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INCLUSION PROPERTIES FOR CLASSES OF UNIVALENT FUNCTIONS

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Abstract. In this paper, using operator Mowafy et. al. [8], which modified the operator of Al-Aboudi and Al-Amoudi we defined a new operator from which one can obtain many other new operators using the principle of Hadamard product (or convolution) by taking different values of its parameters. This operator has, three recurrence relations. Using one of its these recurrence relation we defined four classes related to univalent starlike and convex functions of order α, close-to-convex and quasi-convex functions of order η and type α ($0 \leq \alpha, \eta < 1$), which in turn generalize many other classes for different values of parameter. For these classes and by using Miller and Mocanu lemma we can obtain many inclusion results. Also some inclusion resulls for Libera integral operator are obtained. We have proved our results here by using the first recurrence relation. Using similar argument by applying the second and third recurrence relation one can get new results.

1. Introduction

Let S be the class of functions

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic and univalent in $U = \{z : z \in \mathbb{C},\ |z| < 1\}$. For $f(z)$ given by (1.1) and $g(z) \in S$ given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \tag{2}$$

the Hadamard product (or convolution) is

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \tag{3}$$

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For $f \in S$ denote by $S^*(\alpha)$ and $K(\alpha)$ $(0 \leq \alpha < 1)$ the classes of univalent starlike and convex of order α respectively (Patil and Thakare [12, *when* p=1], Owa [11, when $p = 1$] satisfying

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha. \tag{4}$$

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha. \tag{5}$$

It follows from (1) and (5) that

$$f(z) \in K(\alpha) \Leftrightarrow zf'(z) \in S^*(\alpha). \tag{6}$$

Note that $K(0) = K$ and $S^*(0) = S^*$ the classes of univalent convex and starlike functions, respectively (see Goodman [5, when $p = 1$]). Also,denote by $C(\eta, \alpha)$ and $C^*(\eta, \alpha)$ the classes of univalent close-to-convex and quasi-convex functions of order η and type α $(0 \leq \alpha, \eta < 1)$ which satisfying, respectively(see [7, *at* p=1], Aouf [2, *at* p=1]

$$\operatorname{Re}\left(\frac{zf'(z)}{g(z)}\right) > \eta. \tag{7}$$

$$\operatorname{Re}\left\{\frac{(zf'(z))'}{g'(z)}\right\} > \eta. \tag{8}$$

It follows from (7) and (8) that

$$f(z) \in C^*(\eta, \alpha) \Leftrightarrow zf'(z) \in C(\eta, \alpha). \tag{9}$$

Mowafy et. al. ([8])defined the operator

$$D^n_{\lambda,\mu} f(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\mu)}{\Gamma(k+1-\mu)} \left[1 + (k-1)\lambda\right]^n a_k z^k, (\lambda \geq 0,\ 0 \leq \mu < 1 \ \text{ and } \ n \in \mathbb{N}_0)$$

which modified the operator of Al-Aboudi and Al-Amoudi ([1]).

Setting

$$\Delta^n_{\lambda,\mu}(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\mu)}{\Gamma(k+1-\mu)} \left[1 + (k-1)\lambda\right]^n z^k.$$

We defined a function $\Delta^{n*}_{\lambda,\mu,\delta}$ by

$$\Delta^n_{\lambda,\mu} * \Delta^{n,*}_{\lambda,\mu,\delta} = \frac{z}{(1-z)^{\delta}} = \sum_{k=2}^{\infty} \frac{(\delta)_{k-1}}{(1)_{k-1}} z^k \ \ (\delta > 0),$$

where

$$(d)_k = \begin{cases} 1 & (k = 0;\ d \in \mathbb{C}^* = \mathbb{C}\setminus\{0\}) \\ d(d+1)...(d+k-1) & (k \in \mathbb{N}_0;\ d \in \mathbb{C}) \end{cases}.$$

and the operator $I^{n,\delta}_{\lambda,\mu} f(z) : S \to S$ as follows:

$$\begin{aligned} I^{n,\delta}_{\lambda,\mu} f(z) &= \Delta^{n*}_{\lambda,\mu,\delta} * f(z) \\ &= z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1-\mu)}{\Gamma(k+1)\Gamma(2-\mu)} \left[\frac{1}{1+(k-1)\lambda}\right]^n \frac{(\delta)_{k-1}}{(1)_{k-1}} a_k z^k, \end{aligned} \tag{10}$$

