

## Stability for the motion for a symmetric rigid body having a filled hollow with viscous fluid.

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### ABSTRACT

The dynamical rotatory motion of a rigid, symmetrical body about a dynamically symmetric axis is addressed in this study. It is considered that this body has a single-rotor and a filled spherical hollow with highly viscous fluid exhibits spherical motions around its center of mass under the influence of one component of a gyrostatic moment vector. The controlling system of motion is obtained, stationary movements are identified, and analyses of their stabilities are performed. The numerical method of Runge-Kutta from fourth-order (RK-4) is used to calculate the numerical results of this system and they are displayed graphically. Based on the feedback principle, numerous controls that stabilize both stable and unstable stationary motions of the body to asymptotically stable have been identified. The astonishing applications in the industries of gyroscopes and submarines are where this work's significance lies..

### 1. Introduction

Several studies have focused on the spinning motion of a rigid body (RB) about a given point, e.g. [1-13]. The authors considered that the body is subjected to unsettling torques of different types, such as gravitational, electromagnetic, aerodynamic, etc. The study of this problem has received widespread praise for its excellent applications in a variety of sectors that utilize gyroscopic techniques.

When the body's motion is addressed in relation to its center of mass besides the pressure's observation and moments of resistive forces, the rapid motion of an asymmetrical spacecraft is examined in [6]. The averaging system for the Euler-Poisson's equations is achieved and examined. A closer problem was examined in [7], specifically for the scenario of a subject asymmetrical satellite to gravitational action, which is supposed to be a function of angular velocity. The authors addressed the issue of assessing the average duration of a dynamic system under low perturbation that is in a stable region of phase variables [8].

In [9] and [10], the motion of a gyrostic which is considered to be symmetric is studied, where the body rotates initially with a high speed about the axis of dynamic symmetry.

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In light of the averaging method (AM) [11], the desired solutions are obtained in [12] for an electromagnetic gyrostic when the influence of a Newtonian field is considered. The controlling system of motion is transformed into an averaging one. Several transformations are utilized in [13] to obtain the solutions to Euler's dynamic equations. However, the generalization of this problem is examined in [14] when the action of GM is taken into mind. Moreover, some limited cases are examined.

The stability requirements for a symmetric RB rotating around a fixed point with an ellipsoid cavity completely filled with perfect liquid are investigated in [15]. The consistency of a similar issue is investigated in [16-18] for the situation of an ideal incompressible fluid flow filled in the cavity, whether the motion is smooth or not. In [19], the dynamical movement of asymmetric RB link transporting a moving mass and experiencing viscous resistance is examined. For further information on connected matters, see [20-22]. For a few unique circumstances, the motion of an object moves freely as an interior mass in a RB is examined in [23] and [24], while a combined impact of a moving mass and a viscous liquid inside a cavity for the motion of a symmetric body is investigated in [25].

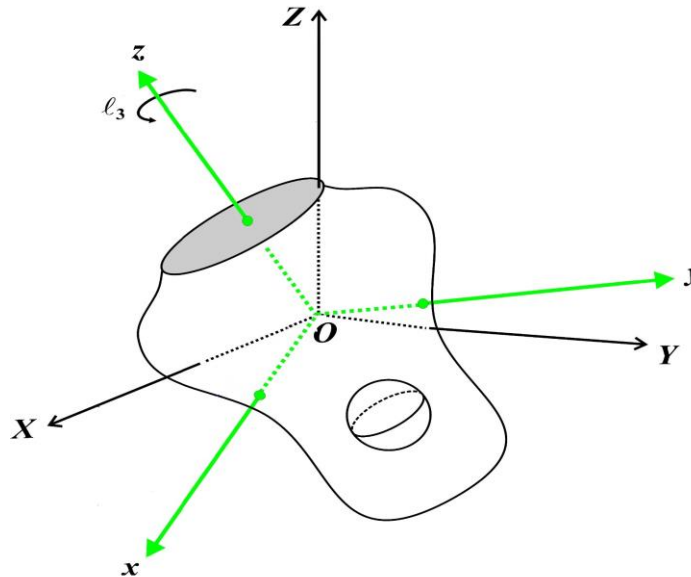
In [26], it is examined how quickly an asymmetric satellite with a cavity rotates in relation to its center of mass when subjected to the combined effects of gravity and light pressure, in which a viscous fluid with a low Reynolds number fills the cavity. The AM [11, 27] is used to obtain the Euler-Poinsot system's averaging system, which is then

