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ON THE OSCILLATION OF SOLUTIONS OF Ψ-HILFER GENERALIZED PROPORTIONAL FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this piece of work, we establish some sufficient conditions that lead to oscillation for Ψ-Hilfer generalized proportional fractional differential equations. Through Young's inequality and the Volterra integral equation, we develop new oscillation criteria for the above problem. As a result of this study, we generalize and regain some existing results in the literature because of the suitable selection of the kernel Ψ. Additionally, we provide two examples to demonstrate the usefulness of our findings.

1. Introduction

Nowadays, fractional calculus has become known as a development of classical calculus. Due to its applications in plenty of fields, fractional calculus has gained more attention recently, see [18, 21, 13, 25, 8, 22]. Numerous studies over the past thirty years have concentrated on the qualitative characteristics of differential equation solutions, like oscillation, existence, uniqueness, stability, etc. Over the last thirty years, many researchers have focused on the qualitative analysis of differential equations. One of the remarkable research area is the study of oscillatory behavior. The monographs by Ladde [19], Agarwal [5] and Erbe [10] include further details regarding the oscillatory behavior of integer order differential equations.

Recently, several new fractional derivatives and integrals have grown because each operator includes different kernels, which enlarge the number of definitional possibilities, see [4, 17, 14, 15, 16, 23, 6]. Later, in 2021, Ishfag et al. [20] introduced

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Ψ-Hilfer generalized proportional fractional (Ψ-HGPF) derivative of a function interms of another function, which generalize HGPF derivative provided in [6]. It combines the Riemann-Liouville type proportional fractional operators and Caputo type proportional fractional operators interms of another function as described by the author Jarad et al. [16]. The main benefits of Ψ-HGPF operator is the flexibility in selecting the kernel Ψ and the ability to combine and retrieve previous discoveries in literature.

Then, in 2012, Grace et al. [11] first suggested the idea of the study on oscillatory behavior of fractional initial value problem of the form

$$\begin{cases} \mathcal{D}_b^{\vartheta} y(\omega) + f_1(\omega, y(\omega)) = v(\omega) + f_2(\omega, y(\omega)), \quad \omega > b \geq 0, \\ \lim\limits_{\omega \to b^+} \mathcal{I}_b^{1-\vartheta} y(\omega) = b, \end{cases}$$

and

$$\begin{cases} {}^C\mathcal{D}_b^{\vartheta} y(\omega) + f_1(\omega, y(\omega)) = v(\omega) + f_2(\omega, y(\omega)), \quad \omega > b \geq 0, \\ y^{(k)}(b) = b_j, \quad j = 0, 1, 2, \ldots, n-1, \end{cases}$$

Here $\vartheta \in (0,1), \mathcal{D}_b^{\vartheta}$ indicates fractional derivative of order ϑ of Riemann-Liouville type, $\mathcal{I}_b^{1-\vartheta}$ signifies fractional integral of order $1-\vartheta$ of Riemann-Liouville type, ${}^C\mathcal{D}_b^{\vartheta}$ denotes the fractional derivative of order ϑ ($\vartheta \in (n-1, n]$) of Caputo type, $n \geq 1$ is an integer, $b \in \mathbb{R}$, $f_i \in C([b,\infty) \times \mathbb{R}, \mathbb{R})$.

Followed by, many researchers are interested in analysing and establishing oscillation criteria for fractional order differential equations, see [9, 24, 2, 1, 3, 7].

To the best of our knowledge, we have not found any results regarding the oscillatory behavior of fractional differential equations via the Ψ-HGPF derivative.

In this study, motivated by the above literature, we investigate the oscillatory behavior of Ψ-HGPF inital value problem as follows:

$$\begin{cases} \mathcal{D}_{b^+}^{\vartheta,\varphi,\gamma,\Psi} y(\omega) + g_1(\omega, y) = r(\omega) + g_2(\omega, y), \quad \omega > b \geq 0, \\ \mathcal{I}_{b^+}^{1-\delta,\gamma,\Psi} y(b) = \sum\limits_{l=1}^{m} \kappa_l \; y(\eta_l), \;\; \eta_l \in (b, \omega), \quad \kappa_l \in \mathbb{R}, \end{cases} \tag{1}$$

where $\vartheta \in (0,1)$, $\varphi \in [0,1]$ and $\gamma \in (0,1]$, $\mathcal{D}_{b^+}^{\vartheta,\varphi,\gamma,\Psi}$ represents the left sided Ψ-HGPF derivative of order ϑ and type φ of the function y interms of another function Ψ, $\mathcal{I}_{b^+}^{1-\delta,\gamma,\Psi}$ denotes the left GPF integral of order $1-\delta$ of the function y interms of another function Ψ with $\delta = \vartheta + \varphi(1-\varphi)$, $\eta_l \in (b, \omega)$ satisfying $b < \eta_1 < \eta_2 < \cdots < \eta_m < \omega$ for $l = 1, 2, \cdots, m$, $r \in \mathbb{C}([b, +\infty), \mathbb{R})$, $g_j \in \mathbb{C}([b, +\infty) \times \mathbb{R}, \mathbb{R})$, $j = 1, 2$.

A nontrivial function $y \in \mathbb{C}([b, +\infty), \mathbb{R})$ is said to be a solution of problem (1) if it satisfies (1) for $\omega > b$ and $\mathcal{D}_{b^+}^{\vartheta,\varphi,\gamma,\Psi} y \in \mathbb{C}([b, +\infty), \mathbb{R})$ exists. A solution of problem (1) is said to be oscillatory if it has arbitrarily large zeros on $(0, \infty)$; otherwise it is called nonoscillatory. An equation is said to be oscillatory if all its solutions are oscillatory.

Remaining sections are structured as follows: In Section 2, we present some basic concepts to demonstrate our main results. In Section 3, we illustrate our major findings regarding oscillation criteria. In Section 4, we provide a couple of examples for clarification of the significance of our findings. We highlight the importance of our primary findings for the specific set of parameters in Section 5, and describe how they coincide with Grace's [11] prior findings.

2. Preliminaries

In preliminary, we provide essential concepts to illustrate our main results.

