Electronic Journal of Mathematical Analysis and Applications
Vol. 12(1) Jan 2024, No. 3
ISSN: 2090-729X (online)
ISSN: 3009-6731(print)
http://ejmaa.journals.ekb.eg/

ON THE EXISTENCE OF GLOBAL STRONG SOLUTIONS TO 1D BILAYER SHALLOW WATER MODEL.

ROAMBA BRAHIMA[1,2],ZONGO JULIEN[1], BAMOGO MOHAMED BASSIROU[1], ZONGO YACOUBA[3] AND ZABSONRE JEAN DE DIEU[1,2]

Abstract. Our study focuses on 1D viscous bilayer shallow water model. The model considered is represented by two superposed immiscible fluids with different physical properties. Each layer is governed by the shallow water equations in one dimension. A regularized model of the considered model has been the subject of some recent studies. Our contribution is to extend the results of the work carried out in [*Nonlinear Analysis, vol* **(14)2, 1216-1124, (2013)**] by proving the existence of global strong solutions of the considered model.

1. Introduction

This paper is devoted to the existence of global strong solutions to 1D bilayer shallow water model. The model studied is as follows:

$$\partial_t h_1 + \partial_x(h_1 u_1) = 0, \tag{1.1}$$

$$\partial_t(h_1 u_1) + \partial_x(h_1 {u_1}^2) - \nu_1 \partial_x(h_1 \partial_x u_1) + grh_1\partial_x h_2 + gh_1 \partial_x h_1 = 0, \tag{1.2}$$

$$\partial_t h_2 + \partial_x(h_2 u_2) = 0, \tag{1.3}$$

$$\partial_t(h_2 u_2) + \partial_x(h_2 {u_2}^2) - \nu_2 \partial_x(h_2 \partial_x u_2) + grh_2\partial_x h_1 + gh_2 \partial_x h_2 = 0, \tag{1.4}$$

where $(t, x) \in (0, T) \times \Omega$, with Ω a periodic domain in one dimension. The index 1 corresponds to the layer located below and the index 2 the one located at the top. Thus we note h_1, h_2, u_1 and u_2 respectively the water heights for each layer and velocity for each layer. The constant $g > 0$ is the gravity number and ν_1 and ν_2 are the kinematic viscosity. We also note ρ_1, ρ_2 the densities associated with each layer, then we define the quotient of densities by $r = \frac{\rho_2}{\rho_1} < 1$.

2010 *Mathematics Subject Classification.* 35Q30, 76B15.

Key words and phrases. shallow water equations, bilayer model, viscous.
Submitted Feb. 2023.

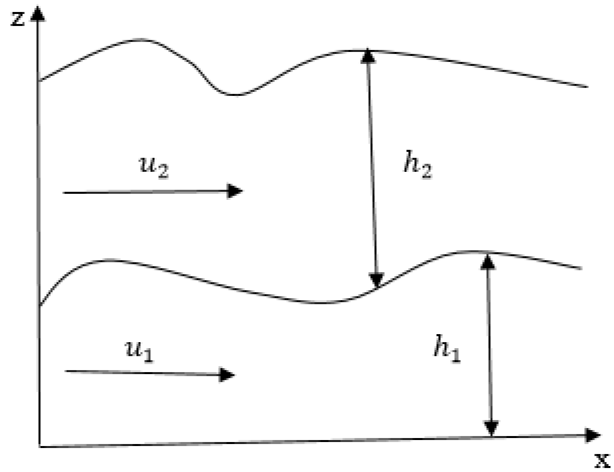

Figure 1: Notation for the bilayer model

This work takes its inspiration from the work done in [1]. Note that in [1], the authors showed the existence of global strong solutions of a regularized model of the model studied in this paper by adding regularizing terms at the level of the momentums equations of each layer of the form $\frac{\varepsilon}{\beta}\nu_i\partial_x(h_i^\beta\partial_x u_i))$ with β belongs to $\left(0, \frac{1}{2}\right)$ and ε is a small parameter. Also the studied model is associated with the initial energies:

$$\mathcal{E}_0 = \frac{1}{2}\int_\Omega h_{1_0,\varepsilon}|v_{1_0,\varepsilon}|^2 + \frac{g(1-r)}{2}\int_\Omega h_{2_0,\varepsilon}|v_{2_0,\varepsilon}|^2 + \frac{rg}{2}\int_\Omega |h_{1_0,\varepsilon} + h_{2_0,\varepsilon}|^2 \leq C\varepsilon^2 \leq C$$

and

$$\mathcal{F}_0 = \frac{1}{2}\int_\Omega \left|\nu_1\frac{\partial_x\varphi_\varepsilon(h_{1_0,\varepsilon})}{\sqrt{h_{1_0,\varepsilon}}}\right|^2 + \frac{1}{2}\int_\Omega \left|\nu_2\frac{\partial_x\varphi_\varepsilon(h_{2_0,\varepsilon})}{\sqrt{h_{2_0,\varepsilon}}}\right|^2 \leq C\varepsilon^2 \leq C.$$

But we noticed that when ε tends towards 0, they obtain the existence of a strong solution of the stationary model. In this paper we study the evolutionary model of the model studied in [1] when ε tends towards 0.

