T_2$. Terminate the experiment at time point T_1 if the r^{th} failure takes place before T_1. Terminate the experiment at the moment of the failure, $Z_{r:n}$, if the r^{th} failure happens between T_1 and T_2. Terminate the experiment at time T_2 if the r^{th} failure takes place after that point. Under such G-Type-II HCS, we will perform one of the following types of observations:

Case I: $\{Z_{1:n} < Z_{2:n} <\ldots< Z_{r:n} <\ldots< Z_{D_1} \le T_1\}$, $if\ \ Z_{r:n} < T_1$,

Case II : $\{Z_{1:n} < Z_{2:n} <\ldots< T_1 <\ldots< Z_{r:n}\}$, $if\ T_1 < Z_{r:n} < T_2$,

Case III: $\{Z_{1:n} < Z_{2:n} <\ldots< T_1 <\ldots< Z_{D_2} \le T_2\}$, $if\ \ Z_{r:n} < T_2$,

where, D_i denote the number of failures up to time T_i , $i \in (1,2)$. Then, the likelihood functions for the three various instances mentioned above under a generalized type-II hybrid censored sample are as follows:

Case I: $\frac{n!}{(n-r)!}\left(\prod_{i=1}^{D_1} f(z_{(i)})\right)(1-F(T_1))^{n-D_1}, D_1 = r, r+1, \ldots, n$,

Case II: $\frac{n!}{(n-r)!}\left(\prod_{i=1}^{r} f(z_{(i)})\right)(1-F(x_r))^{n-z_r}, D_2 = r$,

Case III: $\frac{n!}{(n-r)!}\left(\prod_{i=1}^{D_2} f(z_{(i)})\right)(1-F(T_2))^{n-D_2}$, $D_2 = 0, 1, \ldots, r-1$.

Assume n items with lifetime distribution that are i.i.d. a linear failure rate random variables with cdf (1) and pdf (2), put on a test. Assume also that we subject the experiment to G Type II HCS described in this section. Let D_1, D_2 denotes the number of failures that occur by time point T_1, T_2 respectively, then based on the G TypeII HCS, the likelihood functions of γ and θ are given by:

Case I:

$$L_I(\gamma,\theta) = \frac{n!}{(n-r)!}\left(\prod_{i=1}^{D_1}(\gamma+\theta z_i)\, e^{-\gamma z_i - \frac{\theta}{2}{z_i}^2}\right)\left(e^{-\gamma T_1 - \frac{\theta}{2}{T_1}^2}\right)^{n-D_1},$$

Case II:

$$L_{II}(\gamma,\theta) = \frac{n!}{(n-r)!}\left(\prod_{i=1}^{r}(\gamma+\theta z_i)\, e^{-\gamma z_i - \frac{\theta}{2}{z_i}^2}\right)\left(e^{-\gamma z_r - \frac{\theta}{2}{z_r}^2}\right)^{n-r},$$

Case III:

$$L_{III}(\gamma,\theta) = \frac{n!}{(n-r)!}\left(\prod_{i=1}^{D_2}(\gamma+\theta z_i)\, e^{-\gamma z_i - \frac{\theta}{2}{z_i}^2}\right)\left(e^{-\gamma T_2 - \frac{\theta}{2}{T_2}^2}\right)^{n-D_2}.$$

Furthermore, the following are the corresponding log likelihood functions:

Case I:

$$L_I(\gamma,\theta) = \sum_{i=1}^{D_1}\ln(\gamma+\theta z_i) + \sum_{i=1}^{D_1}\left(-\gamma z_i - \left(\frac{\theta}{2}\right) z_i^2\right) + (n-D_1)\left(-\gamma T_1 - \frac{\theta}{2}T_1^2\right),$$

Case II:

$L_{II}(\gamma,\theta) =$

$\sum_{i=1}^{r} \ln(\gamma+\theta z_i) + \sum_{i=1}^{r}(-\gamma z_i - \left(\frac{\theta}{2}\right) z_i^2) + (n-r)(-\gamma z_r - \frac{\theta}{2} {z_r}^2)$,

Case III:

$L_{III}(\gamma,\theta) =$

$\sum_{i=1}^{D_2} \ln(\gamma+\theta z_i) + \sum_{i=1}^{D_2} \left(-\gamma z_i - \left(\frac{\theta}{2}\right) z_i^2\right) + (n-D_2)\left(-\gamma T_2 - \frac{\theta}{2} T_2^2\right)$.

