Nile Journal of Basic Science

Volume 3, Number 1, October 2023

Journal Webpage: https://njbs.journals.ekb.eg

Solution of Nonlinear Fractional Differential Equations containing Caputo Derivative

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Abstract

In this paper, Adomian decomposition method (ADM) will apply to solve nonlinear fractional differential equations (FDEs) of Caputo sense. These type of equations is very important in engineering applications. Existence and uniqueness of the solution will prove. The convergence of the series solution and the error analysis will discuss. Some numerical examples will solve to test the validity of the method and the given theorems.

Keywords: *Fractional Differential Equation; Adomian Method; Existence; Uniqueness; Error Analysis.*

1. Introduction

Fractional Differential equations (FDEs) have many applications in engineering and science [1-6], including electrical networks, fluid flow, control theory, fractals theory, electromagnetic theory, viscoelasticity, potential theory, chemistry, biology, optical and neural network systems [7-13]. In this paper, Adomian decomposition method (ADM) [14-19] will use to solve nonlinear FDEs of Caputo sense. This method has many advantages; it is efficiently works with different types of linear and nonlinear equations in deterministic or stochastic fields and gives an analytic solution for all these types of equations without linearization or discretization [20-23]. The paper will organize as follows: In section two ADM will apply to the problem under consideration. In section three uniqueness, convergence and error analysis will discuss. Finally, two numerical examples present by using MATHEMATICA package.

2. Formulation of the Problem

Consider the nonlinear FDE,

$$ {}_{0}\mathrm{D}_{t}^{\alpha} y(t) + a(t) f\big(y(t)\big) = x(t), \quad \mathrm{n}-1 \leq \alpha \leq \mathrm{n}, \tag{1} $$

Subject to the initial conditions,

$$y^{(j-1)}(0) = c_j \quad , j = 1,2,\dots,n. \tag{2}$$

The fractional derivative is of Caputo sense. In the applications, the Caputo sense are preferred to use because the initial conditions of $y(t)$ and its derivatives will be of integer orders and has a physical meaning. Now performing the fractional integration of order α this reduces the problem (1)-(2) to the fractional integral equation (FIE):

$$y(t) = \frac{c_j}{\Gamma(\alpha)} t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} x(\tau) d\tau - \frac{1}{\Gamma(\alpha)} \int_0^t a(\tau)(t-\tau)^{\alpha-1} f(y(\tau)) d\tau \tag{3}$$

Assume that $x(t)$ bounded $\forall\, t \in J = [0,T], T \in R^+$, $|a(\tau)| \le M \;\forall\; 0 \le \tau \le t \le T, M$ is a finite constant and $f(y)$ is Lipschitz continuous with Lipschitz constant L such as,

$$|f(y) - f(z)| \le L|y - z| \tag{4}$$

Which has Adomian polynomials representation,

$$f(y) = \sum_{n=0}^{\infty} A_n\,(y_0, y_1, \dots, y_n) \tag{5}$$

Where

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[f\left(\sum_{i=0}^{\infty} \lambda^i y_i \right) \right]_{\lambda=0} \tag{6}$$

Substitute from equation (5) into equation (3) we get,

$$y(t) = \frac{c_j}{\Gamma(\alpha-1)} t^{\alpha} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} x(\tau) d\tau - \frac{1}{\Gamma(\alpha)} \int_0^t a(\tau)(t-\tau)^{\alpha-1} \sum_{n=0}^{\infty} A_n\, d\tau \tag{7}$$

Let $y(t) = \sum_{n=0}^{\infty} y_n(t)$ in (7) and applying ADM, we get the following recursive relations,

$$y_0(t) = \frac{c_j}{\Gamma(\alpha-1)} t^{\alpha} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} x(\tau) d\tau, \tag{8}$$

$$y_i(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t a(\tau)(t-\tau)^{\alpha-1} A_{i-1} d\tau, \; i \ge 1. \tag{9}$$

Finally, the solution is,

$$y(t) = \sum_{i=0}^{\infty} y_i(t) \tag{10}$$

2.1. Existence and Uniqueness

Theorem 1: If $0 < \alpha < 1$ where $\alpha = \frac{LMT^{\alpha}}{\Gamma(\alpha+1)}$, then the series (10) is the solution of the problem (1)-(2) and this solution is unique.