As a reminder, it should be noted that the authors in [1] have proven that the regularized approximate system verifies the BD entropy, which gives the lower bound for the water heights. This allows them to have the existence of global strong solutions for the approximate system by using the regularity theorem for smooth data given in [2] and manages to pass to the limit. The mathematical entropy named BD entropy was introduced firstly in [3]. This work followed an improvement in [4, 5] where the authors extended the result to the more general Navier-Stokes equations with an algebraic relationship between the coefficients of shear and viscosity in bulk. We also note that many researchers have used this entropy in their work, we can mention among others [6, 8, 9, 10] who thanks to this entropy have proven the existence of global weak solutions for shallow water and viscous compressible Navier-Stokes equations.

Several works have been carried out on the existence of strong solutions in shallow water and Navier-Stokes equations. Other examples include [8] where the authors have proven the existence of strong solutions for one-dimensional compressible

Navier-Stokes equations under the hypothesis that the initial datum is smooth and the initial density is bounded below by a positive constant. In [1] the authors proved according to the ideas developed in [8], the existence of strong solutions of one dimensional regularized bilayer model. The existence of global strong solutions to the Cauchy problem for a shallow water system in dimension $N \geq 2$ has been proven in [11]. In [12] the authors proved the existence of global strong solutions for the compressible Navier-Stokes equations with degenerate viscosity coefficient in 1D. The key ingredient of their proof resulting from the control of a new effective velocity (see [12]) in $L^\infty((0,T); L^\infty(\mathbb{R}))$ and this control allowed them to have control of the inverse of the density $1/\rho$ in the same space. Our result draws inspiration from their work.

Our contribution to this work is to extend the results obtained by the authors in [1]. Indeed the authors have noted the existence of global strong solutions of the regularized model using a test function depending on the unknown. For our part, we prove the existence of global strong solutions of the studied model in [1] when the regularizing terms will tend towards zero by following the approach proposed in [12].

In section 2 we will give the theorem of the existence of global strong solutions of bilayer shallow water model in one dimension. To prove this result of existence, we have to use some intermediary results to achieve it, which will be the subject of the third section. We will give the proof of some results in the last section.

2. Main results

In this section we give the initial data, the initial energy associated with the system (1.1) – (1.4) and the existence of strong solutions theorem of the model. Consider the initial data,

$$h_{1_0} = h_{1|t=0}, \quad h_{2_0} = h_{2|t=0}, \quad u_{1_0} = u_{1|t=0} \quad \text{ and } u_{2_0} = u_{2|t=0}$$

testing the following assumptions

$$\begin{gathered} 0 < \underline{c}_{1_0} \leq h_{1_0} \leq \bar{c}_{1_0}, \quad 0 < \underline{c}_{2_0} \leq h_{2_0} \leq \bar{c}_{2_0} \\ h_{1_0} \in H^1(\Omega), \quad u_{1_0} \in H^1(\Omega), \quad h_{2_0} \in H^1(\Omega), \quad u_{2_0} \in H^1(\Omega) \end{gathered} \tag{2.1}$$

where $\underline{c}_{1_0}$, $\underline{c}_{2_0}$, $\bar{c}_{1_0}$ and $\bar{c}_{2_0}$ are some positive constants. We also assume that the viscosities ν_1 and ν_2 verify the following relation:

$$\frac{2\nu_1}{\nu_1+\nu_2} \geq r, \quad \frac{2\nu_2}{\nu_1+\nu_2} \geq r \quad \text{ with } \quad 0 < r < 1. \tag{2.2}$$

We further assume that the following quantities are finished:

$$\begin{aligned} \frac{1}{2}\int_\Omega \Big[& h_{1_0}|u_{1_0}|^2 + h_{2_0}|u_{2_0}|^2 + g(1-r)|h_{1_0}|^2 + g(1-r)|h_{2_0}|^2 \\ & + rg|h_{1_0}+h_{2_0}|^2 \Big] dx \leq C_1, \end{aligned} \tag{2.3}$$

$$\frac{1}{2}\int_\Omega \Big[h_{1_0}|u_{1_0} + \partial_x\varphi(h_{1_0})|^2 + h_{2_0}|u_{2_0} + \partial_x\varphi(h_{2_0})|^2 \Big] dx \leq C_2 \tag{2.4}$$

where C_1, C_2 are real constants and $\varphi(h_i) = \nu_i \log h_i, \quad i = \{1,2\}$.

Theorem 2.1. *The system* $(1.1)-(1.4)$ *admits a strong solution* (h_1, h_2, u_1, u_2) *such that*

$$\begin{aligned}
&h_1 \quad \text{is bounded in} \quad L^\infty(0,T;H^1(\Omega)),\\
&h_2 \quad \text{is bounded in} \quad L^\infty(0,T;H^1(\Omega)),\\
&u_1 \quad \text{is bounded in} \quad L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)),\\
&u_2 \quad \text{is bounded in} \quad L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)),\\
&\partial_t u_1 \quad \text{is bounded in} \quad L^2(0,T;L^2(\Omega)),\\
&\partial_t u_2 \quad \text{is bounded in} \quad L^2(0,T;L^2(\Omega)).
\end{aligned} \tag{2.5}$$

Moreover for every $T > 0$*, there exists constants* $\alpha_1(T), \alpha_2(T), \beta_1(T)$ *and* $\beta_2(T)$ *such that:*

$$\begin{aligned}
&0 < \alpha_1(T) \le h_1(t,x) \le \beta_1(T), \quad \forall (t,x) \in (0,T) \times \Omega,\\
&0 < \alpha_2(T) \le h_2(t,x) \le \beta_2(T), \quad \forall (t,x) \in (0,T) \times \Omega.
\end{aligned} \tag{2.6}$$

In the following section, we will give some results that will help prove the previous theorem.