Definition 2.1. [16] *For $\gamma \in [0,1]$, consider $\phi_0 : [0,1] \times \mathbb{R} \to [0,\infty)$ and $\phi_1 : [0,1] \times \mathbb{R} \to [0,\infty)$ are continuous functions such that $\forall\ \omega \in \mathbb{R}$, we have*

$$\lim_{\gamma \to 0^+} \phi_0(\gamma,\omega) = 0,\ \lim_{\gamma \to 0^+} \phi_1(\gamma,\omega) = 1,\ \lim_{\gamma \to 1^-} \phi_0(\gamma,\omega) = 1,\ \lim_{\gamma \to 1^-} \phi_1(\gamma,\omega) = 0.$$

For $\gamma \in (0,1]$ and $\gamma \in [0,1)$, we have

$$\phi_0(\gamma,\omega) \neq 0,$$
$$\phi_1(\gamma,\omega) \neq 0.$$

Let $\Psi(\omega)$ be a strictly increasing positive continuous function. Then the proportional derivative of a function $f(\omega)$ of order γ interms of another function $\Psi(\omega)$ is given by

$$\mathcal{D}^{\gamma,\Psi} f(\omega) = \phi_1(\gamma,\omega) f(\omega) + \phi_0(\gamma,\omega) \frac{f^{'}(\omega)}{\Psi'(\omega)}. \tag{2}$$

Particularly, if $\phi_0(\gamma,\omega) = \gamma$ and $\phi_1(\gamma,\omega) = 1-\gamma$, then equation (2) becomes

$$\mathcal{D}^{\gamma,\Psi} f(\omega) = (1-\gamma) f(\omega) + \gamma \frac{f^{'}(\omega)}{\Psi'(\omega)}. \tag{3}$$

The integral operator regarding to proportional derivative (3) is defined by

$$\mathcal{I}_b^{1,\gamma,\Psi} f(\omega) = \frac{1}{\gamma} \int_b^\omega e^{\frac{\gamma-1}{\gamma}(\Psi(\omega)-\Psi(\nu))} \Psi'(\nu) f(\nu) d\nu, \tag{4}$$

where we assume that $\mathcal{I}_b^{0,\gamma,\Psi} f(\omega) = f(\omega)$.

The generalized proportional integral operator of order m regarding to proportional derivative $\mathcal{D}^{m,\gamma,\Psi}$ is expressed as follows:

$$(\mathcal{I}_b^{m,\gamma,\Psi} f)(\omega) = \frac{1}{\gamma^m \Gamma(m)} \int_b^\omega e^{\frac{\gamma-1}{\gamma}(\Psi(\omega)-\Psi(\nu))} (\Psi(\omega)-\Psi(\nu))^{m-1} \Psi'(\nu) f(\nu) d\nu,$$

where, $\mathcal{D}_b^{m,\gamma,\Psi} = \underbrace{\mathcal{D}^{\gamma,\Psi} \cdot \mathcal{D}^{\gamma,\Psi} \cdot \mathcal{D}^{\gamma,\Psi} \cdots \mathcal{D}^{\gamma,\Psi}}_{m-times}$.

Definition 2.2. [16] *The left-sided generalized fractional proportional integral of order ϑ of the function f interms of another function Ψ is defined as follows:*

$$(\mathcal{I}_{b^+}^{\vartheta,\gamma,\Psi} f)(\omega) = \frac{1}{\gamma^\vartheta \Gamma(\vartheta)} \int_{b^+}^\omega e^{\frac{\gamma-1}{\gamma}(\Psi(\omega)-\Psi(\nu))} (\Psi(\omega)-\Psi(\nu))^{\vartheta-1} \Psi'(\nu) f(\nu) d\nu,\ \omega > b,$$

where $\gamma \in (0,1]$ *and* $\vartheta \in \mathbb{C}$ *with* $Re(\vartheta) > 0$.

Definition 2.3. [16] *The generalized left proportional fractional derivative of order* ϑ *of the function* f *interms of another function* Ψ *is defined as follows:*

$$(\mathcal{D}_{b^+}^{\vartheta,\gamma,\Psi} f)(\omega) = \frac{\mathcal{D}_{\omega}^{m,\gamma,\Psi}}{\gamma^{m-\vartheta}\Gamma(m-\vartheta)} \int_{b^+}^{\omega} e^{\frac{\gamma-1}{\gamma}(\Psi(\omega)-\Psi(\nu))}(\Psi(\omega)-\Psi(\nu))^{m-\vartheta-1}\Psi'(\nu) f(\nu) d\nu,$$

where $\gamma \in (0,1]$, $\vartheta \in \mathbb{C}$, $Re(\vartheta) \geq 0$, $\Psi \in \mathbb{C}[a,b]$, $\Psi^{'}(\nu) > 0$, $\Gamma(\cdot)$ *is the gamma function and* $m = [Re(\vartheta)] + 1$.

Definition 2.4. [20] *Let* $I = [a,b]$, *where* $-\infty \leq a < b \leq \infty$ *be an interval and* $f \in \mathbb{C}^m[a,b]$, $\Psi \in \mathbb{C}^m[a,b]$ *be continuous functions such that* Ψ *is positive, strictly increasing and* $\Psi^{'}(\omega) \neq 0$, *for all* $\omega \in I$. *The* Ψ*-Hilfer generalized proportional fractional derivatives (left-sided/right-sided) of order* ϑ *and type* φ *of* f *interms of another function* Ψ *are defined by*

$$(\mathcal{D}_{b^{\pm}}^{\vartheta,\varphi,\gamma,\Psi} f)(\omega) = \Big(\mathcal{I}_{b^{\pm}}^{\varphi(m-\vartheta),\gamma,\Psi}(\mathcal{D}^{m,\gamma,\Psi})\ \mathcal{I}_{b^{\pm}}^{(1-\varphi)(m-\vartheta),\gamma,\Psi} f\Big)(\omega), \tag{5}$$

where $\vartheta \in (m-1,m)$, $\varphi \in [0,1]$ *with* $m \in \mathbb{N}$ *and* $\gamma \in (0,1]$.