Therefore, Cases I, II and III can be combined and can be written as:

$L \ \ (\gamma,\theta) =$

$$\sum_{i=1}^{W} \ln(\gamma+\theta z_i) + \sum_{i=1}^{W} \left(-\gamma z_i - \left(\frac{\theta}{2}\right) z_i^2\right) + (n-W)\left(-\gamma K - \frac{\theta}{2} K^2\right), \ (٥)$$

where, $W = D_1$ and $K = T_1$ for Case I, $W = r$ and $K = z_r$ for Case II and $W = D_2$ and $K = T_2$ for Case III.

Taking derivatives with respect to γ and θ of (5)

$$\frac{\partial l}{\partial \gamma} = \sum_{i=1}^{W} \frac{1}{(\gamma+\theta z_i)} - \sum_{i=1}^{W} z_i - (n-W)K,$$

$$\frac{\partial l}{\partial \theta} = \sum_{i=1}^{W} \frac{z_i}{(\gamma+\theta z_i)} - \sum_{i=1}^{W} \frac{{z_i}^2}{2} - (n-W)\frac{K^2}{2},$$

We numerically solve these equations because they cannot be solved analytically to get the maximum likelihood estimates of $\hat{\gamma}$ and $\hat{\theta}$ for γ and θ respectively. The MLE of the Rényi entropy ($\hat{H}_{RMLE}$) is obtained after we have the MLE, $\hat{\gamma}$ and $\hat{\theta}$ as:

$$\hat{H}_{RMLE} = \frac{1}{1-\alpha}\left[\log\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \hat{\gamma}^{\alpha-j} \binom{\alpha}{j} \frac{\alpha^j \hat{\theta}^{i-2}}{j!} \Gamma(\mathrm{i}+\mathrm{j}+1)\right)\right].$$

2.2 Bayes estimation

In this section, the Bayes estimator for the Rényi entropy of the LFR distribution will be derived. To get the Bayes estimator of the entropy, we first specify the prior distributions of the scale (γ) and shape (θ) parameters before calculating their combined prior distribution. The next step is to determine the joint density of γ , θ and the random variable Z. The posterior distribution of θ, γ given Z, will then be obtained. Ultimately, Rényi entropy Bayes estimates will be obtained.

2.2.1 Prior and posterior distribution

In the experimental data, the Bayesian estimation needs the selection of suitable priors for the unknown parameters. We assume that γ and θ are independent unknown parameters in the case of an informative prior and that their corresponding joint prior distribution of γ and θ (see, Salem (1992)) is calculated as:

$$\pi(\gamma,\theta) = \gamma^{a_1-1}\theta^{a_2-1}e^{-b_1\gamma-b_2\theta},\ a_1 > 0, a_2 > 0, b_1 > 0, b_2 > 0$$

Then, the joint density of the γ, θ, and Z is

$$L(\gamma,\theta|Z) = \gamma^{a_1-1}\theta^{a_2-1}e^{-b_1\gamma-b_2\theta}(\gamma+\theta z)\ e^{-\gamma z-\frac{\theta}{2}z^2}.$$

To get the Bayes estimators for the entropy function of the LFR distribution under (G-Type-II HCS) scheme. We obtain estimator under LFF loss function defined as

$$LLF\colon L_l\big(\hat{\beta},\beta\big) = e^{c(\hat{\beta}-\beta)} - c\big(\hat{\beta}-\beta\big) - 1.$$

We can then obtain the Bayes estimator of Rényi entropy under LLF ($\widehat{H}_{RBL}$). It is derived as

$$\widehat{H}_{RBL} = -\frac{1}{c}\log\left[\frac{\int_0^\infty \exp\left\{\frac{-c}{1-\alpha}\left[\log\left(\sum_{i=0}^\infty\sum_{j=0}^\infty \hat{\gamma}^{\alpha-j}\binom{\alpha}{j}\frac{\alpha^j\hat{\theta}^{i-2}}{j!}\Gamma(i+j+1)\right)\right]\right\}L(\gamma,\theta|Z)\pi(\gamma,\theta)d\gamma d\theta}{\int_0^\infty\int_0^\infty L(\gamma,\theta|Z)\pi(\gamma,\theta)d\gamma d\theta}\right].$$

2.2.2 Tierney and Kadane approximation

In this subsection, we produced a Bayesian estimate of the LFR's entropy based on the (G-Type-II HCS). The LLF are used to build this Bayesian estimator. It is obvious that this estimate has the form of the ratio of two integrals, for which there exist no simplified closed forms. In order to approximate Bayesian estimator, we use the Tierney and Kadane approximation approach, see lee (20١٧). The following introduces this method with some details.