3. Energies inequalties

We start this section with the energy equality associated with the system $(1.1)-(1.4)$

Proposition 3.1. *For* (h_1, h_2, u_1, u_2) *smooth solution of the system* $(1.1)-(1.4)$ *with boundary conditions* (2.1),(2.2) *and* (2.3)*, then the following classical equality holds:*

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left[h_1|u_1|^2 + h_2|u_2|^2 + g(1-r)|h_1|^2 + g(1-r)|h_2|^2 + rg|h_1+h_2|^2\right]dx$$
$$+\nu_1 \int_\Omega h_1|\partial_x u_1|^2 + \nu_2 \int_\Omega h_2|\partial_x u_2|^2 dx = 0. \tag{3.1}$$

From this energy estimate (3.1), we deduce the following result:

Corollary 3.2. *Let* (h_1, h_2, u_1, u_2) *be a solution of model* $(1.1)-(1.4)$*. We have the following uniform bounds:*

$$\sqrt{h_1}u_1 \text{ is bounded in } L^\infty(0,T;L^2(\Omega)), \sqrt{h_2}u_2 \text{ is bounded in } L^\infty(0,T;L^2(\Omega)),$$

$$h_1 \text{ is bounded in } L^\infty(0,T;L^2(\Omega)), h_2 \text{ is bounded in } L^\infty(0,T;L^2(\Omega)),$$

$$\sqrt{h_1}\partial_x u_1 \text{ is bounded in } L^2(0,T;L^2(\Omega)), \sqrt{h_2}\partial_x u_2 \text{ is bounded in } L^2(0,T;L^2(\Omega)).$$

We need additional estimates on the unknown h_1, h_2, u_1 and u_2. The following proposition will allow us to have some of them.

Proposition 3.3. *Let* (h_1, h_2, u_1, u_2) *be a smooth solution of* $(1.1)-(1.4)$*, then the following mathematical BD entropy inequality holds:*

$$\frac{1}{2}\frac{d}{dt}\int_\Omega h_1|u_1 + \partial_x\varphi(h_1)|^2 dx + \frac{1}{2}\frac{d}{dt}\int_\Omega h_2|u_2 + \partial_x\varphi(h_2)|^2 dx$$
$$+\frac{1}{2}g\frac{d}{dt}\int_\Omega \left[(1-r)|h_1|^2 + (1-r)|h_2|^2 + r|h_1+h_2|^2\right]dx$$
$$+g(\nu_1 - \frac{1}{2}r(\nu_1+\nu_2))\int_\Omega |\partial_x h_1|^2 + g(\nu_2 - \frac{1}{2}r(\nu_1+\nu_2))\int_\Omega |\partial_x h_2|^2 \leq 0. \tag{3.2}$$

The BD mathematical entropy inequality allows us to find the estimates given in the following Corrolary.

Corollary 3.4. *Let* (h_1, h_2, u_1, u_2) *be a solution of model* $(1.1)-(1.4)$ *verifying the inequality given in* (3.2)*. We have the following uniform bounds:*

$$h_1 \quad \textit{is bounded in} \quad L^\infty(0,T;H^1(\Omega)), \quad h_2 \quad \textit{is bounded in} \quad L^\infty(0,T;H^1(\Omega)),$$
$$u_1 \quad \textit{is bounded in} \quad L^\infty(0,T;H^1(\Omega)), \quad u_2 \quad \textit{is bounded in} \quad L^\infty(0,T;H^1(\Omega)),$$
$$\partial_x h_1 \textit{ is bounded in } L^2(0,T;L^2(\Omega)), \partial_x h_2 \textit{ is bounded in } L^2(0,T;L^2(\Omega)),$$
$$\partial_x\sqrt{h_1} \textit{ is bounded in } L^\infty(0,T;L^2(\Omega)), \partial_x\sqrt{h_2} \textit{ is bounded in } L^\infty(0,T;L^2(\Omega)).$$

Remark 3.5. The sobolev embedding allows us to deduce that:

$$h_i \text{ is bounded in } L^\infty(0,T;L^\infty(\Omega)),$$

which leads to the existence of a constant $\bar{\beta}_i(T) \quad \forall T > 0$ such that:

$$0 \leq h_i(t,x) \leq \bar{\beta}_i(T), \quad \forall (t,x) \in (0,T)\times\Omega; \quad i = \{1,2\}.$$

This assures us the upper bound of the heights in the **theorem**. To lower limit the height, we need the following result.

Lemma 3.6. $\forall\varepsilon > 0$ *small enough, we have for* $\gamma = \frac{1}{2} + \varepsilon$ *:*

$$h_i^\gamma u_i \quad \textit{is bounded in} \quad L^2(0,T;L^\infty(\Omega)), \quad i = \{1,2\}.$$

Proposition 3.7. *For* (h_1, h_2, u_1, u_2) *solution of the system* $(1.1)-(1.4)$*, we have the following estimates:*

$$u_1 \quad \textit{is bounded in} \quad L^2(0,T;H^2(\Omega)), \quad \partial_t u_1 \quad \textit{is bounded in} \quad L^2(0,T;L^2(\Omega)),$$
$$u_2 \quad \textit{is bounded in} \quad L^2(0,T;H^2(\Omega)), \quad \partial_t u_2 \quad \textit{is bounded in} \quad L^2(0,T;L^2(\Omega)).$$

Lemma 3.8. *For* $i = \{1,2\}$*, we have the following bounds:*

$$v_i = u_i + \nu_i\partial_x\varphi(h_i) \quad \textit{is bounded in} \quad L^\infty(0,T);L^\infty(\Omega)).$$

For every $T > 0$*, there exists a continuous fonction* α *and* $c > 0$ *such that for all* $T < T_0$*, we have:*

$$h_i(t,x) \geq \alpha(T) \geq c > 0.$$

The proofs of the **Proposition 3.1** , **Proposition 3.3**, **Proposition 3.7** , **Lemma 3.6** ,**Lemma 3.8** and the **Remark 3.5** assures us the proof of the theorem.
In the next section, we will give the proofs of the above **Propositions**.