Particularly, if $m = 1$, *then* $\vartheta \in (0,1)$ *and* $\varphi \in [0,1]$, *so* (5) *becomes,*

$$(\mathcal{D}_{b^{\pm}}^{\vartheta,\varphi,\gamma,\Psi} f)(\omega) = \Big(\mathcal{I}_{b^{\pm}}^{\varphi(1-\vartheta),\gamma,\Psi}(\mathcal{D}^{1,\gamma,\Psi})\ \mathcal{I}_{b^{\pm}}^{(1-\varphi)(1-\vartheta),\gamma,\Psi} f\Big)(\omega).$$

Theorem 2.1. [20] *For* $\xi \in \mathbb{R}$ *such that* $\xi > m$, *then the image of the function* $f(\omega) = e^{\frac{\gamma-1}{\gamma}(\Psi(\omega)-\Psi(b))}(\Psi(\omega)-\Psi(b)))^{\xi-1}$ *under the operator* $\mathcal{D}_{b^+}^{\vartheta,\varphi,\gamma,\Psi}$ *is defined by*

$$\mathcal{D}_{b^+}^{\vartheta,\varphi,\gamma,\Psi} f(\omega) = \frac{\gamma^{\vartheta}\Gamma(\xi)}{\Gamma(\xi-\vartheta)} e^{\frac{\gamma-1}{\gamma}(\Psi(\omega)-\Psi(b))}(\Psi(\omega)-\Psi(b))^{\xi-\vartheta-1},$$

where $\vartheta \in (m-1,m)$, $m \in \mathbb{N}$, $\varphi \in [0,1]$, $\gamma \in (0,1]$ *and* $\delta = \vartheta + \varphi(m-\vartheta)$.

Lemma 2.1. [20] *If* $f \in \mathbb{C}_{\delta}[a,b]$ *and* $\mathcal{I}_{b^+}^{m-\delta,\gamma,\Psi} f \in \mathbb{C}_{\delta,\Psi}^{m}[a,b]$, *then*

$$\mathcal{I}_{b^+}^{\vartheta,\gamma,\Psi}\mathcal{D}_{b^+}^{\vartheta,\varphi,\gamma,\Psi} f(\omega) = f(\omega) - \sum_{l=1}^{m} \frac{e^{\frac{\gamma-1}{\gamma}(\Psi(\omega)-\Psi(b))}(\Psi(\omega)-\Psi(b))^{\delta-k}}{\gamma^{\delta-k}\Gamma(\delta-l+1)} (\mathcal{I}_{b^+}^{k-\delta,\gamma,\Psi} f)(b),$$

where $\vartheta \in (m-1,m)$, $m \in \mathbb{N}$, $\gamma \in (0,1], \varphi \in [0,1]$, *with* $\delta = \vartheta + \varphi(m-\vartheta)$ *and* $\delta \in (m-1,m)$.

Lemma 2.2. [12] *For* $U \geq 0$ *and* $V > 0$, *we have*

$$\begin{aligned} (I)\ U^{\lambda} + (\lambda-1)V^{\lambda} - \lambda U V^{\lambda-1} &\geq 0,\ \lambda > 1, \\ (II)\ U^{\lambda} - (1-\lambda)V^{\lambda} - \lambda U V^{\lambda-1} &\leq 0,\ \lambda < 1, \end{aligned}$$

where (I) *and* (II) *holds if and only if* $U = V$.

3. Main Results

In this section, we established sufficient conditions for the solutions of Ψ-HGPF problem (1) to be oscillatory. The following notations are used throughout:

$$\Theta(\nu) = \Psi(\omega) - \Psi(\nu),$$

$$\mathcal{Z}(\nu) = \Psi(\eta_i) - \Psi(\nu).$$

By applying Lemma 2.1, the solution representation of the Ψ-HGPF problem (1) can be expressed as follows:

$$\begin{aligned} y(\omega) &= \frac{\Lambda}{\gamma^{\vartheta}\Gamma(\vartheta)} e^{\frac{\gamma-1}{\gamma}(\Theta(b))}(\Theta(b))^{\delta-1} \\ &\quad \times \sum_{l=1}^{m} \kappa_l \int_{b^+}^{\eta_i} e^{\frac{\gamma-1}{\gamma}(\mathcal{Z}(\nu))}(\mathcal{Z}(\nu))^{\vartheta-1}\Psi'(\nu)\ H(\nu)\ d\nu \\ &\quad + \frac{1}{\gamma^{\vartheta}\Gamma(\vartheta)} \int_{b^+}^{M} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu)\ H(\nu)\ d\nu \\ &\quad + \frac{1}{\gamma^{\vartheta}\Gamma(\vartheta)} \int_{M}^{\omega} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu)\ H(\nu)\ d\nu, \end{aligned} \tag{6}$$

where

$$\Lambda = \frac{1}{\gamma^{\delta-1}\Gamma(\delta) - \sum_{l=1}^{m} \kappa_l\ e^{\frac{\gamma-1}{\gamma}(\mathcal{Z}(b))}(\mathcal{Z}(b))^{\delta-1}}$$

and

$$H(\nu) = r(\nu) + g_2(\nu, y) - g_1(\nu, y).$$

Let us assume the following conditions to prove our results.

(A1) $y.g_j(\omega, y) > 0,\ j = 1, 2,\ y \neq 0,\ \omega \geq 0,$

(A2) $|g_1(\omega, y)| \geq q_1(\omega)|y|^{\mu_1}$ and $|g_2(\omega, y)| \leq q_2(\omega)|y|^{\mu_2},\ \omega \geq 0,$

where $q_1, q_2 \in \mathbb{C}([b, +\infty), \mathbb{R}^+)$, $\mu_1, \mu_2 > 0$ are real numbers.

For our convenience, we define

$$\begin{aligned} \Phi(\omega) &= \Lambda\ e^{\frac{\gamma-1}{\gamma}(\Theta(b))}(\Theta(b))^{\delta-1} \\ &\quad \times \sum_{l=1}^{m} \kappa_l \int_{b^+}^{\eta_i} e^{\frac{\gamma-1}{\gamma}(\mathcal{Z}(\nu))}(\mathcal{Z}(\nu))^{\vartheta-1}\Psi'(\nu)\ H(\nu)\ d\nu \end{aligned} \tag{7}$$

and

$$\Omega(\omega, M) = \int_{b^+}^{M} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu)\ H(\nu)\ d\nu. \tag{8}$$

Theorem 3.2. *Let $g_2 = 0$ and the assumption $(A1)$ holds. If*

$$\lim_{\omega\to\infty} \inf(\Psi(\omega))^{1-\vartheta} \int_{b^+}^{\omega} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu)\ r(\nu)\ d\nu = -\infty \tag{9}$$

and

$$\lim_{\omega\to\infty} \sup(\Psi(\omega))^{1-\vartheta} \int_{b^+}^{\omega} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu)\ r(\nu)\ d\nu = +\infty, \tag{10}$$

for every sufficiently large M, then every solution of problem (1) is oscillatory.