solved analytically. Whereas [28] examines a coupled system consisting of a rigid structure with a hole that is totally filled with a viscous liquid. According to [29], the space is thought to be filled with a highly viscous fluid. Recently, the movement of a body contains a spherical cavity completely filled with a viscous liquid is studied in [30]. It is supposed that a movable mass connects to this cavity in which it is attached in double elastically to a located point on the axis of dynamic symmetry and exerting a viscous friction for the motion of this body.

This paper focuses on the dynamical rotational motion of a symmetric RB about a dynamically symmetric axis. This body is thought to have a single rotor, and it exhibits spherical motions around its centre of mass when one component of a gyrostatic moment (GM) vector acts on it. The motion's governing system is found, stationary motions are determined, and investigations of their stabilities are carried out. The numerical results of this system are computed applying RK-4, and they are graphically demonstrated. Many controls that stabilise both stable and unstable stationary motions of the body to asymptotically stable have been found, all of which are based on the feedback principle. The importance of the obtained results rests in its astounding applications in the submarine and gyroscope industries.

## 2. Statement of the problem

In present section, a full description of the studied problem is presented. To achieve this aim, let us consider the rotational movement of RB which is considered to be symmetric around its symmetric axis  $Oz$ . This body contains a carrier which has a filled cavity with highly viscous fluid and a single rotor which they are symmetric about the same axis of rotation. Therefore, we consider that  $(A_1, B_1, C_1)$  and  $(A_2, B_2, C_2)$  are, respectively, the principal inertia moments of the carrier and the rotor, where  $A_1 = B_1 \neq C_1$  and  $A_2 = B_2 \neq C_2$ . In order to attribute the body to fixed and moving coordinate axes, we will assume that the fixed one is  $OXYZ$  and the moving one is  $Oxyz$ , in which it is fixed to the body and rotates with it, as explored in Fig. (1). It should be emphasised that the body's fixed point  $O$ , which is situated on an axis of dynamic symmetry for the RB and rotor, is considered to coincide with the system's centre of mass. The rotation angle  $\sigma$  around the axis  $Oz$  provides a description of the rotor's revolution about the body and the body's movement is assessed in the consistent with a GM  $\ell_3$  about the same axis.



**Fig. (1):** The physical model of the problem

Consequently, the regulating system of motion can be identified as follows [31]

$$\begin{aligned} A\dot{\omega}_1 + (C - B)\omega_2\omega_3 + \omega_2\ell_3 + C_2\omega_2\sigma &= m_x, \\ B\dot{\omega}_2 + (A - C)\omega_1\omega_3 - \omega_1\ell_3 + C_2\omega_1\sigma &= m_y, \\ C\dot{\omega}_3 + (B - A)\omega_1\omega_2 + C_2\dot{\sigma} &= m_z, \end{aligned} \quad (1)$$