4. Appendix

Proof. **Proposition 3.1**

We multiply the momentum equations (1.2) and (1.4) respectively by u_1 and u_2 and we obtain:

$$\int_\Omega \Big[(\partial_t h_1 u_1) + \partial_x(h_1 u_1^2)\Big] u_1 dx + g\int_\Omega \Big[h_1\partial_x h_1 + r h_1 \partial_x h_2\Big] u_1 dx$$
$$-\nu_1 \int_\Omega u_1 \partial_x(h_1\partial_x u_1) dx = 0,$$

and

$$\int_\Omega \Big[(\partial_t h_2 u_2) + \partial_x(h_2 u_2^2)\Big] u_2 dx + g\int_\Omega \Big[h_2\partial_x h_2 + r h_2 \partial_x h_1\Big] u_2 dx$$
$$-\nu_2 \int_\Omega u_2 \partial_x(h_2\partial_x u_2) dx = 0.$$

Look at the terms :

$$\int_\Omega \Big[(\partial_t h_1 u_1) + \partial_x(h_1 u_1^2)\Big] u_1 dx = \frac{1}{2}\frac{d}{dt}\int_\Omega h_1|u_1|^2 dx, \tag{4.1}$$

$$\int_\Omega \Big[(\partial_t h_2 u_2) + \partial_x(h_2 u_2^2)\Big] u_2 dx = \frac{1}{2}\frac{d}{dt}\int_\Omega h_2|u_2|^2 dx. \tag{4.2}$$

Furthermore,

$$g\int_\Omega \Big[h_1 u_1 \partial_x h_1 + h_2 u_2 \partial_x h_2 + r\big(h_1 u_1 \partial_x h_2 + h_2 u_2 \partial_x h_1\big)\Big] dx$$
$$= \frac{1}{2} g(1-r)\frac{d}{dt}\int_\Omega |h_1|^2 dx + \frac{1}{2} g(1-r)\frac{d}{dt}\int_\Omega |h_2|^2 dx$$
$$+\frac{1}{2} r g \frac{d}{dt}\int_\Omega |h_1+h_2|^2 dx. \tag{4.3}$$

We have also

$$-\nu_1\int_\Omega u_1\partial_x(h_1\partial_x u_1)dx - \nu_2\int_\Omega u_2\partial_x(h_2\partial_x u_2)$$
$$= \nu_1\int_\Omega h_1|\partial_x u_1|^2 + \nu_2\int_\Omega h_2|\partial_x u_2|^2 dx. \tag{4.4}$$

Now we add the equations (4.1) – (4.4) to find the proclaimed equality. □

Proof. **Proposition 3.3**

The system (1.1) – (1.4) can be written as follows: for $i, j = 1, 2$ with $i \neq j$

$$(S_i)\begin{cases} \partial_t h_i + \partial_x(h_i u_i) = 0, \\ \\ \partial_t(h_i u_i) + \partial_x(h_i {u_i}^2) - \nu_i \partial_x(h_i \partial_x u_i) + g r h_i \partial_x h_j + \dfrac{1}{2} g \partial_x h_i^2 = 0. \end{cases}$$

Following the idea proposed in [12], we set $v_i = u_i + \nu_i \partial_x \log h_i = u_i + \partial_x \varphi(h_i)$ and we can rewrite the system (S_i) as follows:

$$(S_i')\begin{cases} \partial_t h_i + \partial_x(h_i v_i) - \nu_i \partial_x^2 h_i = 0, \\ \\ h_i \partial_t(v_i) + h_i u_i \partial_x(v_i) + g r h_i \partial_x h_j + \dfrac{1}{2} g \partial_x h_i^2 = 0, \end{cases}$$

for $i, j = 1, 2$ with $i \neq j$.
We multiply the second equation of (S_i') by v_i and integrate on Ω, for $i = 1, 2$.

We have for each layer:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega h_1|u_1+\partial_x\varphi(h_1)|^2dx + rg\int_\Omega h_2\partial_t h_1 dx + rg\nu_1\int_\Omega \partial_x h_1\partial_x h_2 dx$$
$$+\frac{1}{2}g\int_\Omega \partial_t h_1^2 dx + g\nu_1\int_\Omega |\partial_x h_1|^2 = 0, \tag{4.5}$$

And

$$\frac{1}{2}\frac{d}{dt}\int_\Omega h_2|u_2+\partial_x\varphi(h_2)|^2dx + rg\int_\Omega h_1\partial_t h_2 dx + rg\nu_2\int_\Omega \partial_x h_1\partial_x h_2 dx$$
$$+\frac{1}{2}g\int_\Omega \partial_t h_2^2 dx + \nu_2 g\int_\Omega |\partial_x h_2|^2 = 0. \tag{4.6}$$