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Proof For existence,

$$y(t) = \sum_{i=0}^{\infty} y_i(t)$$

$$= y_0(t) + \sum_{i=1}^{\infty} y_i(t)$$

$$= y_0(t) - \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{\infty} \int_0^t a(\tau)(t-\tau)^{\alpha-1} A_{i-1} d\tau$$

$$= y_0(t) - \frac{1}{\Gamma(\alpha)} \int_0^t a(\tau)(t-\tau)^{\alpha-1} \sum_{i=1}^{\infty} A_{i-1} d\tau$$

$$= y_0(t) - \frac{1}{\Gamma(\alpha)} \int_0^t a(\tau)(t-\tau)^{\alpha-1} \sum_{i=0}^{\infty} A_i d\tau$$

$$= \frac{c_j}{\Gamma(\alpha-1)} t^{\alpha} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} x(\tau) d\tau - \frac{1}{\Gamma(\alpha)} \int_0^t a(\tau)(t-\tau)^{\alpha-1} f(y(\tau)) d\tau$$

Then the Adomian's series solution satisfy equation (3) which is the equivalent FIE to the problem (1)-(2).

For uniqueness of the solution: Assume that y and z are two different solutions to the problem (1)-(2) and hence,

$$|y-z| = \left| \frac{1}{\Gamma(\alpha)} \int_0^t a(\tau)(t-\tau)^{\alpha-1} [f(y) - f(z)] d\tau \right|$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} |a(\tau)| |f(y) - f(z)| d\tau$$

$$\leq \frac{LM}{\Gamma(\alpha)} |y-z| \int_0^t (t-\tau)^{\alpha-1} d\tau$$

$$\leq \frac{LMT^{\alpha}}{\Gamma(\alpha+1)} |y-z|$$

Let $\frac{LMT^{\alpha}}{\Gamma(\alpha+1)} = \beta$ where, $0 < \beta < 1$ then,

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$$|y-z| \le \beta\,|y-z|$$
$$(1-\beta)\,|y-z| \le 0$$

However, $(1-\beta)\,|y-z| \ge 0$ and since, $(1-\beta) \ne 0$ then, $|y-z| = 0$ this imply that, $y = z$ and this completes the proof.

2.2. Proof of Convergence

Theorem 2: The series solution (10) of the problem (1)-(2) using ADM converges if $|y_1| < \infty$ and $0 < \beta < 1$, $\beta = \frac{LMT^{\alpha}}{\Gamma(\alpha+1)}$.

Proof Define the Banach space $(C[J], \|\cdot\|)$, the space of all continuous functions on J with the norm $\|y(t)\| = \max_{t\in J}|y(t)|$. Define the sequence $\{S_n\}$ such that, $S_n = \sum_{i=0}^{n} y_i\,(t)$ the sequence of partial sums from the series solution $\sum_{i=0}^{\infty} y_i\,(t)$ since,

$$f(y) = f\left(\sum_{i=0}^{\infty} y_i\,(t)\right) = \sum_{i=0}^{\infty} A_i\,(y_0, y_1, \dots, y_i)$$

So,

$$f(y_0) = f(S_0) = A_0,$$
$$f(y_0+y_1) = f(S_1) = A_0 + A_1,$$
$$f(y_0+y_1+y_2) = f(S_2) = A_0 + A_1 + A_2,$$
$$\vdots$$
$$f(S_n) = \sum_{i=0}^{n} A_i\,(y_0, y_1, \dots, y_i).$$

Let, S_n and S_m be two arbitrary partial sums with $n \ge m$. Now, we are going to prove that $\{S_n\}$ is a Cauchy sequence in this Banach space.

$$\|S_n - S_m\| = \max_{t\in J}|S_n - S_m| = \max_{t\in J}\left|\sum_{i=m+1}^{n} y_i\,(t)\right|$$

$$= \max_{t\in J}\left|\sum_{i=m+1}^{n} -\frac{1}{\Gamma(\alpha)}\int_0^t a\,(\tau)(t-\tau)^{\alpha-1}A_{i-1}d\tau\right|$$