$$C_2(\dot{\omega}_3 + \dot{\sigma}) = M_z. \quad (2)$$

Here  $A, B,$  and  $C$  represent the body's principal moments of inertia such that  $A = A_1 + A_2, B = B_1 + B_2,$  and  $C = C_1 + C_2$  along the inertia principal axes  $Ox, Oy,$  and  $Oz,$  respectively, and the over dot is the differentiation regarding time  $t$ . Whereas  $\omega_1, \omega_2,$  and  $\omega_3$  represent the projections of the angular velocity  $\underline{\Omega} = (\omega_1, \omega_2, \omega_3)^T$  on the same axes and the notation  $( )^T$  denotes transposition. Equation (2) indicates the relative rotation of the rotor, where  $M_z$  represents the torque generated by the carrier and acting on the rotor. We'll also take into account the gyrostator motions under the assumption that this torque is zero. Thus Eq. (2) is therefore given as follows

$$\dot{\sigma} = -\dot{\omega}_3, \quad (3)$$

and it can be integrated to yield  $\omega_3(t) + \sigma(t) = \omega_{30} + \sigma_0,$  in which  $\omega_{30} + \sigma_0$  is the integration's constant.

The right side in the previous equation represents the components of the torque forces acting on the carrier from the fluid-filled cavity in the direction of the principal axes. Based on the presented model in [32], they are determined as follows

$$\underline{m} = -\left(\frac{d\underline{L}}{dt} + \underline{\Omega} \times \underline{L}\right), \quad (4)$$

where  $\underline{L}$  is the fluid's moment for spherical cavity which can be defined as follows [32]

$$\underline{L} = -\frac{\rho}{\nu} P \underline{\dot{\Omega}} = -\frac{\rho}{\nu} p E \underline{\dot{\Omega}}. \quad (5)$$

Here,  $p = 8\pi a^7 / 525$  is a coefficient that considers the cavity to have the shape of a sphere with radius  $a,$   $\nu$  represents the fluid's kinematic viscosity,  $\rho$  denotes its density,  $\underline{\dot{\Omega}}$  is the carrier's angular acceleration,  $E$  is defined as the identity matrix. Suppose that the filled fluid in the cavity is highly viscous  $\nu^{-1} \ll 1$ . Therefore, one can rewrite Eq. (4) as follows

$$\underline{m} = -\frac{\rho}{\nu} P \underline{g}, \quad \underline{g} = \begin{bmatrix} \ddot{\omega}_1 + \omega_2 \dot{\omega}_3 - \omega_3 \dot{\omega}_2 \\ \ddot{\omega}_2 + \omega_3 \dot{\omega}_1 - \omega_1 \dot{\omega}_3 \\ \ddot{\omega}_3 + \omega_1 \dot{\omega}_2 - \omega_2 \dot{\omega}_1 \end{bmatrix}, \quad (6)$$

In accordance with the works [32,33], let's formulate the vector components of the carrier's angular acceleration  $\underline{\dot{\Omega}} = (\dot{\omega}_1, \dot{\omega}_2, \dot{\omega}_3)^T$  according to Eq. (1), leaving out  $m = (m_x, m_y, m_z)^T$  due to the tiny value of  $\nu^{-1} \ll 1$ :

$$\begin{aligned} \dot{\omega}_1 &\approx -A^{-1}[(C - B)\omega_2\omega_3 + C_2\omega_2\sigma + \omega_2\ell_3], \\ \dot{\omega}_2 &\approx -B^{-1}[(A - C)\omega_1\omega_3 + C_2\omega_1\sigma - \omega_1\ell_3], \\ \dot{\omega}_3 &\approx -C_1^{-1}(B - A)\omega_1\omega_2. \end{aligned} \quad (7)$$

Differentiating these equations to obtain

$$\begin{aligned} \ddot{\omega}_1 &\approx -A^{-1}[(C - B)(\dot{\omega}_2\omega_3 + \omega_2\dot{\omega}_3) + C_2(\dot{\omega}_2\sigma + \omega_2\dot{\sigma}) + \dot{\omega}_2\ell_3], \\ \ddot{\omega}_2 &\approx -B^{-1}[(A - C)(\dot{\omega}_1\omega_3 + \omega_1\dot{\omega}_3) + C_2(\dot{\omega}_1\sigma + \omega_1\dot{\sigma}) - \dot{\omega}_1\ell_3], \\ \ddot{\omega}_3 &\approx -C_1^{-1}(B - A)(\dot{\omega}_1\omega_2 + \omega_1\dot{\omega}_2). \end{aligned} \quad (8)$$