let G be smooth, positive function on the parameter space. As a result, the posterior expectation of G(γ,θ) is calculated:

$$\hat{G} = E(G(\gamma,\theta)|Z)) = \iint G(\gamma,\theta)\pi(\gamma,\theta|Z)d\gamma d\theta$$

$$= \frac{\iint e^{n\vartheta^*(\gamma,\theta)}d\gamma d\theta}{\iint e^{n\vartheta(\gamma,\theta)}d\gamma d\theta} ,$$

where

$$\vartheta(\gamma,\theta) = \frac{\log L(\gamma,\theta)+\log\pi(\gamma,\theta)}{n} \quad \text{and} \quad \vartheta^*(\gamma,\theta) = \vartheta(\gamma,\theta) + \frac{\log G(\gamma,\theta)}{n}.$$

The Bayes estimator for the (γ,θ) can be constructed as follows using the Tierney and Kadane approximation of G(γ,θ):

$$\hat{G} = \sqrt{\frac{|\Sigma^*|}{|\Sigma\ |}} e^{n\vartheta^*(\hat{\gamma}_{\vartheta^*},\hat{\theta}_{\vartheta^*})-n\vartheta(\hat{\gamma}_{\vartheta},\hat{\theta}_{\vartheta})},$$

where $(\hat{\gamma}_{\vartheta,}\hat{\theta}_{\vartheta})$ and $(\hat{\gamma}_{\vartheta^*},\hat{\theta}_{\vartheta^*})$ maximize the $\vartheta(\gamma,\theta)$ and $\vartheta^*(\gamma,\theta)$, respectively. $|\Sigma^*|$ and $|\Sigma\ \ |$ denote the minus of inverse of Hessians of the $\vartheta(\gamma,\theta)$ and $\vartheta^*(\gamma,\theta)$ at $\vartheta(\hat{\gamma}_{\vartheta,}\hat{\theta}_{\vartheta})$ and $(\hat{\gamma}_{\vartheta^*},\hat{\theta}_{\vartheta^*})$ respectively. In our issue, we note that:

$$\vartheta(\gamma,\theta) = \frac{1}{n}\Big[\sum_{i=1}^{W}\ln(\gamma+\theta z_i) + \sum_{i=1}^{W}\Big(-\gamma z_i - \Big(\frac{\theta}{2}\Big)z_i^2\Big) + (n - W)\Big(-\gamma K - \frac{\theta}{2}K^2\Big) + (a_1-1)\log\gamma + (a_2-1)\log\theta - (b_1\gamma + b_2\theta)\Big].$$

Hence, by resolving the following equations, $(\hat{\gamma}_{\vartheta,}\hat{\theta}_{\vartheta})$ is calculated.

$$\frac{\partial\vartheta(\gamma,\theta)}{\partial\gamma} = \frac{1}{\sum_{i=1}^{W}(\gamma+\theta z_i)} - \sum_{i=1}^{W} z_i - (n-W)K + \frac{a_1-1}{\gamma} - b_1 = 0,$$

and

$$\frac{\partial\vartheta(\gamma,\theta)}{\partial\theta} = \sum_{i=1}^{W}\frac{z_i}{(\gamma+\theta z_i)} - \frac{1}{2}\sum_{i=1}^{W} z_i - (n-W)\frac{K^2}{2} + \frac{a_2-1}{\theta} - b_2 = 0.$$

Furthermore, we calculate $|\Sigma|$ and it is given by

$$|\Sigma| = \left[\frac{\partial^2\vartheta(\gamma,\theta)}{\partial\gamma^2}\frac{\partial^2\vartheta(\gamma,\theta)}{\partial\theta^2} - \frac{\partial\vartheta^2(\gamma,\theta)}{\partial\gamma\partial\theta}\frac{\partial\vartheta^2(\gamma,\theta)}{\partial\theta\partial\gamma}\right]^{-1},$$

where

$$\frac{\partial^2\vartheta(\gamma,\theta)}{\partial\gamma^2} = \frac{1}{n}\left[\sum_{i=1}^{W}\frac{-1}{(\gamma+\theta z_i)^2} - \frac{a_1-1}{\gamma^2}\right], \quad \frac{\partial^2\vartheta(\gamma,\theta)}{\partial\theta^2} = \frac{1}{n}\left[\sum_{i=1}^{W}\frac{-z_i}{(\gamma+\theta z_i)^2} - \frac{a_2-1}{\theta^2}\right].$$

and

$$\frac{\partial\vartheta^2(\gamma,\theta)}{\partial\gamma\partial\theta} = \frac{1}{n}\left[\sum_{i=1}^{W}\frac{-z_i}{(\gamma+\theta z_i)^2}\right].$$