Proof. By contradiction, let us assume that $y(\omega)$ be a nonoscillatory solution of problem (1) with $g_2 = 0$. Consequently, without loss of generality, we may assume that $M > b$ be large enough such that $y(\omega) > 0$ for all $\omega \geq M$. According to the assumption $(A1)$, it is clearly shows that $g_1(\omega, y) > 0$ for $\omega \geq M$. Then from (6), we obtain

$$\begin{aligned} y(\omega) \quad \leq \quad & \frac{\Lambda}{\gamma^{\vartheta}\Gamma(\vartheta)} e^{\frac{\gamma-1}{\gamma}(\Theta(b))}(\Theta(b))^{\delta-1} \\ & \times \sum_{l=1}^{m} \kappa_l \int_{b^+}^{\eta_i} e^{\frac{\gamma-1}{\gamma}(\mathcal{Z}(\nu))}(\mathcal{Z}(\nu))^{\vartheta-1}\Psi'(\nu)\ H(\nu)\ d\nu \\ & + \frac{1}{\gamma^{\vartheta}\Gamma(\vartheta)} \int_{b^+}^{M} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu)\ H(\nu)\ d\nu \\ & + \frac{1}{\gamma^{\vartheta}\Gamma(\vartheta)} \int_{M}^{\omega} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu)\ r(\nu)\ d\nu. \end{aligned} \tag{11}$$

Multiplying the above inequality (11) by $\gamma^{\vartheta}\Gamma(\vartheta)(\Psi(\omega))^{1-\vartheta}$, we get

$$\begin{aligned} \gamma^{\vartheta}\Gamma(\vartheta)(\Psi(\omega))^{1-\vartheta}y(\omega) \quad \leq \quad & (\Psi(\omega))^{1-\vartheta}\Lambda\ e^{\frac{\gamma-1}{\gamma}(\Theta(b))}(\Theta(b))^{\delta-1} \\ & \times \sum_{l=1}^{m} \kappa_l \int_{b^+}^{\eta_i} e^{\frac{\gamma-1}{\gamma}(\mathcal{Z}(\nu))}(\mathcal{Z}(\nu))^{\vartheta-1}\Psi'(\nu)\ H(\nu)\ d\nu \\ & + (\Psi(\omega))^{1-\vartheta} \int_{b^+}^{M} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu)\ H(\nu)\ d\nu \\ & + (\Psi(\omega))^{1-\vartheta} \int_{M}^{\omega} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu)\ r(\nu)\ d\nu, \end{aligned}$$

which implies that

$$\begin{aligned} 0 < \gamma^{\vartheta}\Gamma(\vartheta)(\Psi(\omega))^{1-\vartheta}y(\omega) \quad \leq \quad & (\Psi(\omega))^{1-\vartheta}\Phi(\omega) + (\Psi(\omega))^{1-\vartheta}\Omega(\omega, M) + (\Psi(\omega))^{1-\vartheta} \\ & \times \int_{M}^{\omega} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\ \Psi'(\nu)\ r(\nu)\ d\nu, \end{aligned} \tag{12}$$

where $\Phi(\omega)$ and $\Omega(\omega, M)$ are respectively, defined in (7) and (8).

Take $M_1 \geq M$.

Since $|e^{\frac{\gamma-1}{\gamma}(\Psi(\omega))}| \leq 1$, we have

$$\begin{aligned}(\Psi(\omega))^{1-\vartheta}\ \Phi(\omega) &\leq (\Psi(\omega))^{1-\vartheta}|\Lambda|\sum_{l=1}^{m}\kappa_l\int_{b^+}^{\eta_i}(\mathcal{Z}(\nu))^{\vartheta-1}(\Psi(\nu)-\Psi(b))^{\delta-1}\Psi'(\nu)|H(\nu)|d\nu \\ &= \mathcal{B}(\delta,\vartheta)|\Lambda|\ |H(\nu)|\sum_{l=1}^{m}\kappa_l\left(\frac{\mathcal{Z}(b)}{\Psi(\omega)}\right)^{\vartheta-1} \quad \text{for} \quad \omega \geq M_1,\end{aligned}$$

where $\mathcal{B}(\delta,\vartheta)=\int_0^1 v^{\delta-1}(1-v)^{\vartheta-1}dv$ is the beta function with $Re(\delta), Re(\vartheta)>0$.

Using the monotonicity of Ψ on (b,ω) and $h_1(\omega)=\left(\frac{(\Theta(b))^{\delta}}{\Psi(\omega)}\right)^{\vartheta-1}$ is decreasing function for $0<\vartheta<1$, we have

$$\begin{aligned}(\Psi(\omega))^{1-\vartheta}\ \Phi(\omega) &< \mathcal{B}(\delta,\vartheta)|\Lambda|\ |H(\nu)|\sum_{l=1}^{m}\kappa_l\left(\frac{(\Theta(b))^{\delta}}{\Psi(\omega)}\right)^{\vartheta-1} \\ &\leq \mathcal{B}(\delta,\vartheta)|\Lambda|\ |H(\nu)|\sum_{l=1}^{m}\kappa_l\left(\frac{(\Psi(M_1)-\Psi(b))^{\delta}}{\Psi(M_1)}\right)^{\vartheta-1} \\ &:= C(M_1) \quad \text{for} \quad \omega \geq M_1. \end{aligned} \tag{13}$$

Again since $|e^{\frac{\gamma-1}{\gamma}(\Psi(\omega))}| \leq 1$ and $h_2(\omega)=\left(\frac{\Theta(\nu)}{\Psi(\omega)}\right)^{\vartheta-1}$ is decreasing function for $0<\vartheta<1$, we obtain

$$\begin{aligned}(\Psi(\omega))^{1-\vartheta}\Omega(\omega,M) &\leq (\Psi(\omega))^{1-\vartheta}\int_{b^+}^{M}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu)\ |H(\nu)|\ d\nu \\ &= \int_{b^+}^{M}\left(\frac{\Theta(\nu)}{\Psi(\omega)}\right)^{\vartheta-1}\Psi'(\nu)\ |H(\nu)|\ d\nu \\ &\leq \int_{b^+}^{M}\left(\frac{\Psi(M_1)-\Psi(\nu)}{\Psi(M_1)}\right)^{\vartheta-1}\Psi'(\nu)\ |H(\nu)|\ d\nu \\ &:= C(M,M_1) \quad \text{for} \quad \omega \geq M_1. \end{aligned} \tag{14}$$