The substitution of the previous equations into system (1), inserting the following new variable [33]

$$s = [(C - A)\omega_3 + C_2\sigma + \ell_3]A^{-1}, \quad (9)$$

and considering the integrating of Eq. (3) to achieve the following system of three equations

$$\begin{aligned}\dot{\omega}_1 + s\omega_2 &= \frac{\rho P}{\nu A(C_1 - A)} s\omega_1 [C_1 s + C_2(\omega_{30} + \sigma_0) + \ell_3], \\ \dot{\omega}_2 - s\omega_1 &= \frac{\rho P}{\nu A(C_1 - A)} s\omega_2 [C_1 s + C_2(\omega_{30} + \sigma_0) + \ell_3], \\ \frac{C_1}{C_1 - A} \dot{s} &= \frac{\rho P}{\nu A} s(\omega_1^2 - \omega_2^2).\end{aligned}\quad (10)$$

These equations represent the dynamical equations for the motion of the examined body, which they are analogous to the Euler's kinematic equations.

Let's now look at the stationary motions of the body that has a cavity filled with viscous fluid and analyse their stability. Through the use of active control and the feedback principle, we shall convert unstable and stable stationary motions into asymptotically stable ones.

### 3. Stationary motion stability

The goal of the current section is to examine the stability of stationary solutions. To accomplish this goal, the projections of the angular velocity vector  $\underline{\Omega}$  on the principal inertia axes  $Ox, Oy,$  and  $Oz$  must be equal constants, i.e.,  $\dot{\omega}_1 = \dot{\omega}_2 = \dot{\omega}_3 = 0$ . Therefore, the substitution of these equalities into (10) and the solutions of the produced system, yields the following multiplicities of stationary body's motion

$$\begin{aligned}\omega_1 = \omega_2 = 0, \quad s = u = const, \\ \omega_1 = a = const, \quad \omega_2 = b = const, \quad s = 0.\end{aligned}\quad (11)$$

A closer look at the first multiplicity in Eq. (11) and the presence of Eq. (3) makes it clear that there is a manifold of carrier rotation and the body's rotor around the dynamic symmetry axis, where the angular velocities are considered to be arbitrary by means of the modulus and the sign. It is determined in relation to the starting values  $\omega_{30}$  and  $\sigma_0$  of  $\omega_3$  and  $\sigma$ . The second multiplicity in Eq. (11) depicts the body's rotations with the constancy of angular velocities besides the components  $\omega_1 = a = const,$   $\omega_2 = b = const,$  and  $\omega_3 = \omega_{30} = const$ . The rotor revolves in this scenario with a constant angular velocity  $\sigma = \sigma_0 = [(A - C)\omega_{30} - \ell_3]/C_2$ .

Since the second multiplicity was studied in [34], we are going to investigate the stability of the first ones. Therefore, let us inserting the deviations  $x = (x_1, x_2, x_3)^T$  in view of the below formulas

$$\begin{aligned}\omega_1 &= x_1, \\ \omega_2 &= x_2, \\ s &= u + x_3,\end{aligned}\quad (12)$$

one formulates the perturbed equations of the corresponding motion of a body's single-rotor and the equations of first approximation as follows

$$\begin{aligned}\dot{x}_1 &= Mx_1 - ux_2, \\ \dot{x}_2 &= ux_1 + Mx_2, \\ \dot{x}_3 &= 0,\end{aligned}\quad (13)$$

where

$$M = \frac{\rho P}{\nu A(C_1 - A)} [C_1 u^2 + C_2 u(\omega_{30} + \sigma_0) + \ell_3 u].\quad (14)$$