Bayes estimator of entropy function under LLF is calculated by taking into $g(\gamma,\theta) = e^{\frac{-\mathrm{c}}{(1-\alpha)}\left[\log\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\hat{\gamma}^{\alpha-j}\binom{\alpha}{j}\frac{\alpha^j\hat{\theta}^{i-2}}{j!}\Gamma(\mathrm{i}+\mathrm{j}+1)\right)\right]}$. After that, $\vartheta_L^*(\gamma,\theta)$ is obtained as

$$\vartheta_L^*(\gamma,\theta) = \vartheta(\gamma,\theta) - \frac{\mathrm{c}}{\mathrm{n}(1-\alpha)}\left[\log\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\hat{\gamma}^{\alpha-j}\binom{\alpha}{j}\frac{\alpha^j\hat{\theta}^{i-2}}{j!}\Gamma(\mathrm{i}+\mathrm{j}+1)\right)\right].$$

Currently, calculate the following equation

$$\frac{\partial\vartheta_L^*(\gamma,\theta)}{\partial\gamma} = \frac{\partial\vartheta(\gamma,\theta)}{\partial\gamma} - \frac{c(\alpha-i)}{n(1-\alpha)\gamma} = 0,$$

and

$$\frac{\partial\vartheta_L^*(\gamma,\theta)}{\partial\theta} = \frac{\partial\vartheta(\gamma,\theta)}{\partial\theta} - \frac{c(i-2)}{n(1-\alpha)\theta} = 0,$$

we get $\left(\gamma_{\vartheta_L^*},\theta_{\vartheta_L^*}\right)$. Next, we calculate $|\Sigma_L^*|$, and the result is given by

$$|\Sigma_L^*| = \left[\frac{\partial^2\vartheta_L^*(\gamma,\theta)}{\partial\gamma^2}\frac{\partial^2\vartheta_L^*(\gamma,\theta)}{\partial\theta^2} - \frac{\partial\vartheta_L^*(\gamma,\theta)}{\partial\gamma\partial\theta}\frac{\partial\vartheta_L^*(\gamma,\theta)}{\partial\theta\partial\gamma}\right]^{-1},$$

where

$$\frac{\partial^2\vartheta_L^*(\gamma,\theta)}{\partial\gamma^2} = \frac{\partial^2\vartheta(\gamma,\theta)}{\partial\gamma^2} + \frac{c(\alpha-i)}{n\gamma^2(1-\alpha)}, \quad \frac{\partial^2\vartheta_L^*(\gamma,\theta)}{\partial\theta^2} = \frac{\partial^2\vartheta(\gamma,\theta)}{\partial\theta^2} + \frac{c(i-2)}{n\theta^2(1-\alpha)}$$

and

$$\frac{\partial^2\vartheta_L^*(\gamma,\theta)}{\partial\gamma\partial\theta} = \frac{1}{n}\left[\sum_{i=1}^{W}\frac{-z_i}{(\gamma+\theta z_i)^2}\right].$$

Following that, the Bayes estimate of the entropy function under LLF is produced by

$$\hat{H}_{RBL} = -\frac{1}{c}\log\left[\sqrt{\frac{|\Sigma^*|}{|\Sigma\ |}}\, e^{n\vartheta_L^*\left(\gamma_{\vartheta_L^*},\theta_{\vartheta_L^*}\right)-n\vartheta(\hat{\gamma}_{\vartheta},\hat{\theta}_{\vartheta})}\right].$$

3. Illustrative examples

Two real data sets are investigated for illustrative purposes and to evaluate the statistical performances of the MLEs and Bayesian methods for Rényi entropy estimates in the case of the LFR distribution under G-Type-II HCS schemes.

3,1 First real data set

The first real life data set is from the data on the strengths of 1.5 cm glass fibres, measured at the National Physical Laboratory, England (Mahmoudi and Jafari, 2017). Mahmoudi and Jafari examined the goodness-of-fit of the previous data to the LFR distribution graphically and found that LFR distribution fits the data very well. The ordered data are:

0.39,	0.85	1.08,	1.25,	1.47,	1.57,	1.61,	1.61,	1.69,	1.8,
1.84,	1.87,	1.89,	2.03,	2.03,	2.05,	2.12,	2.35,	2.41,	2.43,
2.48,	2.5,	2.53,	2.55,	2.55,	2.56,	2.59,	2.67,	2.73,	2.74,
2.79,	2.81,	2.82,	2.85,	2.87,	2.88,	2.93,	2.95,	2.96,	2.97,
3.09,	3.11,	3.11,	3.15,	3.15,	3.19,	3.22,	3.22,	3.27,	3.28,
3.31,	3.31,	3.33,	3.39,	3.39,	3.56,	3.6,	3.65,	3.68,	3.7,
3.75,	4.2,	4.38,	4.42,	4.7	4.9				

We considered applying G-Type-II HCS to this data. We consider case I ($T_1 = 2, T_2 = 2.5$ and $\mathrm{r} = 10$), case II ($T_1 = 3, T_2 = 4,$ and $\mathrm{r} = 50$), and case III ($T_1 = 3, T_2 = 3.5$ and $\mathrm{r} = 60$). Table 1 presents the estimation of the entropy of the G-Type-II HCS. The Bayesian estimates based on the LLF with c = 2 is also included. We found that the Bayesian estimates of entropy (H_{RBL}) using Tierney and Kadane approximation produced under the LLF are somewhat less than the corresponding MLE of entropy (H_{RMLE}). Furthermore, we observed that Bayes estimates are superior to the respective MLE in terms of relative bias (RBias) and relative mean square error (RMSE) values.