Substituting (13) and (14) in (12), we obtain for $\omega \geq M_1$,

$$\begin{aligned}0<\gamma^{\vartheta}\Gamma(\vartheta)(\Psi(\omega))^{1-\vartheta}y(\omega) &\leq C(M_1)+C(M,M_1)+(\Psi(\omega))^{1-\vartheta} \\ &\quad \times\int_{M}^{\omega}e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu)\ r(\nu)\ d\nu, \end{aligned} \tag{15}$$

which implies that

$$\begin{aligned}(\Psi(\omega))^{1-\vartheta}\int_{M}^{\omega}e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu)\ r(\nu)\ d\nu &\geq -[C(M_1)+C(M,M_1)] \\ &> -\infty. \end{aligned} \tag{16}$$

Taking the limit infimum of both sides of inequality (16) as $\omega \to \infty$, we arrive contradiction to condition (9). Similarly, when y is eventually negative, we get a contradiction to condition (10). Consequently, proof is completed. □

Theorem 3.3. *Let the assumptions (A1) and (A2) hold with $\mu_1 > 1$ and $\mu_2 = 1$. If*

$$\lim_{\omega\to\infty} \inf(\Psi(\omega))^{1-\vartheta} \int_{b^+}^{\omega} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu) \left[r(\nu) + H_{\mu_1}(\nu)\right] d\nu = -\infty \quad (17)$$

and

$$\lim_{\omega\to\infty} \sup(\Psi(\omega))^{1-\vartheta} \int_{b^+}^{\omega} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu) \left[r(\nu) + H_{\mu_1}(\nu)\right] d\nu = +\infty, \quad (18)$$

where

$$H_{\mu_1}(\nu) = (\mu_1 - 1)\ (\mu_1)^{\frac{\mu_1}{(1-\mu_1)}}\ q_1^{\frac{1}{(1-\mu_1)}}(\nu)\ q_2^{\frac{\mu_1}{(\mu_1-1)}}(\nu), \quad (19)$$

for every sufficiently large M, then every solution of problem (1) is oscillatory.

Proof. By contradiction, let us assume that $y(\omega)$ be a nonoscillatory solution of problem (1). Consequently, without loss of generality, we may assume that $M > b$ be large enough such that $y(\omega) > 0$ for all $\omega \geq M$. By using the assumptions $(A1)$ and $(A2)$ with $\mu_1 > 1$ and $\mu_2 = 1$, we get

$$g_2(\nu, y) - g_1(\nu, y) \leq q_2(\nu)y(\nu) - q_1(\nu)y^{\mu_1}(\nu). \quad (20)$$

Applying the inequality (20) in (6), we obtain

$$\begin{aligned} y(\omega) \leq\ & \frac{\Lambda}{\gamma^{\vartheta}\Gamma(\vartheta)} e^{\frac{\gamma-1}{\gamma}(\Theta(b))}(\Theta(b))^{\delta-1} \\ & \times \sum_{l=1}^{m} \kappa_l \int_{b^+}^{\eta_i} e^{\frac{\gamma-1}{\gamma}(\mathcal{Z}(\nu))}(\mathcal{Z}(\nu))^{\vartheta-1}\Psi'(\nu)\ H(\nu)\ d\nu \\ & + \frac{1}{\gamma^{\vartheta}\Gamma(\vartheta)} \int_{b^+}^{M} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu)\ H(\nu)\ d\nu \\ & + \frac{1}{\gamma^{\vartheta}\Gamma(\vartheta)} \int_{M}^{\omega} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu) \left[r(\nu)\right] d\nu \\ & + \frac{1}{\gamma^{\vartheta}\Gamma(\vartheta)} \int_{M}^{\omega} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu) \\ & \times \left[q_2(\nu)y(\nu) - q_1(\nu)y^{\mu_1}(\nu)\right] d\nu. \end{aligned} \quad (21)$$

Take

$$\lambda = \mu_1,\ U = q_1^{\frac{1}{\mu_1}} \ \text{ and } \ V = (q_2 q_1^{\frac{-1}{\mu_1}}/\mu_1)^{\frac{1}{(\mu_1-1)}}.$$

Then by using Lemma 2.2 (I), we have

$$q_2(\nu)y(\nu) - q_1(\nu)y^{\mu_1}(\nu) \leq (\mu_1 - 1)\ (\mu_1)^{\frac{\mu_1}{1-\mu_1}}\ q_1^{\frac{1}{1-\mu_1}}(\nu)\ q_2^{\frac{\mu_1}{\mu_1-1}}(\nu). \quad (22)$$

Thus, (21) becomes

$$\begin{aligned} y(\omega) \quad \leq \quad & \frac{\Lambda}{\gamma^{\vartheta}\Gamma(\vartheta)} e^{\frac{\gamma-1}{\gamma}(\Theta(b))}(\Theta(b))^{\delta-1} \\ & \times \sum_{l=1}^{m} \kappa_l \int_{b^+}^{\eta_i} e^{\frac{\gamma-1}{\gamma}(\mathcal{Z}(\nu))}(\mathcal{Z}(\nu))^{\vartheta-1}\Psi'(\nu)\ H(\nu)\ d\nu \\ & + \frac{1}{\gamma^{\vartheta}\Gamma(\vartheta)} \int_{b^+}^{M} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu)\ H(\nu)\ d\nu \\ & + \frac{1}{\gamma^{\vartheta}\Gamma(\vartheta)} \int_{M}^{\omega} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu)\ \Big[r(\nu)\Big]\ d\nu \\ & + \frac{1}{\gamma^{\vartheta}\Gamma(\vartheta)} \int_{M}^{\omega} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu) \\ & \times\Big[(\mu_1-1)\ (\mu_1)^{\frac{\mu_1}{1-\mu_1}}\ q_1^{\frac{1}{1-\mu_1}}(\nu)\ q_2^{\frac{\mu_1}{\mu_1-1}}(\nu)\Big]d\nu. \end{aligned}$$

Multiplying the above inequality by $\gamma^{\vartheta}\Gamma(\vartheta)(\Psi(\omega))^{1-\vartheta}$, we have

$$\begin{aligned} \gamma^{\vartheta}\Gamma(\vartheta)(\Psi(\omega))^{1-\vartheta}y(\omega) \quad \leq \quad & (\Psi(\omega))^{1-\vartheta}\Lambda e^{\frac{\gamma-1}{\gamma}(\Theta(b))}(\Theta(b))^{\delta-1} \\ & \times \sum_{l=1}^{m} \kappa_l \int_{b^+}^{\eta_i} e^{\frac{\gamma-1}{\gamma}(\mathcal{Z}(\nu))}(\mathcal{Z}(\nu))^{\vartheta-1}\Psi'(\nu)\ H(\nu)\ d\nu \\ & + (\Psi(\omega))^{1-\vartheta} \int_{b^+}^{M} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu)\ H(\nu)\ d\nu \\ & + (\Psi(\omega))^{1-\vartheta} \int_{M}^{\omega} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu) \\ & \times\Big[r(\nu)+H_{\mu_1}(\nu)\Big]d\nu, \end{aligned}$$

where $H_{\mu_1}(\nu)$ is defined in (19). We ignore the remaining proof as it is similar to the proof of Theorem 3.2. □