which has the recurrence relations:

$$\lambda z\left(I_{\lambda,\mu}^{n+1,\delta}f(z)\right)^{'}=I_{\lambda,\mu}^{n,\delta}f(z)-(1-\lambda)\,I_{\lambda,\mu}^{n+1,\delta}f(z), \tag{11}$$

$$z\left(I_{\lambda,\mu+1}^{n,\delta}\left(\delta_1\right)f(z)\right)^{'}=(1-\mu)\,I_{\lambda,\mu}^{n,\delta}f(z)+\mu I_{\lambda,\mu+1}^{n,\delta}f(z), \tag{12}$$

and

$$z\left(I_{\lambda,\mu}^{n,\delta}f(z)\right)^{'}=\delta I_{\lambda,\mu}^{n,\delta+1}f(z)-(\delta-1)\,I_{\lambda,\mu}^{n,\delta}f(z). \tag{13}$$

Using the operator $I_{\lambda,\mu}^{n,\delta}f\left(z\right)$ we introduce the following subclasses of univalent analytic functions for $n\in\mathbb{N}_0$, $\lambda\geq 0$, $0\leq\alpha,\eta,\mu<1$:

$$S_{\lambda,\mu}^{*}\left(n,\delta,\alpha\right)=\left\{f\in S:I_{\lambda,\mu}^{n,\delta}f(z)\in S^{*}\left(\alpha\right)\right\}, \tag{14}$$

$$K_{\lambda,\mu}\left(n,\delta,\alpha\right)=\left\{f\in S:I_{\lambda,\mu}^{n,\delta}f(z)\in K\left(\alpha\right)\right\}, \tag{15}$$

$$C_{\lambda,\mu}\left(n,\delta,\eta,\alpha\right)=\left\{f\in S:I_{\lambda,\mu}^{n,\delta}f(z)\in C\left(\eta,\alpha\right)\right\}. \tag{16}$$

and

$$C_{\lambda,\mu}^{*}\left(n,\delta,\eta,\alpha\right)=\left\{f\in S:I_{\lambda,\mu}^{n,\delta}f(z)\in C^{*}\left(\eta,\alpha\right)\right\}. \tag{17}$$

We not that:

$$f\left(z\right)\in K_{\lambda,\mu}\left(n,\delta,\alpha\right)\Leftrightarrow zf'\left(z\right)\in S_{\lambda,\mu}^{*}\left(n,\delta,\alpha\right), \tag{18}$$

and

$$f\left(z\right)\in C_{\lambda,\mu}^{*}\left(n,\delta,\eta,\alpha\right)\Leftrightarrow zf'\left(z\right)\in C_{\lambda,\mu}\left(n,\delta,\eta,\alpha\right). \tag{19}$$

In this paper, we obtain several inclusion properties of these subclasses and investigate integral operator in them. For more study of the inclusion (see [2, 3, when $p=1$] and [6])

2. Main results

In the reminder we shall assume that $\lambda\geq 0$, $0\leq\alpha,\eta,\mu<1$, $1-\lambda+\lambda\alpha>0$ and $n\in\mathbb{N}_0$.

In order to prove our results, we shall require the following lemma.

Lemma 2.1. ([9]). Let $\varphi(u,v)$ be complex-valued function such that,

$$\varphi:D\longrightarrow\mathbb{C},\ \left(D\subset\mathbb{C}\times\mathbb{C}\right)$$

$\mathbb{C}$ being the complex plane and let $u=u_1+iu_2,\ v=v_1+iv_2$. Suppose that $\varphi(u,v)$ satisfies the following conditions:

i) $\varphi(u,v)$ is continuous in D;

ii) $(1,0)\in D$ and $\mathrm{Re}\left\{\varphi\left(1,0\right)\right\}>0$;

iii) $\mathrm{Re}\left\{\varphi\left(iu_2,v_1\right)\right\}\leq 0$ for all $\left(iu_2,v_1\right)\in D$ and such that $v_1\leq-\frac{1+u_2^2}{2}$.