Table1. Estimates of entropy for first real data set

	T_1	T_2	r	$\hat{H}_{RMLE}$	RBais $\hat{H}_{RMLE}$	RMSE $\hat{H}_{RMLE}$	$\hat{H}_{RBL}$	RBais $\hat{H}_{RBL}$	RMSE $\hat{H}_{RBL}$
Case I	2	2.5	10	11.795	0.4267	0.0538	11.3628	0.1562	0.0197
Case II	3	4	50	11.032	0.2173	0.0274	11.2873	0.0450	0.0057
Case III	3	3.5	60	11.090	0.2132	0.0269	11.3443	0.0404	0.0051

3.2 Second real data set

We examine the second real data set using the proposed estimators from Section 2. We use a set of data supplied by W.B. Nelson in 1970. The data set describes the outcomes of a life test experiment in which patterns of a sort of electrical insulating fluid were treated to a continual voltage shock. The time in minutes it took for each unit to fail was as follows: 0.27, 0.4, 0.69, 0.79, 2.75, 3.91, 9.88, 13.95, 15.93, 27.8, 53.24, 82.85, 89.29, 100.58, and 215.1. We investigated using G-type-II HCS on this data. We will consider the case I ($T_1 = 4, T_2 = 15$ and $r = 5$), case II ($T_1 = 3, T_2 = 30$, and $r = 9$) and case III ($T_1 = 3, T_2 = 60$ and $r = 12$). Table2 presents the estimation of the entropy under the G-Type-II HCS. The Bayesian estimates based on the LLF with c = 2 is also included. We discovered that the H_{RBL} created under the LLF using the Tierney and Kadane approach are less than the comparable H_{RMLE}. In terms of RMSE and RBais values, we also found that Bayes estimates are superior to the corresponding MLE.

Table2. Estimates of entropy for second real data set

	T_1	T_2	r	$\hat{H}_{RMLE}$	RBais $\hat{H}_{RMLE}$	RMSE $\hat{H}_{RMLE}$	$\hat{H}_{RBL}$	RBais $\hat{H}_{RBL}$	RMSE $\hat{H}_{RBL}$
Case I	4	15	5	8.7365	0.3802	0.0982	8.5643	0.2597	0.0671
Case II	3	30	9	8.5149	0.7188	0.1856	8.4944	0.2090	0.0540
Case III	3	60	12	8.2328	0.7855	0.2028	8.1366	0.2706	0.0699

4. Simulation

In this section, we evaluate how well the entropy estimations based on simulated data under GHCS Type II performed. Different sample sizes, linear failure rate distribution parameter values, and time point T_2 were included in the simulation, all of which used the identical T_1 value. Each time, the procedure is repeated 10000 times. Newton-Raphson is used to compute the corresponding MLEs. With regard to the diffuse prior distribution ($a_1 = a_2 = b_1 = b_2 = 0.0001$), all Bayes estimates are computed by using the Mathematica ® 13 software. Entropy estimates based on Bayes are produced for LLF. Additionally, Bayes estimates' approximate closed forms have been derived using the Tierney and Kadane approximation approach. Bayes estimates for c = 2 is produced under LLF. Additionally, several methods have been considered for calculating RBias and RMSE of each estimate. And tables 3-10 present these findings in tabular form. On the basis of the RMSEs and RBias, we provide discussions. In tables 3-10, RMSE and RBias values of all estimates of entropy are presented for various choices of T_{1}, T_2, n and generalized type-II hybrid censoring schemes. We have tabulated RMSE and RBias values of the respective MLE in the sixth and seventh columns of the table. The last two columns correspond to the RBias and RMSE of entropy using Lindley's approximation based on informative prior. Bayes estimates based on informative prior under the LLF function. In general, we observed that the RMSE values decrease as the sample size n increases. For a fixed sample size, the RMSE values decrease generally as the number of generalized type-II hybrid censored samples T_1 increases. For a fixed $\gamma, \theta, \alpha, r$ and T_1, it seems that the RBias values increase as the stopping time T_2 increase. The RBais and RMSE values of the entropy estimates a $\gamma = .3$, $\theta = 0.5$ and $\alpha = 10$ have the smallest values among other values use. For a fixed θ the RBias values decrease in general as the shape parameter γ increase. For a fixed $\gamma, \theta, \alpha, r$ and T_1 the RBias values decrease in general as the parameter α increase Furthermore, we observed that Bayes estimates are superior to the respective MLE in terms of RMSE and RBias values. In particular, respective Bayes estimates under LLF of entropy are better than the corresponding MLE. For estimating the entropy, the choice c = 2 seems to be a reasonable choice under LLF.