Theorem 3.4. *Let the assumptions* (*A*1) *and* (*A*2) *hold with* $\mu_1 = 1$ *and* $\mu_2 < 1$. *If*

$$\lim_{\omega\to\infty} \inf(\Psi(\omega))^{1-\vartheta} \int_{b^+}^{\omega} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu)\ \Big[r(\nu)+H_{\mu_2}(\nu)\Big]\ d\nu = -\infty \quad (23)$$

and

$$\lim_{\omega\to\infty} \sup(\Psi(\omega))^{1-\vartheta} \int_{b^+}^{\omega} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu)\ \Big[r(\nu)+H_{\mu_2}(u)\Big]\ d\nu = +\infty, \quad (24)$$

where

$$H_{\mu_2}(\nu) = (1-\mu_2)\ (\mu_2)^{\frac{\mu_2}{(1-\mu_2)}}\ q_1^{\frac{\mu_2}{(\mu_2-1)}}(\nu)\ q_2^{\frac{1}{(1-\mu_2)}}(\nu), \quad (25)$$

for every sufficiently large M, *then every solution of problem* (1) *is oscillatory.*

Proof. By contradiction, let us assume that $y(\omega)$ be a nonoscillatory solution of problem (1). Consequently, without loss of generality, we may assume that $M > b$ be large enough such that $y(\omega) > 0$ for all $\omega \geq M$. Applying the assumptions $(A1)$ and $(A2)$ with $\mu_1 = 1$ and $\mu_2 < 1$, we get

$$g_2(\nu, y) - g_1(\nu, y) \leq q_2(\nu)y^{\mu_2}(\nu) - q_1(\nu)y(\nu). \tag{26}$$

Then, using inequality (26) in (6), we obtain

$$\begin{aligned} y(\omega) \leq\ & \frac{\Lambda}{\gamma^{\vartheta}\Gamma(\vartheta)} e^{\frac{\gamma-1}{\gamma}(\Theta(b))}(\Theta(b))^{\delta-1} \\ & \times \sum_{l=1}^{m} \kappa_l \int_{b^+}^{\eta_i} e^{\frac{\gamma-1}{\gamma}(\mathcal{Z}(\nu))}(\mathcal{Z}(\nu))^{\vartheta-1}\Psi'(\nu)\ H(\nu)\ d\nu \\ & + \frac{1}{\gamma^{\vartheta}\Gamma(\vartheta)} \int_{b^+}^{M} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu)\ H(\nu)\ d\nu \\ & + \frac{1}{\gamma^{\vartheta}\Gamma(\vartheta)} \int_{M}^{\omega} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu)\ \Big[r(\nu)\Big]\ d\nu \\ & + \frac{1}{\gamma^{\vartheta}\Gamma(\vartheta)} \int_{M}^{\omega} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu) \\ & \times \Big[q_2(\nu)y^{\mu_2}(\nu) - q_1(\nu)y(\nu)\Big] d\nu. \end{aligned} \tag{27}$$

Take

$$\lambda = \mu_2,\ U = q_2^{\frac{1}{\mu_2}} \ \text{ and } \ V = (q_1 q_2^{\frac{-1}{\mu_2}}/\mu_2)^{\frac{1}{(\mu_2-1)}}.$$

Then by using Lemma 2.2 (II), we get

$$q_2(\nu)y(\nu) - q_1(\nu)y^{\mu_1}(\nu) \leq (1-\mu_2)\ (\mu_2)^{\frac{\mu_2}{(1-\mu_2)}}\ q_1^{\frac{\mu_2}{(\mu_2-1)}}(\nu)\ q_2^{\frac{1}{(1-\mu_2)}}(\nu). \tag{28}$$

Thus, (27) becomes

$$\begin{aligned} y(\omega) \leq\ & \frac{\Lambda}{\gamma^{\vartheta}\Gamma(\vartheta)} e^{\frac{\gamma-1}{\gamma}(\Theta(b))}(\Theta(b))^{\delta-1} \\ & \times \sum_{l=1}^{m} \kappa_l \int_{b^+}^{\eta_i} e^{\frac{\gamma-1}{\gamma}(\mathcal{Z}(\nu))}(\mathcal{Z}(\nu))^{\vartheta-1}\Psi'(\nu)\ H(\nu)\ d\nu \\ & + \frac{1}{\gamma^{\vartheta}\Gamma(\vartheta)} \int_{b^+}^{M} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu)\ H(\nu)\ d\nu \\ & + \frac{1}{\gamma^{\vartheta}\Gamma(\vartheta)} \int_{M}^{\omega} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu)\ \Big[r(\nu)\Big]\ d\nu \\ & + \frac{1}{\gamma^{\vartheta}\Gamma(\vartheta)} \int_{M}^{\omega} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu) \\ & \times \Big[(1-\mu_2)\ (\mu_2)^{\frac{\mu_2}{(1-\mu_2)}}\ q_1^{\frac{\mu_2}{(\mu_2-1)}}(\nu)\ q_2^{\frac{1}{(1-\mu_2)}}(\nu)\Big] d\nu. \end{aligned}$$

Multiplying the above inequality by $\gamma^{\vartheta}\Gamma(\vartheta)(\Psi(\omega))^{1-\vartheta}$, we have

$$\begin{aligned}\gamma^{\vartheta}\Gamma(\vartheta)(\Psi(\omega))^{1-\vartheta}y(\omega) \leq\ & (\Psi(\omega))^{1-\vartheta}\Lambda e^{\frac{\gamma-1}{\gamma}(\Theta(b))}(\Theta(b))^{\delta-1} \\ & \times\sum_{l=1}^{m}\kappa_l\int_{b^+}^{\eta_i} e^{\frac{\gamma-1}{\gamma}(\mathcal{Z}(\nu))}(\mathcal{Z}(\nu))^{\vartheta-1}\Psi'(\nu)\ H(\nu)\ d\nu \\ & +(\Psi(\omega))^{1-\vartheta}\int_{b^+}^{M} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu)\ H(\nu)\ d\nu \\ & +(\Psi(\omega))^{1-\vartheta}\int_{M}^{\omega} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu) \\ & \times\Big[r(\nu)+H_{\mu_2}(\nu)\Big]d\nu.\end{aligned}$$

where $H_{\mu_2}(\nu)$ is defined in (25). We ignore the remaining proof as it is similar to the proof of Theorem 3.2. □