Let

$$h\left(z\right)=1+h_1z+h_2z^2+... \tag{20}$$

be regular in $\mathbb{U}$ such that $\left(h\left(z\right),\ zh^{'}\left(z\right)\right)\in D$ for all $z\in\mathbb{U}$. If

$$\mathrm{Re}\left\{\varphi\left(h\left(z\right),\ zh^{'}\left(z\right)\right)\right\}>0,$$

then

$$\mathrm{Re}\left\{h\left(z\right)\right\}>0.$$

Theorem 2.1. Let $f \in S$. Then

$$S^{*}_{\lambda,\mu}(n,\ \delta;\alpha) \subset S^{*}_{\lambda,\mu}(n+1,\ \delta;\alpha).$$

Proof. For $f \in S^{*}_{\lambda,\mu}(n,\ \delta\ ;\alpha)$, and

$$\frac{z\left(I^{n+1,\delta}_{\lambda,\mu}f(z)\right)'}{I^{n+1,\delta}_{\lambda,\mu}f(z)} = \alpha + (1-\alpha)\,h(z), \tag{21}$$

where $h(z)$ given by (20). Applying (11) in (21), we have

$$\frac{I^{n,\delta}_{\lambda,\mu}f(z)}{I^{n+1,\delta}_{\lambda,\mu}f(z)} = 1-\lambda+\lambda\alpha+\lambda(1-\alpha)\,h(z). \tag{22}$$

Differentiating (22), we have

$$\frac{z\left(I^{n,\delta}_{\lambda,\mu}f(z)\right)'}{I^{n,\delta}_{\lambda,\mu}f(z)} = \frac{z\left(I^{n+1,a}_{\lambda,\mu}f(z)\right)'}{I^{n+1,a}_{\lambda,\mu}f(z)} + \frac{\lambda(1-\alpha)\,zh'(z)}{1-\lambda+\lambda\alpha+\lambda(1-\alpha)\,h(z)},$$

which, in view of (21), leads to

$$\frac{z\left(I^{n,\delta}_{\lambda,\mu}f(z)\right)'}{I^{n,\delta}_{\lambda,\mu}f(z)} - \alpha = (1-\alpha)\,h(z) + \frac{\lambda(1-\alpha)\,zh'(z)}{1-\lambda+\lambda\alpha+\lambda(1-\alpha)\,h(z)}. \tag{23}$$

Let

$$\varphi(u,\upsilon) = (1-\alpha)\,u + \frac{\lambda(1-\alpha)\,\upsilon}{1-\lambda+\lambda\alpha+\lambda(1-\alpha)\,u}, \tag{24}$$

with $h(z) = u = u_1 + iu_2,\ zh'(z) = \upsilon = \upsilon_1 + i\upsilon_2$. Then

i) $\varphi(u,\upsilon)$ is continuous in $D = \mathbb{C} \diagdown \left\{\frac{1-\lambda+\lambda\alpha}{\lambda(\alpha-1)}\right\} \times \mathbb{C}$,

ii) $(1,0) \in D$ and $\mathrm{Re}\{\varphi(1,0)\} = 1-\alpha$,

iii) $\mathrm{Re}\{\varphi(iu_2,\upsilon_1)\} \le 0$ for all $(iu_2,\upsilon_1) \in D$ and such that $\upsilon_1 \le -\frac{(1+u_2^2)}{2}$,

$$\begin{aligned}
\mathrm{Re}\{\varphi(iu_2,\upsilon_1)\} &= \mathrm{Re}\left\{\frac{\lambda(1-\alpha)\,\upsilon_1}{1-\lambda+\lambda\alpha+\lambda(1-\alpha)\,iu_2}\right\} \\
&= \frac{(1-\lambda+\lambda\alpha)\,\lambda(1-\alpha)\,\upsilon_1}{(1-\lambda+\lambda\alpha)^2+\lambda^2(1-\alpha)^2u_2^2} \\
&< 0,
\end{aligned}$$

for $\upsilon_1 < 0$. Therefore, the function $\varphi(u,\upsilon)$ satisfies the conditions in Lemma 2.1. Thus we have $\mathrm{Re}\{h(z)\} > 0\ \ (z \in \mathbb{U}\ )$, that is, $f \in S^{*}_{\lambda,\mu}(n+1,\ \delta;\alpha)$.