We examine the following real components of the roots by solving the characteristic equation of system (13)

$$\lambda_1 = 0, \quad \text{Re}(\lambda_2) = \text{Re}(\lambda_3) = M.\quad (15)$$

According to Lyapunov's theorem on asymptotic stability, if  $M < 0$ , then it is impossible to draw any conclusions regarding the stability or instability of the corresponding stationary movements by the first approximation [35] due to the existence of  $\lambda_1 = 0$ . Other tools for stability theory should be used to examine such motions. The motions become unstable if  $M$  is greater than 0, i.e. for the case  $M > 0$ . Keeping in mind that, in (9), the equality  $u$  defines the constant  $u = [(C - A)\omega_{30} + C_2\sigma_0 + \ell_3]A^{-1}$ . It is clear from Eq. (14) that the sign of  $M$  is estimated by the sign of starting values  $\omega_{30}$  and  $\sigma_0$ , and the sign of the body's inertia moments and its.

#### 4. Stability of the body's stationary motions

At various  $M$  values, we will address a solution for the problem of stabilizing the initial unstable stationary solution of Eq. (11), in which the second solution was examined in [33]. Let's choose the linear procedure to establish the control utilizing the feedback concept [35]. Let's determine the multiplicity of values for the elements of the matrix  $B$  and apply the linear control deviation  $u = Bx$  to stabilise both stable and unstable stationary movements to asymptotic stability.

Let us consider the choice two conditions for the matrix's structure  $B$ . First, we shall begin by taking the greatest number of zero components to facilitate control. Second, matrix  $B$  needs to satisfy the requirements of fully autonomous linear systems [35]. Therefore, we select  $B$  as follows

$$B = \begin{bmatrix} b_{11} & 0 & 0 \\ b_{21} & 0 & 0 \\ 0 & 0 & b_{33} \end{bmatrix}. \quad (16)$$

The first approximation of equations that correspond to equations (15) takes the form:

$$\begin{aligned} \dot{x}_1 &= (M + b_{11})x_1 - ux_2, \\ \dot{x}_2 &= (u + b_{21})x_1 + Mx_2, \\ \dot{x}_3 &= b_{33}x_3. \end{aligned} \quad (17)$$

Based on the principle of decomposition [35], equations (17) can be divided into two subsystems, in which they can be individually merged in its independently own subspace of each other. First, one can examine the third equation in the system (17), in which  $\lambda = b_{33}$  is the root of characteristic equation. It must be noted that when  $b_{33} < 0$ , the asymmetrically stable case regarding  $x_3$ -coordinate is produced. It is sufficient for the real roots of the following characteristic equation to be negative for the controlled motion to be asymptotically stable with regard to the other variables

$$\lambda^2 - \lambda(2M + b_{11}) + M^2 + u^2 + Mb_{11} + ub_{21} = 0, \quad (18)$$

The criteria of Routh-Hurwitz stability can define a domain of allowable values for the control coefficients  $b_{11}$  and  $b_{21}$  which permits asymptotic stabilization. Therefore, one obtain the below two inequalities

$$\begin{aligned} -2M - b_{11} &> 0, \\ M^2 + F^2 + Mb_{11} + Fb_{21} &> 0. \end{aligned} \quad (19)$$

The following remarks must be taken into account when we deal with the solutions of (19). First, if  $M < 0$  and  $b_{33} < 0$ , we find that the steady motion of the system (10) is asymptotically stable due to the three negative roots. As a result, we are able to assume that  $b_{11} = b_{21} = 0$  (control is only required for the third coordinate). Further system (19) will be analysed at  $M \geq 0$ .