Theorem 3.5. *Let the assumptions* $(A1)$ *and* $(A2)$ *hold with* $\mu_1 > 1$ *and* $\mu_2 < 1$. *If*

$$\lim_{\omega\to\infty}\inf(\Psi(\omega))^{1-\vartheta}\int_{b^+}^{\omega} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu)\ \Big[r(\nu)+H_{\mu_1,\mu_2}(\nu)\Big]\ d\nu=-\infty \quad (29)$$

and

$$\lim_{\omega\to\infty}\sup(\Psi(\omega))^{1-\vartheta}\int_{b^+}^{\omega} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu)\ \Big[r(\nu)+H_{\mu_1,\mu_2}(\nu)\Big]\ d\nu=+\infty, \quad (30)$$

where

$$\begin{aligned}H_{\mu_1,\mu_2}(\nu) =\ & (\mu_1-1)(\mu_1)^{\frac{\mu_1}{(1-\mu_1)}}\ q_1^{\frac{1}{(1-\mu_1)}}(\nu)\ \zeta^{\frac{\mu_1}{(\mu_1-1)}}(\nu) \\ & +(1-\mu_2)\ (\mu_2)^{\frac{\mu_2}{(1-\mu_2)}}\ \zeta^{\frac{\mu_2}{(\mu_2-1)}}(\nu)\ q_2^{\frac{1}{(1-\mu_2)}}(\nu), \end{aligned} \quad (31)$$

for every sufficiently large M, *then every solution of problem* (1) *is oscillatory.*

Proof. By contradiction, let us assume that $y(\omega)$ be a nonoscillatory solution of problem (1). Consequently, without loss of generality, we may assume that $M > b$ be large enough such that $y(\omega) > 0$ for all $\omega \geq M$. By using the assumptions $(A1)$ and $(A2)$ with $\mu_1 > 1$ and $\mu_2 < 1$, we get

$$g_2(\nu,y)-g_1(\nu,y)\leq q_2(\nu)y^{\mu_2}(\nu)-q_1(\nu)y^{\mu_1}(\nu). \quad (32)$$

Then by applying the inequality (32) in (6), we obtain

$$\begin{aligned}
y(\omega) \quad \leq \quad & \frac{\Lambda}{\gamma^{\vartheta}\Gamma(\vartheta)} e^{\frac{\gamma-1}{\gamma}(\Theta(b))}(\Theta(b))^{\delta-1} \\
& \times \sum_{l=1}^{m} \kappa_l \int_{b^+}^{\eta_i} e^{\frac{\gamma-1}{\gamma}(\mathcal{Z}(\nu))}(\mathcal{Z}(\nu))^{\vartheta-1}\Psi'(\nu)\ H(\nu)\ d\nu \\
& + \frac{1}{\gamma^{\vartheta}\Gamma(\vartheta)} \int_{b^+}^{M} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu)\ H(\nu)\ d\nu \\
& + \frac{1}{\gamma^{\vartheta}\Gamma(\vartheta)} \int_{M}^{\omega} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu)\ \Big[r(\nu)\Big]\ d\nu \\
& + \frac{1}{\gamma^{\vartheta}\Gamma(\vartheta)} \int_{M}^{\omega} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu)\ \Big[\zeta(\nu)y(\nu) - q_1(\nu)y^{\mu_1}(\nu)\Big]\ d\nu \\
& + \frac{1}{\gamma^{\vartheta}\Gamma(\vartheta)} \int_{M}^{\omega} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu)\ \Big[q_2(\nu)y^{\mu_2}(\nu) - \zeta(\nu)y(\nu)\Big]\ d\nu. \quad (33)
\end{aligned}$$

Now, by using the inequality (22) with $q_2 = \zeta$, we get

$$\zeta(\nu)y(\nu) - q_1(\nu)y^{\mu_1}(\nu) \leq (\mu_1 - 1)\ (\mu_1)^{\frac{\mu_1}{(1-\mu_1)}}\ q_1^{\frac{1}{(1-\mu_1)}}(\nu)\ \zeta^{\frac{\mu_1}{(\mu_1-1)}}(\nu). \qquad (34)$$

Similarly, by using the inequality (28) with $q_1 = \zeta$, we get

$$q_2(\nu)y^{\mu_2}(\nu) - \zeta(\nu)y(\nu) \leq (1 - \mu_2)\ (\mu_2)^{\frac{\mu_2}{(1-\mu_2)}}\ \zeta^{\frac{\mu_2}{(\mu_2-1)}}(\nu)\ q_2^{\frac{1}{(1-\mu_2)}}(\nu). \qquad (35)$$

Applying the inequalities (34) and (35) in (33), we have

$$\begin{aligned}
y(\omega) \quad \leq \quad & \frac{\Lambda}{\gamma^{\vartheta}\Gamma(\vartheta)} e^{\frac{\gamma-1}{\gamma}(\Theta(b))}(\Theta(b))^{\delta-1} \\
& \times \sum_{l=1}^{m} \kappa_l \int_{b^+}^{\eta_i} e^{\frac{\gamma-1}{\gamma}(\mathcal{Z}(\nu))}(\mathcal{Z}(\nu))^{\vartheta-1}\Psi'(\nu)\ H(\nu)\ d\nu \\
& + \frac{1}{\gamma^{\vartheta}\Gamma(\vartheta)} \int_{b^+}^{M} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu)\ H(\nu)\ d\nu \\
& + \frac{1}{\gamma^{\vartheta}\Gamma(\vartheta)} \int_{M}^{\omega} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu)\ \Big[r(\nu)\Big]\ d\nu \\
& + \frac{1}{\gamma^{\vartheta}\Gamma(\vartheta)} \int_{M}^{\omega} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu)\Big[(\mu_1 - 1)(\mu_1)^{\frac{\mu_1}{(1-\mu_1)}} q_1^{\frac{1}{(1-\mu_1)}}(\nu)\zeta^{\frac{\mu_1}{(\mu_1-1)}}(\nu) \\
& \quad + (1 - \mu_2)(\mu_2)^{\frac{\mu_2}{(1-\mu_2)}} \zeta^{\frac{\mu_2}{(\mu_2-1)}}(\nu) q_2^{\frac{1}{(1-\mu_2)}}(\nu)\Big] d\nu.
\end{aligned}$$