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$$= \max_{t\in J}\left|\frac{1}{\Gamma(\alpha)}\int_0^t a\,(\tau)(t-\tau)^{\alpha-1}\sum_{i=m}^{n-1}A_i\,d\tau\right|$$

$$= \max_{t\in J}\left|\frac{1}{\Gamma(\alpha)}\int_0^t a\,(\tau)(t-\tau)^{\alpha-1}[f(S_{n-1})-f(S_{m-1})]d\tau\right|$$

$$\leq \frac{1}{\Gamma(\alpha)}\max_{t\in J}\int_0^t (t-\tau)^{\alpha-1}|a(\tau)||f(S_{n-1})-f(S_{m-1})|d\tau$$

$$\leq \frac{LM}{\Gamma(\alpha)}\max_{t\in J}|S_{n-1}-S_{m-1}|\int_0^t (t-\tau)^{\alpha-1}d\tau$$

$$\leq \frac{LMT^{\alpha}}{\Gamma(\alpha+1)}\|S_{n-1}-S_{m-1}\|$$

$$\leq \beta\|S_{n-1}-S_{m-1}\|$$

Let $n = m + 1$ then,

$$\|S_{m+1}-S_m\| \leq \beta\|S_m-S_{m-1}\| \leq \beta^2\|S_{m-1}-S_{m-2}\| \leq \cdots \leq \beta^m\|S_1-S_0\|$$

From the triangle inequality we have,

$$\|S_n-S_m\| \leq \|S_{m+1}-S_m\| + \|S_{m+2}-S_{m+1}\| + \cdots + \|S_n-S_{n-1}\|$$

$$\leq [\beta^m+\beta^{m+1}+\cdots+\beta^{n-1}]\|S_1-S_0\|$$

$$\leq \beta^m[1+\beta+\cdots+\beta^{n-m-1}]\|S_1-S_0\|$$

$$\leq \beta^m\left[\frac{1-\beta^{n-m}}{1-\beta}\right]\|y_1(t)\|$$

Since, $0<\alpha<1$ and $n \geq m$ then, $(1-\beta^{n-m}) \leq 1$. Consequently,

$$\|S_n-S_m\| \;\leq\; \frac{\beta^m}{1-\beta}\|y_1(t)\| \;\leq\; \frac{\beta^m}{1-\beta}\max_{t\in J}|y_1(t)|$$

However, $|y_1(t)| \leq \infty$ and as $m \to \infty$ then, $\|S_n-S_m\| \to 0$ and hence, $\{S_n\}$ is a Cauchy sequence in this Banach space so, the series $\sum_{n=0}^{\infty} y_n\,(t)$ converges and the proof is complete.

2.3. Error Analysis

For ADM, we can estimate the maximum absolute truncated error of the Adomian's series solution in the following theorem.

Theorem 3: The maximum absolute truncation error of the series solution (10) to the

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problem (1)-(2) estimate to be,

$$\max_{t\in J}\left|y(t)-\sum_{i=0}^{m}y_i\,(t)\right|\le\frac{\beta^m}{1-\beta}\max_{t\in J}|y_1(t)|$$

Proof. From Theorem 2 we have

$$\|S_n-S_m\|\ \le\ \frac{\beta^m}{1-\beta}\max_{t\in J}|y_1(t)|$$

But, $S_n=\sum_{i=0}^{n}y_i\,(t)$ as $n\to\infty$ then, $S_n\to y(t)$ so,

$$\|y(t)-S_m\|\ \le\ \frac{\beta^m}{1-\beta}\max_{t\in J}|y_1(t)|$$

Therefore, the maximum absolute truncation error in the interval J is,

$$\max_{t\in J}\left|y(t)-\sum_{i=0}^{m}y_i\,(t)\right|\le\frac{\beta^m}{1-\beta}\max_{t\in J}|y_1(t)|$$

Moreover, this completes the proof.

3. Numerical Examples

Example 1. Consider the initial value problem [20],

$$D^{0.5}y=y^2+1, \qquad (11)$$

$$\mathrm{y}(0)=0. \qquad (12)$$

Operating with $J^{0.5}$ on both sides of equation (11) and using the initial conditions (12) we obtain,

$$y(t)=J^{0.5}[1]+J^{0.5}[y^2], \qquad (13)$$

Using ADM and replace the nonlinear term $f(y)=y^2$ by its corresponding Adomian polynomials we have,

$$y_0=J^{0.5}[1], \qquad (14)$$

$$y_n=J^{0.5}[A_{n-1}],n\ge 1. \qquad (15)$$

From the two relations (14) and (15), the ADM series solution is,

$$\mathrm{y(t)}=\sum_{k=0}^{5}y_i(t), \qquad (16)$$

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Figures 1.a-1.b illustrate the comparison between ADM solution ($n = 5$) and the numerical solution using the numerical method given in [2] ($h = 0.01$). The numerical method gives unbounded solution when $t \in [0,1]$, see Figure 1.a, while, ADM gives a bounded solution in the same interval, see Figure 1.b.