Second, the value of  $u = [(C - A)\omega_3 + C_2\sigma + \ell_3]A^{-1} = const.$ , which depend on the different values of the system parameter besides the conditions of initial motion. System (19) is equivalent to the following equations

$$u > 0, \quad b_{11} < -2M, \quad b_{21} > \frac{M^2 + u^2 + Mb_{11}}{u}, \quad (20)$$

$$u = 0, \quad b_{11} < -2M, \quad 0 < M^2 + Mb_{11}, \quad \forall b_{21}, \tag{21}$$

$$u < 0, \quad b_{11} < -2M, \quad b_{21} < \frac{M^2 + u^2 + Mb_{11}}{-u}. \tag{22}$$

Equations (20), (21), and (22) illustrate the coefficients of multiple control  $b_{11}$  and  $b_{21}$  which solve the issue of maintaining the gyrostat's stationary revolutions.

Keep in mind that, system (21) is inconsistent at  $M \geq 0$  and consequently, it impossible to solve the mentioned problem. The opportunity for the most straightforward selection for the components of control matrix  $B$ , which simplifies the structure of stabilising control, is provided by the straightforward analysis of (20) and (22). For solution of system (20) when  $M = 0$ , then we can reach to  $b_{11} < 0$  and  $b_{21} > -u$ . In this scenario, we can choose  $b_{21} = 0$ . If  $M > 0$ , then  $b_{21} = 0$  can be chosen besides satisfaction of the condition  $M^2 + u^2 + Mb_{11} > 0$ . For the solution of system (22) at  $M = 0$ , it is enough to choose  $b_{11} < 0$  and  $b_{21} = 0$ . If  $M > 0$ , then besides  $b_{11} < -2M$ , the zero coefficient  $b_{21} = 0$  can be selected only with the condition  $M^2 + u^2 + Mb_{11} > 0$ .

### 5. Numerical investigation

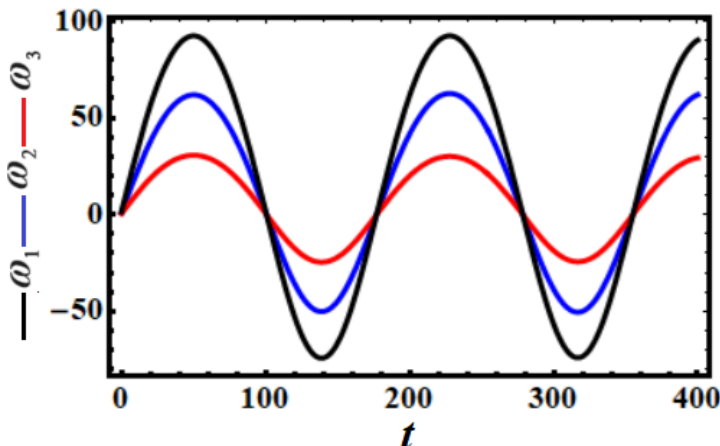
This section's objective is to analysis the graphical representations for the numerical results of the systems of equations (1) and (17) in view of the numerical values for various body's parameters by using the RK-4. Therefore, the below data can be considered

$$\begin{aligned} A &= 10kg.m^2, & B &= 10kg.m^2, & c_1 &= 2kg.m^2, & c_2 &= 6kg.m^2, \\ \sigma &= 50, & \omega_{30} &= 0.001m, & \rho &= 1000, & \nu &= 5000J, & b_{11} &= 0.005, \\ b_{21} &= 0.003, & b_{33} &= -0.0005, & a &= 0.5, & \ell_3 & (= 5, 10, 15)kg.m^2.s^{-1}, \end{aligned} \tag{23}$$

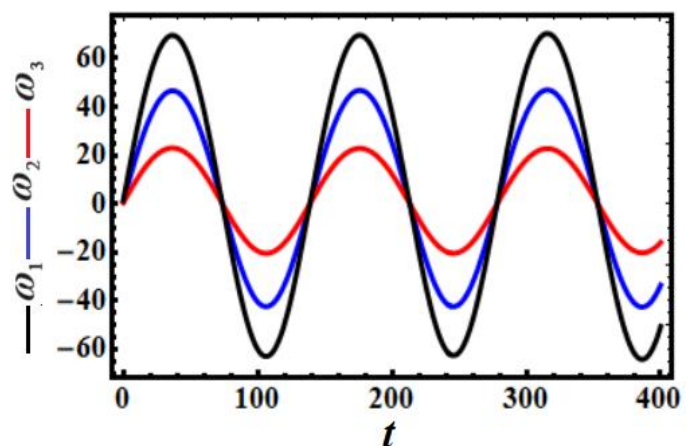
along with the following initial conditions.