We sum up the equations by performing a simple calculation to have:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega h_1|u_1+\partial_x\varphi(h_1)|^2dx + \frac{1}{2}\frac{d}{dt}\int_\Omega h_2|u_2+\partial_x\varphi(h_2)|^2dx$$
$$+\frac{1}{2}g\frac{d}{dt}\int_\Omega\left[(1-r)|h_1|^2+(1-r)|h_2|^2+r|h_1+h_2|^2\right]dx$$
$$+\nu_1 g\int_\Omega|\partial_x h_1|^2 + \nu_2 g\int_\Omega|\partial_x h_2|^2 + rg(\nu_1+\nu_2)\int_\Omega \partial_x h_1\partial_x h_2 = 0. \tag{4.7}$$

We use the inequality $xy \leq \frac{1}{2}(x^2+y^2)$ on the last term to deduce the proclaimed inequality. □

Proof. **Lemma** 3.6 (We follow the ideas proposed in [12])

The proof of the lemma 3.6 is inspired by the work of Boris Haspot in [12]. We first rewrite the equations (1.2) and (1.4) as follows:

$$\partial_t(h_iu_i) + \partial_x(h_i{u_i}^2) - \nu_i\partial_x(h_i\partial_x u_i) + grh_i\partial_x h_j + \frac{1}{2}g\partial_x h_i^2 = 0,$$

for $i, j = 1, 2$ with $i \neq j$. We multiply this momentum equation by $u_i\,|u_i|^p$ with $p > 0$ and we integrate by parts over $[0, T]\times\Omega$ where $T \in (0, T_0)$

$$\frac{1}{p+2}\int_\Omega h_i|u_i|^{p+2}dx + \nu_i(p+1)\int_0^T\int_\Omega h_i|\partial_x u_i|^2|u_i|^p dxdt + g\int_0^T\int_\Omega h_iu_i|u_i|^p\partial_x h_i dxdt$$
$$+gr\int_0^T\int_\Omega h_iu_i|u_i|^p\partial_x h_j dxdt = \frac{1}{p+2}\int_\Omega h_{i_0}|u_{i_0}|^{p+2}dx. \tag{4.8}$$

Next we have by integration by parts and Young inequality:

$$\left|g\int_0^T\int_\Omega h_iu_i|u_i|^p\partial_x h_i dxdt\right| = \left|\frac{g(p+1)}{2}\int_0^T\int_\Omega h_i^2|u_i|^p\partial_x u_i dxdt\right|$$
$$\leq \frac{g(p+1)}{4}\int_0^T\int_\Omega h_i\,|\partial_x u_i|\,|u_i|^p dxdt + \frac{g(p+1)}{4}\int_0^T\int_\Omega h_i^3|u_i|^p dxdt \tag{4.9}$$

Using Hölder inequality and by interpolation, we get for $p \geq 2$:

$$\int_0^T \int_\Omega h_i^3 |u_i|^p dxdt \leq ||h_i||^2_{L^\infty([0,T\times\Omega])} \int_0^T \int_\Omega h_i(t,x)|u_i|^p(t,x)dxdt$$
$$\leq ||h_i||^2_{L^\infty([0,T\times\Omega])} \int_0^T \left|\left|h_i^{1/p+2} u_i(t,.)\right|\right|^{\frac{(p-2)(p+2)}{p}}_{L^{p+2}} \left|\left|\sqrt{h_i}u_i(t,.)\right|\right|^{\frac{4}{p}}_{L^2} dt$$
$$\leq ||h_i||^2_{L^\infty([0,T\times\Omega])} \int_0^T \left(1+\left|\left|h_i^{1/p+2} u_i(t,.)\right|\right|^{p+2}_{L^{p+2}}\right) \left|\left|\sqrt{h_i}u_i(t,.)\right|\right|^{\frac{4}{p}}_{L^2} dt \tag{4.10}$$

Using Young inequality we have for $\eta \geq 0$:

$$\left|gr\int_0^T \int_\Omega u_i|u_i|^p h_i \partial_x h_j dxdt\right|$$
$$\leq gr\left[\frac{\eta}{2}\int_0^T \int_\Omega \left|\partial_x h_j\right|^2 |u_i|^p dxdt + \frac{1}{2\eta}\int_0^T ||h_i(t,.)||^2_{L^\infty}\left(\int_\Omega |u_i|^{p+2}dx\right)dt\right] \tag{4.11}$$

We recall that since u_{i_0} belongs to $L^\infty(\mathbb{R})\cap L^2(\mathbb{R})$ then u_{i_0} is any $L^{p+2}(\mathbb{R})$. Plugging (4.9)-(4.11) in (4.8) and using Gronwall lemma we deduce that $h_i^{\frac{1}{p}} u_i$ is bounded in $L^\infty(0,T_0;L^p(\Omega))$ for any $p\geq 4$ and by interpolation for any $p \geq 2$.

Using the ideas that in [12], we now put the term $\partial_x\left(h_i^\gamma u_i\right)$ in the following form:

$$\partial_x\left(h_i^\gamma u_i\right) = h_i^{\gamma-\frac{1}{2}} h_i^{\frac{1}{2}}\partial_x u_i + (2\gamma)\, h_i^{\gamma-\frac{1}{2}} u_i \partial_x\left(h_i^{\frac{1}{2}}\right)$$

Taking $\gamma = \varepsilon + \dfrac{1}{2}$ with ε small enough, we have

$$\partial_x\left(h_i^\gamma u_i\right) = h_i^{\varepsilon} h_i^{\frac{1}{2}}\partial_x u_i + 2\beta h_i^{\varepsilon} u_i \partial_x\left(h_i^{\frac{1}{2}}\right).$$