Notices:

1. All computations and figures made using MATHEMATICA software for all the given examples.
2. In all figures, the solid curve represents ADM solution, while the other curve for the other method.

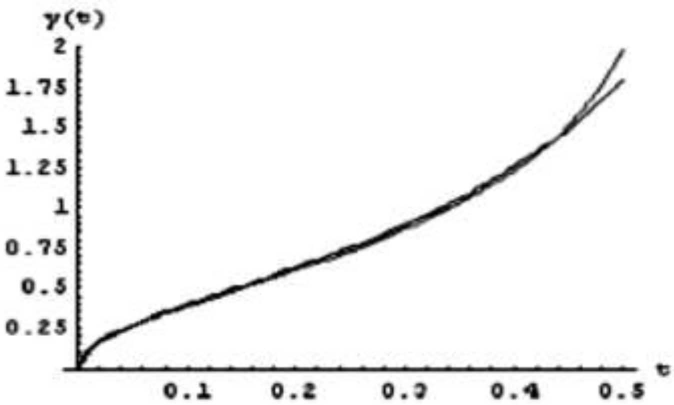

Figure 1.a: ADM and Numerical Sol. [$\mu = 0.5$]

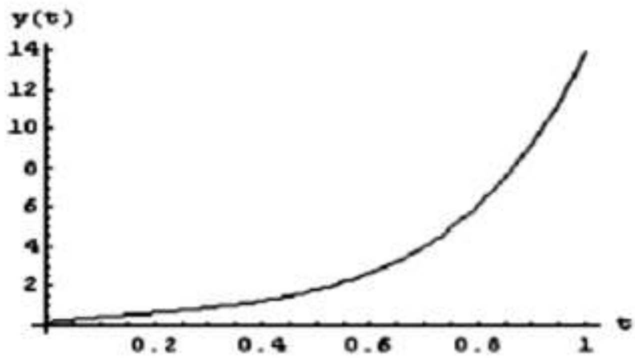

Figure 1.b: ADM Sol. [$\mu = 0.5$]

Example 2. Consider the following nonlinear FDE with nonhomogeneous initial conditions,

$$D^{1.5}y = \frac{9}{4}\sqrt{y} + y, t \geq 0, \tag{17}$$

$$y(0) = 1, y'(0) = 2.$$

This problem was solved by Nabil Shawagfeh in [20] by using ADM but the given solution was incorrect. Here, we give the correct solution.

Operating with $J^{1.5}$ on both sides of (17), we get

$$y = 1 + 2t + \frac{9}{4}J^{1.5}(\sqrt{y}) + J^{1.5}(y). \quad (18)$$

Using ADM and Adomian polynomials to the equation (18) and since the computation of A_n depends heavily on y_0 we will use a slight modification [24]. This will ease the computations considerably. Thus,

$$y_0 = 1, \quad (19)$$

$$y_1 = 2t + \frac{9}{4}J^{1.5}(A_0) + J^{1.5}(y_0), \quad (20)$$

$$y_n = \frac{9}{4}J^{1.5}(A_{n-1}) + J^{1.5}(y_{n-1}), n \geq 2. \quad (21)$$

Using relations (19)-(21), the first three-terms of the series solution for $\mu = 1.5$ are,

$$y(t) = 1 + 2t + 2.44482t^{1.5} + 1.27883t^{2.5} + 1.15104t^3 + \cdots, \quad (22)$$

Figure 2 shows ADM solution ($n = 5$). In Shawaghfeh's study [20], the mistake was ignoring the third term in equation (20).

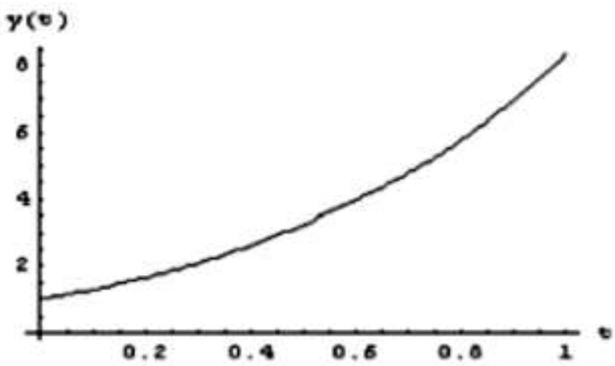

Figure 2: ADM Sol. [$\mu = 1.5$]

4. Conclusion

In this paper, an interesting method (ADM) used to solve fractional differential equations.