Curves of Figs. (2), (3), and (4) show the variations for the numerical solutions  $\omega_1, \omega_2$ , and  $\omega_3$  for the system of equations (1), when  $\ell_3$  equals  $5kg.m^2.s^{-1}$ ,  $10kg.m^2.s^{-1}$ , and  $15kg.m^2.s^{-1}$ , respectively. Periodic standing waves with some nodes are plotted in these figures. The number of waves increases and the corresponding wavelengths decrease with the increase of the values of  $\ell_3$ . In each figure, we conclude that the amplitude for the waves of the solution  $\omega_3$  is less than the amplitudes of the solutions  $\omega_2$  and  $\omega_1$ . The reason backs to the mathematical formulation of equations (1). As a result of this analysis, one says that the behavior of these solutions is stable.

On the other hand, the numerical results of system (17) are graphed in Figs. (5), (6), and (7) to show the behavior of the waves describing  $x_1, x_2$ , and  $x_3$ , respectively. The plotted curves in these figures constitute with the formula of system (17).



**Fig. (2):** Variation of  $\omega_1(t), \omega_2(t)$ , and  $\omega_3(t)$  at  $\ell_3 = 5kg.m^2.s^{-1}$ .



**Fig. (3):** Variation of  $\omega_1(t), \omega_2(t)$ , and  $\omega_3(t)$  at  $\ell_3 = 10kg.m^2.s^{-1}$ .

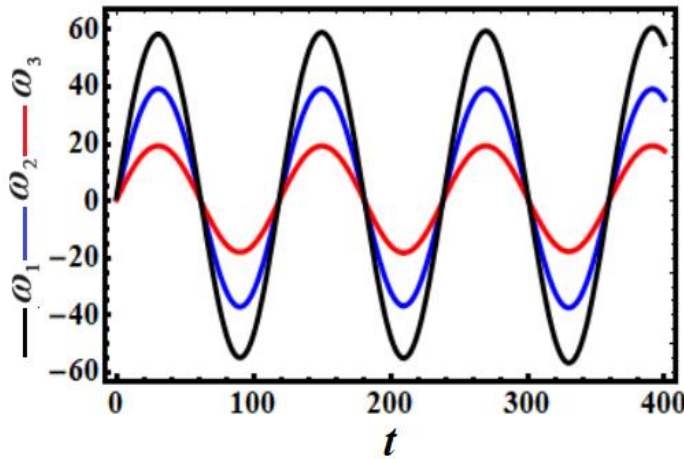


Fig. (4): Variation of  $\omega_1(t), \omega_2(t),$  and  $\omega_3(t)$  at  $\ell_3 = 15kg.m^2.s^{-1}$ .