Table 3. The relative MSEs and biases of entropy estimators with MLE and the Bayes for selected values $\gamma = .1, \theta = .5, \alpha = 5, r = 50, T_1 = 0.7\ and\ T_2$

$T_1 = 0.7$								
					$\widehat{H}_{RMLE}$		$\widehat{H}_{RBL}$	
n	T_2	γ	θ	α	RBais	RMSE	RBais	RMSE
	1.5	.1	.5	5	0.671715	0.094995	0.555489	0.078558
50	1.7				0.771525	0.10911	0.508473	0.071909
	2				0.91317	0.129142	0.473349	0.066942
	1.5				0.61524	0.06152	0.518146	0.05181
100	1.7				0.701571	0.07016	0.480699	0.04807
	2				0.872209	0.087221	0.451245	0.04512
	1.5				0.574175	0.0468812	0.339833	0.0277473
150	1.7				0.514245	0.0419879	0.324757	0.0265163
	2				0.680005	0.0555222	0.344319	0.0281135
	1.5				0.547321	0.0419776	0.330066	0.0253149
170	1.7				0.481571	0.0369348	0.320637	0.0245918
	2				0.641891	0.0492308	0.32562	0.0249739
	1.5				0.539	0.0381131	0.323529	0.0228769
200	1.7				0.459175	0.0324686	0.316863	0.0224056
	2				0.626041	0.0442678	0.319178	0.0225693

Table 4. The relative MSEs and biases of entropy estimators with MLE and the Bayes for selected values $\gamma = .1, \theta = .5, \alpha = 10, r = 50, T_1 = 0.7\ and\ T_2$

$T_1 = 0.7$								
					$\widehat{H}_{RMLE}$		$\widehat{H}_{RBL}$	
n	T_2	γ	θ	α	RBais	RMSE	RBais	RMSE
	1.5	.1	.5	10	0.54742	0.077416	0.503773	0.0712443
50	1.7				0.566	0.080046	0.464712	0.0657202
	2				0.684683	0.096829	0.441924	0.0669417
	1.5				0.49538	0.04954	0.404318	0.040432
100	1.7				0.562541	0.05625	0.421152	0.042115
	2				0.680649	0.06806	0.422229	0.04222
	1.5				0.471643	0.0385095	0.332712	0.0271658
150	1.7				0.488831	0.0399129	0.308559	0.0251937
	2				0.671634	0.0548387	0.330088	0.0269516
	1.5				0.395966	0.0303692	0.327673	0.0251314
170	1.7				0.423242	0.0324612	0.307622	0.0235935
	2				0.66542	0.0510216	0.323413	0.0248046
	1.5				0.334379	0.0236442	0.322718	0.0228196
200	1.7				0.393901	0.07853	0.304206	0.0215106
	2				0.656436	0.046417	0.294766	0.0210675

Table 5. The relative MSEs and biases of entropy estimators with MLE and the Bayes for selected values $\gamma = .3, \theta = .5, \alpha = 5, r = 50, T_1 = 0.7\ and\ T_2$

$T_1 = 0.7$								
					$\widehat{H}_{RMLE}$		$\widehat{H}_{RBL}$	
n	T_2	γ	θ	α	RBais	RMSE	RBais	RMSE
	1.5	0.3	0.5	5	0.549166	0.077664	0.426213	0.0602756
50	1.7				0.604201	0.0854469	0.489442	0.0692175
	2				0.717828	0.101516	0.484953	0.0544406
	1.5				0.500019	0.05	0.256533	0.025653
100	1.7				0.584096	0.05840	0.245078	0.0508
	2				0.7075	0.07076	0.244217	0.024421
	1.5				0.461982	0.0377207	0.252364	0.0206054
150	1.7				0.536624	0.0438152	0.227127	0.0185449
	2				0.695026	0.0567487	0.234135	0.019117
	1.5				0.443884	0.00340444	0.243685	0.0196898
170	1.7				0.412448	0.0316334	0.216564	0.0166097
	2				0.688577	0.0528114	0.219268	0.0168171
	1.5				0.421927	0.0298348	0.226253	0.0159985
200	1.7				0.374635	0.0264907	0.210029	0.0148513
	2				0.68712	0.0485867	0.215591	0.0152446