Multiplying the above inequality by $\gamma^{\vartheta}\Gamma(\vartheta)(\Psi(\omega))^{1-\vartheta}$, we have

$$\begin{aligned}\gamma^{\vartheta}\Gamma(\vartheta)(\Psi(\omega))^{1-\vartheta}y(\omega) \quad \leq \quad & (\Psi(\omega))^{1-\vartheta}\Lambda e^{\frac{\gamma-1}{\gamma}(\Theta(b))}(\Theta(b))^{\delta-1} \\ & \times\sum_{l=1}^{m}\kappa_l\int_{b^+}^{\eta_i} e^{\frac{\gamma-1}{\gamma}(\mathcal{Z}(\nu))}(\mathcal{Z}(\nu))^{\vartheta-1}\Psi'(\nu)\ H(\nu)\ d\nu \\ & +(\Psi(\omega))^{1-\vartheta}\int_{b^+}^{M} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu)\ H(\nu)\ d\nu \\ & +(\Psi(\omega))^{1-\vartheta}\int_{M}^{\omega} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu) \\ & \times\Big[r(\nu)+H_{\mu_1,\mu_2}(\nu)\Big]\ d\nu,\end{aligned}$$

where $H_{\mu_1,\mu_2}(\nu)$ is defined in (31). We ignore the remaining proof as it is similar to the proof of Theorem 3.2. □

4. Examples

In this section, we present a couple of examples to highlight its importance of our findings.

Example 1. *Let us take the problem below:*

$$\begin{cases} \mathcal{D}_{0^+}^{\frac{1}{4},\frac{3}{4},1,\Psi}y(\omega)+y(\omega)=\cos(\omega),\ \omega\geq 0, \\ \mathcal{I}_{0^+}^{1-\delta,1,\Psi}x(0)=0. \end{cases} \tag{36}$$

Here, $b=0,\ \vartheta=\frac{1}{4},\ \varphi=\frac{3}{4},\ \gamma=1,\ g_1(\omega,y)=y(\omega),\ r(\omega)=\cos(\omega),\ g_2(\omega,y)=0,$ $m=1,\ \kappa_1=0$ *and* $\eta_1=\frac{1}{6}$.

By choosing $\Psi(\omega)=\omega$, *one can verify that*

$$(\Psi(\omega))^{1-\vartheta}\int_{b^+}^{\omega} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu)\ r(\nu)d\nu=\omega^{\frac{3}{4}}\Big(\int_0^{\omega}(\omega-\nu)^{-\frac{3}{4}}\cos(\nu)d\nu\Big).$$

Therefore,

$$\lim_{\omega\to\infty}\inf(\Psi(\omega))^{1-\vartheta}\int_{b^+}^{\omega} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu)\ r(\nu)d\nu=-\infty$$

and

$$\lim_{\omega\to\infty}\sup(\Psi(\omega))^{1-\vartheta}\int_{b^+}^{\omega} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu)\ r(\nu)d\nu=+\infty.$$

Hence, every solution of the problem (36) *is oscillatory, according to Theorem 3.2.*

Example 2. *Let us take the problem below:*

$$\begin{cases} \mathcal{D}_{0^+}^{\frac{1}{2},\frac{3}{2},1,\Psi}y(\omega)+y^3(\omega)e^{3\omega}=\frac{8}{3\sqrt{\pi}}\omega^{\frac{3}{2}}+\omega^6e^{3\omega}-\omega^2e^{2\omega}+y(\omega)e^{\omega},\ \omega\geq 0, \\ \mathcal{I}_{0^+}^{1-\delta,\gamma,\Psi}x(0)=0. \end{cases} \tag{37}$$

Here, $b = 0,\ \vartheta = \frac{1}{2},\ \varphi = \frac{3}{2},\ \gamma = 1,\ g_1(\omega, y) = y^3(\omega)e^{3\omega},\ r(\omega) = \frac{8}{3\sqrt{\pi}}\omega^{\frac{3}{2}} + \omega^6 e^{3\omega} - \omega^2 e^{2\omega},\ g_2(\omega, y) = y(\omega)e^{\omega},\ m = 1,\ \kappa_1 = 0$ *and* $\eta_1 = \frac{1}{3}$.

By choosing $\mu_1 = 3,\ \mu_2 = 1,\ q_1(\omega) = e^{3\omega}$ *and* $q_2(\omega) = e^{\omega}$, *we get*

$$H_{\mu_1}(\nu) = \frac{2}{3^{\frac{3}{2}}}.$$

In addition, let $\Psi(\omega) = \omega$. *Since* $v(\omega) \geq 0$, *one can verify that*

$$(\Psi(\omega))^{1-\vartheta} \int_{b^+}^{\omega} e^{\frac{\gamma-1}{\gamma}(\Theta(\nu))}(\Theta(\nu))^{\vartheta-1}\Psi'(\nu)\Big[r(\nu) + H_{\mu_1}(\nu)\Big]d\nu$$

$$= (\omega)^{\frac{1}{2}}\left(\int_0^{\omega}(\omega-\nu)^{-\frac{1}{2}}\Big[\frac{8}{3\sqrt{\pi}}\nu^{\frac{3}{2}} + \nu^6 e^{3\nu} - \nu^2 e^{2\nu} + \frac{2}{3^{\frac{3}{2}}}\Big]d\nu\right)$$

$$\geq (\omega)^{\frac{1}{2}}\int_0^{\omega}(\omega-\nu)^{-\frac{1}{2}}\Big(\frac{2}{3^{\frac{3}{2}}}\Big)d\nu$$

$$= \frac{4}{3^{\frac{3}{2}}}\omega.$$

Thus, neither (17) *nor* (18) *is satisfied. Infact, using Theorem 2.1 with* $\xi = 3$, *it is simple to verify that* $y(\omega) = \omega^2$ *is a solution of the problem* (37) *that does not oscillate.*

5. Conclusion

We discussed the oscillatory behavior of Ψ-HGPF initial value problem (1), and established appropriate conditions to ensure the oscillation of solutions to this type of problem. Our contributions using the Ψ-HGPF operator covers the findings mentioned in [11], which are obtained via Riemann-Liouville and Caputo operators. This is particularly true for the specific choice of the parameter $\gamma = 1$ and the kernel $\Psi(\omega) = \omega$. We included additional numerical examples to highlight the importance of the proposed findings.

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