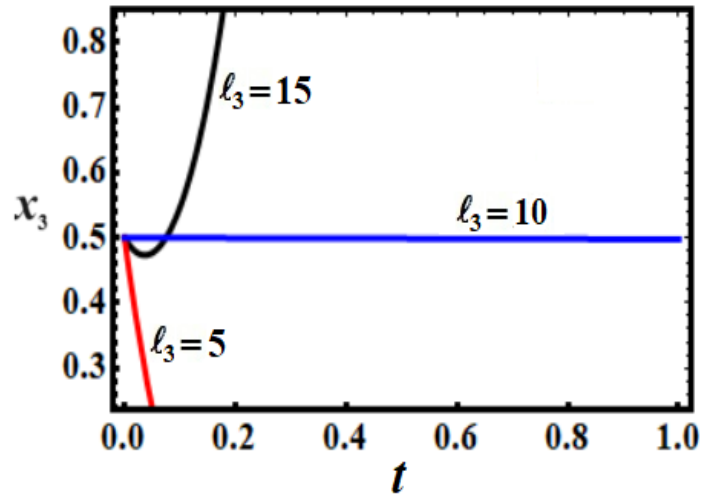


Fig. (7): Variation of  $x_3(t)$  at  $\ell_3 (= 5,10,15)kg.m^2.s^{-1}$ .

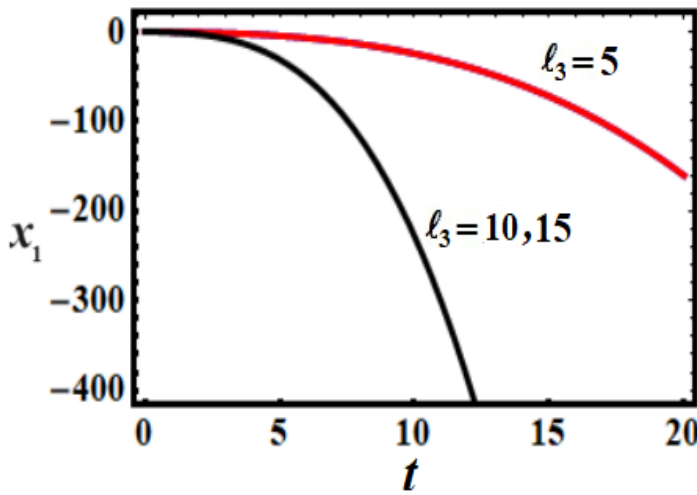


Fig. (5): Variation of  $x_1(t)$  at  $\ell_3 (= 5,10,15)kg.m^2.s^{-1}$ .

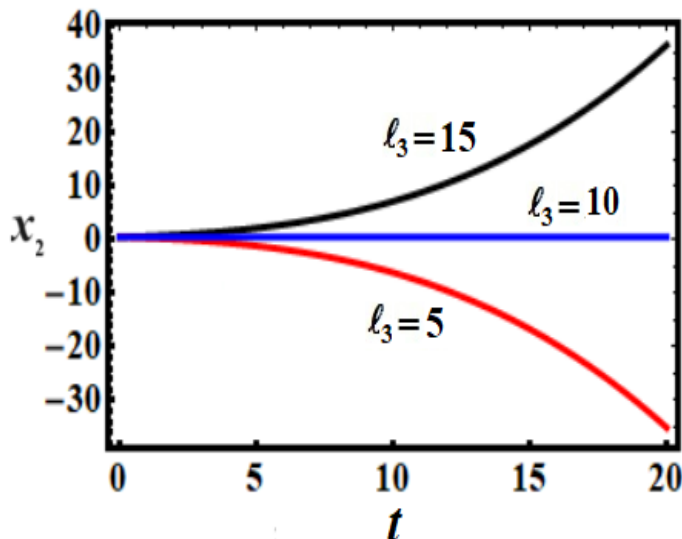


Fig. (6): Variation of  $x_2(t)$  at  $\ell_3 (= 5,10,15)kg.m^2.s^{-1}$ .

6. Conclusions

In this article the dynamic rotational motion of a symmetric RB around a dynamically symmetric axis has been investigated. One component of a GM vector acting on this body, which is considered to have a single rotor, causes it to move spherically around its center of mass. The regulating system of the motion is discovered, stationary movements are identified, and investigations on the stability of those motions proceed. All of the controls that have been discovered to stabilize both stable and unstable stationary motions of the body to asymptotically stable are based on the feedback principle. The significance of the research is found in their amazing applications in the gyroscope and submarine industries.

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