This method gives analytical solution and when we comparing ADM solution with Numerical solution method, we see that, it gives a bounded solution but the numerical method give unbounded solution.

References

[1] K. S. Miller, and B. Ross, (1993), An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley-Interscience, New York.

[2] I. Podlubny, (1999), Fractional Differential Equations, Academic Press, New York.

[3] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, (2006), Theory and Applications of Fractional differential equations, Elsevier, New York.

[4] Sh. A. Abd El-Salam, and A. M. A. El-Sayed, (2007), On the stability of some fractional-order non-autonomous systems, Electronic Journal of Qualitative Theory of Differential Equations, 6, pp. 1-14.

[5] A. M. A. El-Sayed, and Sh. A. Abd El-Salam, (2008), On the stability of a fractional-order differential equation with nonlocal initial condition, Electronic Journal of Qualitative Theory of Differential Equations, 29, pp. 1-8.

[6] D. J. Evans, and K. R. Raslan, (2005), The Adomian decomposition method for solving delay differential equation, International Journal of Computer Mathematics, (UK), 82, pp. 49-54.

[7] D. Zwillinger, (1997), Handbook of Differential Equations, Academic Press.

[8] B. Mensour, and A. Longtin, (1998), Chaos control in multistable delay-differential equations and their singular limit maps, Pysical Review E, 58, pp. 410-422.

[9] J. M. Hefferan, and R. M. Corless, (2005), Solving some delay differential equations with computer algebra, Applied Probability Trust, pp. 1-22.

[10] A. M. A. El-Sayed, E. M. El-Mesiry and H. A. A. El-Saka, (2004), Numerical solution for multi-term fractional (arbitrary) orders differential equations, Comput. and Appl. Math., 23, 1, pp. 33-54.

[11] E. M. El-Mesiry, A. M. A. El-Sayed and H. A. A. El-Saka, (2005), Numerical methods for multi-term fractional (arbitrary) orders differential equations, Appl. Math. and Comput., 160, 3, 683-699.

[12] E. Ahmed, A. M. A. El-Sayed, and H. A. A. El-Saka (2007), Equilibrium points, stability and numerical solutions of fractional-order predator-prey and rabies models, J. Math. Anal. Appl., pp. 542–553.

[13] A. M. A. El-Sayed, Linear differential equations of fractional orders, J. Appl. Math. Comput., (1993), pp. 1-12.

[14] G. Adomian, (1995), Solving Frontier Problems of Physics: The Decomposition Method, Kluwer.

[15] G. Adomian, (1983), Stochastic System, Academic press.

[16] G. Adomian, (1986), Nonlinear Stochastic Operator Equations, Academic press, San Diego.

[17] G. Adomian, (1989), Nonlinear Stochastic Systems: Theory and Applications to Physics, Kluwer.

[18] K. Abbaoui, and Y. Cherruault, (1994), Convergence of Adomian's method applied to differential equations, Computers Math. Applic., 28, pp. 103-109.

[19] Y. Cherruault, G. Adomian, K. Abbaoui, and R. Rach, (1995), Further remarks on convergence of decomposition method, International J. of Bio-Medical Computing., 38, pp. 89-93.

[20] N. T. Shawaghfeh, (2002), Analytical approximate solution for nonlinear fractional differential equations, J. Appl. Math. Comput., 131, pp. 517-529.

[21] I. L. El-kalla, (2008), Convergence of the Adomian method applied to a class of nonlinear integral equations, Applied Mathematics Letters, 21, pp. 372-376.

[22] S. Momani, and K. Al-Khaled (2005), Numerical solutions for systems of fractional differential equations by the decomposition method, J. Appl. Math. Comput. 162, pp. 1351-1365.

[23] H. Jafari, and V. Daftardar-Gejji (2006), Solving a system of nonlinear fractional differential equations using Adomian decomposition method, J. Appl. Math. Comput., 196 (2), pp. 644–651.

[24] A. M. Wazwaz, *A reliable modification of Adomian's decomposition method,* J. Appl. Math. Comput. 102, (1999), pp. 77-86.