We note now that $h_i^\varepsilon u_i$ is bounded in $L^\infty\left(0,T_0;L^{\frac{1}{\varepsilon}}(\Omega)\right)$because h_i is bounded in $L^\infty\left(0,T_0;L^\infty(\Omega)\right)$ (Remark 3.5) and $h_i^{\frac{1}{p}} u_i$ is in $L^\infty\left(0,T_0;L^p(\Omega)\right)$ for any $p\geq 2$. It implies via the estimates of the corollary 3.4 , the existence of a constant $\bar{\beta}_i(T)$ with $\forall T>0$ such that:

$$0 \leq h_i(t,x) \leq \bar{\beta}_i(T)$$

that $\partial_x\left(h_i^\gamma u_i\right)$ is bounded in $L^2(0,T;L^2(\Omega))+L^\infty(0,T;L^p(\Omega))$ (for any $T\in(0,T_0)$) which is embedded in $L^2(0,T;L^2(\Omega))+L^p(\Omega)$ with $\frac{1}{p}=\frac{1}{2}+\varepsilon$. By the Riesz Thorin theorem it implies that the Fourier transform $\mathcal{F}(\partial_x(h_i^\gamma u_i))$ is in $L^2(0,T;L^2(\Omega))+L^q(\Omega)$ with $\frac{1}{p}+\frac{1}{q}=1$. In particular we deduce from Hölder inequality that $\mathcal{F}\left(h_i^\gamma u_i\right)1_{\{|\xi|\geq 1\}}$ is in $L^2(0,T;L^1(\Omega))$ for any $T\in(0,T_0)$. As $h_i^\gamma u_i = h_i^\varepsilon\sqrt{h_i}u_i$, we obtain from Hölder inequality that $h_i^\gamma u_i$ is in $L^2(0,T;L^2(\Omega))$ for any $T\in(0,T_0)$. From Plancherel theorem we can prove that $\mathcal{F}\left(h_i^\gamma u_i\right)1_{\{|\xi|\leq 1\}}$ is in $L^2(0,T;L^1(\Omega))$. It gives that $\mathcal{F}\left(h_i^\gamma u_i\right)$ is in $L^2(0,T;L^1(\Omega))$. We thus get that $h_i^\gamma u_i$ is bounded in $L^2(0,T;L^\infty(\Omega))$. □

Proof. **Proposition 3.7**

We consider the momentum equation for $i,j=\{1,2\}\quad i\neq j$,

$$\partial_t(h_iu_i)+\partial_x(h_i{u_i}^2)-\nu_i\partial_x(h_i\partial_x u_i)+grh_i\partial_x h_j+\frac{1}{2}g\partial_x h_i^2=0$$

We rewrite that as:

$$h_i\partial_t u_i + h_i u_i \partial_x u_i - \nu_i \partial_x(h_i \partial_x u_i) + rgh_i\partial_x h_j + gh_i \partial_x h_i = 0,$$
$$h_i\partial_t u_i + h_i u_i \partial_x u_i - \nu_i h_i \partial_x^2 u_i - \nu_i \partial_x h_i \partial_x u_i + rgh_i\partial_x h_j + gh_i \partial_x h_i = 0,$$
$$\partial_t u_i + u_i \partial_x u_i - \nu_i \partial_x^2 u_i - \nu_i \frac{\partial_x h_i}{h_i}\partial_x u_i + rg\partial_x h_j + g\partial_x h_i = 0,$$
$$\partial_t u_i - \nu_i \partial_x^2 u_i = -rg\partial_x h_j - g\partial_x h_i + (\nu_i \partial_x \log h_i - u_i)\partial_x u_i. \tag{4.12}$$

Thanks to the **corollary** 2, $\partial_x h_i$ is bounded in $L^2(0,T;L^2(\Omega))$. Following the ideas proposed in [12] and [8] using Holder inequality, Gagliardo-Nirenberg inequality and energy estimate, we have:

$$\begin{aligned} &||(\partial_x \varphi(h_i) - u_i)\partial_x u_i||_{L^2(0,T;L^2(\Omega))} \\ &\le ||\partial_x \varphi(h_i) - u_i||_{L^\infty(0,T;L^2(\Omega))} ||\partial_x u_i||_{L^2(0,T;L^\infty(\Omega))} \\ &\le ||\partial_x \varphi(h_i) - u_i||_{L^\infty(0,T;L^2(\Omega))} ||\partial_x u_i||^{\frac{1}{2}}_{L^2(0,T;L^2(\Omega))} ||\partial_x^2 u_i||^{\frac{1}{2}}_{L^2(0,T;L^2(\Omega))} \\ &\le C||\partial_x^2 u_i||^{\frac{1}{2}}_{L^2(0,T;L^2(\Omega))}. \end{aligned}$$

Using regularity results for parabolic equation of the form (4.12) gives for any $T \in (0, T_0)$:

$$||\partial_t u_i||_{L^2(0,T;L^2(\Omega))} + ||\partial_x u_i||_{L^2((0,T;H^1(\Omega))} \le C||\partial_x u_i||^{\frac{1}{2}}_{L^2(0,T;H^1(\Omega))} + C,$$

with C depending on $||u_{i_0}||_{H^1}$ and by boostrap for any $T \in (0, T_0)$:

$$||\partial_t u_i||_{L^2((0,T);L^2(\Omega))} + ||u_i||_{L^2((0,T);H^2(\Omega))} \le C(T)$$