Table 6. The relative MSEs and biases of entropy estimators with MLE and the Bayes for selected values $\gamma = .3, \theta = .5, \alpha = 10, r = 50, T_1 = 0.7\ and\ T_2$

$T_1 = 0.7$								
					$\widehat{H}_{RMLE}$		$\widehat{H}_{RBL}$	
n	T_2	γ	θ	α	RBais	RMSE	RBais	RMSE
	1.5	0.3	0.5	10	0.499118	0.070586	0.324451	0.0458842
50	1.7				0.488872	0.069137	0.329008	0.0465288
	2				0.609661	0.086210	0.343258	0.048544
	1.5				0.442٩89	0.04430	0.29967	0.02997
100	1.7				0.443379	0.04434	0.25569	0.02557
	2				0.603416	0.06034	0.244719	0.024472
	1.5				0.442609	0.0361389	0.244348	0.0199509
150	1.7				0.430237	0.0351287	0.226709	0.0185107
	2				0.589254	0.0481124	0.22169	0.0181009
	1.5				0.32068	0.024595	0.24321	0.019858
170	1.7				0.407714	0.312702	0.215952	0.0165628
	2				0.583811	0.0447763	0.213095	0.0163436
	1.5				0.271615	0.0192061	0.222424	0.0157278
200	1.7				0.374935	0.0265119	0.209018	0.0147798
	2				0.580172	0.0410244	0.201625	0.0149641

Table 7. The relative MSEs and biases of entropy estimators with MLE and the Bayes for selected values $\gamma = .1, \theta = .5, \alpha = 5$, $r = 50, T_1 = 1$ and T_2

$T_1 = 1$								
					$\hat{H}_{RMLE}$		$\hat{H}_{RBL}$	
n	T_2	γ	θ	α	RBais	RMSE	RBais	RMSE
	1.5	.1	.5	5	0.598877	0.0846939	0.521628	0.0737694
50	1.7				0.538437	0.0747323	0.496858	0.0702664
	2				0.746535	0.105576	0.4892	0.0691833
	1.5				0.563221	0.05632	0.456885	0.045689
100	1.7				0.521315	0.05213	0.457632	0.045763
	2				٠.666184	0.06662	0.485116	0.048512
	1.5				0.420875	0.0424748	0.294052	0.024007
150	1.7				0.471728	0.0385165	0.2293821	0.0239904
	2				0.646781	0.0528094	0.292141	0.0238532
	1.5				0.407634	0.0312641	0.292792	0.0224561
170	1.7				0.463624	0.0355583	0.283749	0.0217625
	2				0.626429	0.0480449	0.272462	0.0208969
	1.5				0.403575	0.0285371	0.278625	0.0197017
200	1.7				0.455137	0.0321831	0.282282	0.0199604
	2				0.621807	0.0439684	0.26098	0.0184541

Table 8. The relative MSEs and biases of entropy estimators with MLE and the Bayes for selected values $\gamma = .1, \theta = .5, \alpha = 10$, $r = 50, T_1 = 1$ and T_2

$T_1 = 1$								
					$\hat{H}_{RMLE}$		$\hat{H}_{RBL}$	
n	T_2	γ	θ	α	RBais	RMSE	RBais	RMSE
	1.5	.1	.5	10	0.523968	0.0741003	0.416594	0.0589158
50	1.7				0.504615	0.0713633	0.40025	0.0566039
	2				0.629359	0.0890049	0.409587	0.040958
	1.5				0.506638	0.050664	0.379869	0.037987
100	1.7				0.50196	0.05120	0.370373	0.037037
	2				0.597368	0.05974	0.383128	0.0541825
	1.5				0.404107	0.0397573	0.266653	0.0220415
150	1.7				0.454493	0.0424748	0.27899	0.022445
	2				0.592771	0.0542541	0.2664	0.0217515
	1.5				0.391179	0.030004	0.25894	0.0198598
170	1.7				0.434325	0.0333112	0.261327	0.0200444
	2				0.58601	0.0414371	0.264289	0.0202701
	1.5				0.36303	0.0256701	0.233427	0.0165058
200	1.7				0.398343	0.0281671	0.257063	0.0181771
	2				0.584825	0.0448541	0.238611	0.168723

Table 9. The relative MSEs and biases of entropy estimators with MLE and the Bayes for selected values $\gamma = .3, \theta = .5, \alpha = 5$, $r = 50, T_1 = 1$ and T_2