□

Proof. **Lemma 3.8**

We recall that if (h_1, h_2, u_1, u_2) is a regular solution which verifies the system $(1.1) - (1.4)$ on $(0, T_0)$ then (h_1, h_2, u_1, u_2) is the solution of the system (S_i'):

$$\begin{cases} \partial_t h_i + \partial_x(h_i v_i) - \nu_i \partial_x^2 h_i = 0, \\ \\ h_i \partial_t(v_i) + h_i u_i \partial_x(v_i) + grh_i \partial_x h_j + gh_i \partial_x h_i = 0, \end{cases} \tag{4.13}$$

for $i, j = 1, 2$ with $i \neq j$.
We multiply the momentum equation of (4.13) by $v_i|v_i|^p$ for $p \ge 0$ and integrate over $(0,T) \times \Omega$ where $T \in (0, T_0)$.

$$\int_0^T \int_\Omega \big[h_i \partial_t(v_i) + h_i u_i \partial_x(v_i) + grh_i \partial_x h_j + gh_i \partial_x h_i\big] v_i |v_i|^p dx = 0,$$

we notice that:

$$g(h_i \partial_x h_i) v_i |v_i|^p = \frac{g}{\nu_i} h_i^2 (v_i - u_i)|v_i|^p. \tag{4.14}$$

Using (4.14), we have:

$$\begin{aligned} &\int_0^T \int_\Omega \big[g(h_i(z,x)\partial_x h_i(z,x)) v_i(z,x)|v_i(z,x)|^p\big] dx \\ &= \frac{g}{\nu_i} \int_0^T \int_\Omega h_i^2(z,x)|v_i(z,x)|^{p+2} dx dz \end{aligned}$$

$$+\frac{g}{\nu_i}\int_0^T\int_\Omega h_i^2(z,x)u_i(z,x)v_i(z,x)|v_i(z,x)|^p dxdz.$$

Also, we have:

$$\Big|\int_0^T\int_\Omega h_i^2(z,x)u_i(z,x)v_i(z,x)|v_i(z,x)|^p dxdz\Big|$$

$$=\Big|\int_0^T\int_\Omega h_i^{1+\frac{1}{p+2}}(z,x)u_i h_i^{\frac{p+1}{p+2}}(z,x)v_i(z,x)|v_i(z,x)|^p dxdz\Big|$$

$$\leq \int_0^T\Big[\parallel h_i^{\frac{1}{p+2}}v_i(z,.)\parallel_{L^{p+2}}^{p+1}\parallel h^{1+\frac{1}{p+2}}u_i(z,.)\parallel_{L^{p+2}}\Big]dz$$

$$\leq \int_0^T\Big[\parallel h_i^{\frac{1}{p+2}}v_i\parallel_{L^{p+2}}^{p+1}\times\parallel h_i^{\gamma}u_i\parallel_{L^\infty}^{p}\parallel\sqrt{h_i(z,.)}u_i(z,.)\parallel_{L^2}^2\parallel h_i\parallel_{L^\infty}^{(p-2)(1-\frac{p\gamma}{p+2})},$$

for any γ such that: $\gamma=\dfrac{1}{2}+\varepsilon$ with $\varepsilon>0$ (see [12]).

Furthermore,

$$\Big|\int_0^T\int_\Omega\big[gh_i(z,x)\partial_x h_j(z,x)\big]v_i(z,x)|v_i(z,x)|^p dx\Big|$$

$$\leq g\Big[\int_0^T\int_\Omega h_i(z,x)|v_i(z,x)|^{p+1}dzdx+\int_0^T\int_\Omega|\partial_x h_j(z,x)|^2dzdx\Big]$$

$$\leq g\Big[\int_0^T\int_\Omega h_i(z,x)|v_i(z,x)|^{p+1}1_{\{|v_i(z,x)|\geq 1\}}dzdx$$

$$+\int_0^T\int_\Omega h_i(z,x)|v_i(z,x)|^{p+1}1_{\{|v_i(z,x)|<1\}}dzdx+\int_0^T\int_\Omega|\partial_x h_j(z,x)|^2dzdx\Big]$$

$$\leq g\Big[\int_0^T\int_\Omega h_i(z,x)|v_i(z,x)|^{p+2}1_{\{|v_i(z,x)|\geq 1\}}dzdx$$

$$+\int_0^T\int_\Omega h_i(z,x)|v_i(z,x)|^{2}1_{\{|v_i(z,x)|<1\}}dzdx+\int_0^T\int_\Omega|\partial_x h_j(z,x)|^2dzdx\Big]$$

$$\leq g\Big[2\int_0^T\int_\Omega h_i(z,x)|v_i(z,x)|^{p+2}dzdx$$

$$+2\int_0^T\int_\Omega h_i(z,x)|v_i(z,x)|^{2}dzdx+\int_0^T\int_\Omega|\partial_x h_j(z,x)|^2dzdx\Big].$$

By borrowing the ideas developed in [12] (using Young inequality, the Gronwall lemma, the **Remark 3.5** and the **Lemma 3.6**), we have for all $p\in[0,+\infty[$:

$$\parallel h_i(T,.)^{\frac{1}{p+2}}v_i(t,.)\parallel_{L^{p+2}}\leq C(T)\quad\forall T\in(0,T_0),\tag{4.15}$$

where C is a continuous function independent on p.
By considering the regularization on the initial data such that $v_{i_0}*k_n$ belongs to H^s with $s>\frac{1}{2}$ and k_n the regularizing Kernel, the solution (h_1,h_2,v_1,v_2) verifies:

$$h_i(t,x)\geq\beta_i(t)>0\quad\forall x\in\Omega\text{ and }\quad\parallel v_i(t,.)\parallel_{L^\infty}\leq C_i(t),\quad\forall t\in[0,T_0],$$

with possibility $\beta_i(t)\longrightarrow_{t\to T_0}0$ and $C_i(t)\longrightarrow_{t\to T_0}+\infty$.
We observe that $\forall\varepsilon>0$ sufficiently small (such that $\dfrac{\parallel v_i(t,.)\parallel_{L^\infty}}{2}>\varepsilon$)

we have $\forall p \geq 2, t \in (0, T_0)$:

$$\| h_i^{\frac{1}{p}} v_i(t,.) \|_{L^p} \geq$$
$$\Big[\| v_i(t,.) \|_{L^{\infty-\varepsilon}} \Big] \beta_i(t)^{\frac{1}{p}} \Big| \{x, |v_i(t,x)| \geq \| v_i(t,.) \|_{L^{\infty-\varepsilon}} \} \Big|^{\frac{1}{p}}. \tag{4.16}$$

Since we have $\beta_i(t) > 0$ and $0 < \Big| \{x, |v_i(t,x)| \geq \| v_i(t,.) \|_{L^{\infty-\varepsilon}} \} \Big| < +\infty$, we can pass to the limit when the limit p goes to $+\infty$ in (4.16). It implies that for any $\varepsilon > 0$ small enough, we get using (4.15):

$$\| v_i(t,.) \|_{L^{\infty-\varepsilon}} \leq C(t) \quad \forall t \in (0, T_0).$$

Thanks to the maximum principle, we deduce according to the ideas proposed in [2, 12] that:

$$\frac{1}{h_i} \text{is bounded in } L^\infty(0,T;L^\infty)).$$

□

Conclusion

In this paper, we were interested in the theoretical study of a viscous bilayer shallow water model of one dimension. Indeed, the authors in [*Nonlinear Analysis, vol* **(14)2, 1216-1124, (2013)**], considered an approximate system while justifying the existence of global strong solutions using the regularity theorem for smooth data given in [*R. I. American Mathematical Society, vol* **(23), ((1968)**]. We have improved their work by proving the existence of global strong solutions to the initial model(without the regularizing terms) by getting inspired by the work done in [Math Nachr. **291 (14-15), 2183-2203, (2018)].**

References

[1] Zabsonré J. D. D., Lucas C. and Ouedraogo A. (2013), Strong solutions for a 1D viscous bilayer shallow water model. *Nonlinear Analysis*, **vol(14) 2**, 1216-1224

[2] Ladyzhenskaya O. A., Solonnikov V. A. and Uraltseva N. N. (1968), Linear and quasilinear equations of parabolic type, *AMS translation,Providence,* ***Vol 23***

[3] Bresch D. and Desjardins B.(2003), Existence of global weak solutions for a 2D viscous shallow water equations and convergence to the quasi-geostrophic model, Comm. Math. Phys. 238**(1-3)**, 211-223.

[4] Bresch D. and Desjardins B.(2007), On the existence of global weak solutions to the Navier-Stokes equations for viscous compressible and heat conducting fluids;*J. Maths Pures Appl. vol* **87 (1)**, 57-90.

[5] Bresch D. , Desjardins B. and Lin C.K. (2003), On some compressible fluid models : Korteweg, lubrication and shallow water systems, *Communications in partial differential equations* 28 **(3,4)**, 843-868.

[6] Bresch D., Desjardins B. and Gérard-Varet D.(2007), On compressible Navier-Stokes equations with density dependent viscosities in bounded domains,*J. Math. Pures Appl.* **87(2)** , 227-235.

[7] Desjardins B., Esteban M.J. (2000) *On weak solutions for fluid-rigid structure interaction: compressible and incompressible models.* Comm. Partial Diff. Eqs., 25, No 7-8,263-283.

[8] Mellet A. and Vasseur A.(2008), Existence and uniqueness of global strong solutions for one-dimensional compressible Navier-Stokes equations.*SIAM J. Math,Anal.* **39(4)**, 1344-1365.

[9] Roamba B., Zabsonré J. D. D. and Zongo Y. (2017), Weak solutions of one dimensional pollutant transport model. *Annals of the University of Craiova, Mathematics and Computer Science Series*, Vol 44**(1)**), 137-148.

[10] Zabsonré J. D. D. (2012), On the stability of weak solutions of sediment transport models,*Annals of the University of Craiova, Mathematics and Computer Science Serie,* Vol 39**(1)**, 86-96.

[11] Haspot B. (2017), Global existence of strong solution for viscous shallow water system with large initial data on the irrotational part, *Journal of Differential Equations* **262(10)**, 4931-4978.

[12] Haspot B. (2018), Existence of global strong solution for the compressible Navier-Stokes equations with degenerate viscosity coefficients in 1D,*Math. Nachr.,* Vol. 291 **(14-15)**, 2183-2203.

[1] Laboratoire de Mathématiques d'Informatique et Applications (LaMIA),Université Nazi Boni, 01 BP 1091 Bobo-Dioulasso, Burkina Faso.

Email address: braroamba@gmail.com, zjulino@gmail.com, bachirbamos22@gmail.com, jzabsonre@gmail.com

[2]Institut Universitaire de Technologie (IUT), Université Nazi Boni 01 BP 1091 Bobo-Dioulasso, Burkina Faso.

Email address: braroamba@gmail.com, jzabsonre@gmail.com

[3]Ecole Polytechnique de Ouagadougou, 08 PB 143 Ouaga 08, Burkina Faso.

Email address: zongoyac10@gmail.com