$T_1 = 1$								
					$\widehat{H}_{RMLE}$		$\widehat{H}_{RBL}$	
n	T_2	γ	θ	α	RBais	RMSE	RBais	RMSE
50	1.5	.3	.5	5	0.418915	0.0592436	0.408354	0.05775
	1.7				0.465801	0.0658742	0.451322	0.063827
	2				0.606256	0.0857375	0.373135	0.052769
100	1.5				0.364173	0.36417	0.216643	0.02166
	1.7				0.436948	0.04370	0.203199	0.023120
	2				0.56876	0.05688	0.21606	0.02161
150	1.5				0.355251	0.0290061	0.184074	0.0150296
	1.7				0.398832	0.0328645	0.169076	0.013805
	2				0.542827	0.0443217	0.176790	0.0144282
170	1.5				0.298528	0.0228961	0.173913	0.0133385
	1.7				0.440651	0.0337964	0.167519	0.0128481
	2				0.521007	0.0399594	0.176521	0.0135385
200	1.5				0.30085	0.0212733	0.173072	0.012238
	1.7				0.405022	0.0286394	0.156731	0.0110825
	2				0.508072	0.035923	0.175798	0.0124308

Table 10. The relative MSEs and biases of entropy estimators with MLE and the Bayes for selected values $\gamma = .3, \theta = .5, \alpha = 10$, $r = 50, T_1 = 1$ and T_2

$T_1 = 1$								
					$\widehat{H}_{RMLE}$		$\widehat{H}_{RBL}$	
n	T_2	γ	θ	α	RBais	RMSE	RBais	RMSE
150	1.5	.3	.5	10	0.384587	0.054389	0.317785	0.0449416
	1.7				0.412124	0.058283	0.325124	0.0459795
	2				0.423217	0.059852	0.337523	0.07733
170	1.5				0.371681	0.0372	0.204021	0.02040
	1.7				0.336272	0.03363	0.199346	0.01993
	2				0.365497	0.03655	0.205028	0.020503
150	1.5				0.232351	0.0189714	0.193227	0.0157769
	1.7				0.28256	0.0240709	0.18937	0.015462
	2				0.364087	0.0297276	0.196322	0.0160296
170	1.5				0.223971	0.0171778	0.181869	0.0139487
	1.7				0.270431	0.0207411	0.183132	0.0140456
	2				0.354772	0.0271945	0.019209	0.0147327
200	1.5				0.196629	0.0139038	0.179749	0.0127102
	1.7				0.26903	0.0184588	0.180314	0.0127501
	2				0.353442	0.0249922	0.186948	0.0132192

5. Conclusion

In this article, we take into account both the traditional and Bayesian estimations of an LFR's entropy under the G-Type-II HCS scheme. We use a flexible prior-based Bayes estimation of the entropy. We construct the Bayes estimates by discarding the Tierney and Kadane approximate method under LLF because the Bayes estimators cannot be produced in closed form. Moreover, simulation studies were conducted to evaluate the impact of various censoring parameter selections on the entropy estimations. In terms of RMSEs and RBias, the Bayes estimators of entropy are preferable to the MLE. For Bayes estimation of entropy, LLF appears to be a logical choice.

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تقدير إنتروبي Rényi لتوزيع معدل الفشل الخطي على أساس العينات المعممة من النوع الثاني الهجين الخاضعة للرقابة.

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الملخص

الانتروبي هو مصطلح أساسي في الإحصاء الذي تم تعريفه في الأصل في القانون الثاني للديناميكا الحرارية. في هذا البحث، قمنا بتقدير مقياس انتروبي Rényi لتوزيع معدل الفشل الخطي عندما تكون البيانات معممة من النوع الثاني الهجين. حيث يتم الحصول على التقديرات لطريقة الامكان الأكبر. وبالاعتماد علي دالة خسارة Linex وأيضا تقريب Tierney and Kadane، تم اقتراح تقديرات طريقة بايز. ولتوضيح هذه الأساليب المقترحة تم تحليل مجموعتين من البيانات الحقيقية وأيضا تم عمل دراسة المحاكاة لتقييم أداء التقديرات بأحجام عينات مختلفة. بالإضافة إلى ذلك، تم اقتراح العديد من المعايير لمقارنة أحجام العينات المختلفة وذلك مثل معيار متوسط الخطأ التربيعي النسبي ومعيار التحيز النسبي للعينات الخاضعة للرقابة المختلفة. حيث تبين من خلال التطبيق على البيانات الحقيقية ودراسة المحاكاة أن تقديرات بايز للإنتروبي تفوق طريقة الامكان الأكبر.

الكلمات الافتتاحية

تقدير بايز، البيانات المعممة من النوع الثاني الهجين الخاضعة للرقابة، توزيع معدل الفشل الخطي، تقريب تيرني وكاداني