Theorem 2.2. For $f \in S$, we have

$$K_{\lambda,\mu}(n,\delta,\alpha) \subset K_{\lambda,\mu}(n+1,\delta,\alpha)\ \ (0 \le \alpha < 1).$$

Proof. Applying (18) and using Theorem 2.1, we have

$$\begin{aligned}
f(z) \quad &\in \quad K_{\lambda,\mu}(n,\delta,\alpha) \Longleftrightarrow zf'(z) \in S^{*}_{\lambda,\mu}(n,\delta,\alpha) \\
&\Longrightarrow \quad zf'(z) \in S^{*}_{\lambda,\mu}(n+1,\delta,\alpha) \Longleftrightarrow f(z) \in K_{\lambda,\mu}(n+1,\delta,\alpha).
\end{aligned}$$

Theorem 2.3. For $f(z) \in S$, we have

$$C_{\lambda,\mu}(n,\delta,\eta,\alpha) \subset C_{\lambda,\mu}(n+1,\delta,\eta,\alpha)\ \ (\eta \ge 0,\ \ \alpha < 1).$$

Proof. Let $f(z) \in C_{\lambda,\mu}(n,\delta,\eta,\alpha)$. Then, from (16), there exist a function $g(z) \in S^{*}_{\lambda,\mu}(n,\ \delta;\alpha)$, such that

$$\operatorname{Re}\left\{\frac{z\left(I^{n,\delta}_{\lambda,\mu}f(z)\right)'}{I^{n,\delta}_{\lambda,\mu}g(z)}\right\} > \eta\ . \tag{25}$$

Put

$$\frac{z\left(I^{n+1,\delta}_{\lambda,\mu}f(z)\right)'}{I^{n+1,\delta}_{\lambda,\mu}g(z)} = \eta + (1-\eta)\,h(z), \tag{26}$$

where $h(z)$ given by (20). Applying (11), in (26), differentiating the resulting equation and multiplying by z, we have

$$z\left(I^{n,\delta}_{\lambda,\mu}f(z)\,f(z)\right)' = \{\lambda\eta + \lambda(1-\eta)\,h(z)\}\,z\left(I^{n+1,\delta}_{\lambda,\mu}g(z)\right)'$$

$$+\,\lambda(1-\eta)\,zh'(z)\,I^{n+1,\delta}_{\lambda,\mu}g(z) + (1-\lambda)\,z\left(I^{n+1,\delta}_{\lambda,\mu}f(z)\right)'. \tag{27}$$

Since $g \in S^{*}_{\lambda,\mu}(n,\ \delta;\alpha)$, then by Theorem 2.1, we have $g \in S^{*}_{\lambda,\mu}(n+1,\ \delta;\alpha)$. Let

$$\frac{z\left(I^{n+1,\delta}_{\lambda,\mu}g(z)\right)'}{I^{n+1,\delta}_{\lambda,\mu}g(z)} = \alpha + (1-\alpha)\,\hat{H}(z), \tag{28}$$

applying (11) in (28), we have

$$\frac{I^{n,\delta}_{\lambda,\mu}g(z)}{I^{n+1,\delta}_{\lambda,\mu}g(z)} = 1-\lambda+\lambda\alpha+\lambda(1-\alpha)\,\hat{H}(z),\ \ \operatorname{Re}\left\{\hat{H}(z)\right\} > 0. \tag{29}$$

From (27) and (29), we have

$$\frac{z\left(I^{n,\delta}_{\lambda,\mu}f(z)\right)'}{I^{n,\delta}_{\lambda,\mu})g(z)} - \eta = (1-\eta)\,h(z) + \frac{\lambda(1-\eta)\,zh'(z)}{1-\lambda+\lambda\alpha+\lambda(1-\alpha)\,\hat{H}(z)}.$$

Now, Let

$$(u,\ \upsilon) = (1-\eta)\,u + \frac{\lambda(1-\eta)\,\upsilon}{1-\lambda+\lambda\alpha+\lambda(1-\alpha)\,\hat{H}(z)},$$

with $h(z) = u = u_1 + iu_2,\ zh'(z) = \upsilon = \upsilon_1 + i\upsilon_2$. Then

i) (u,υ) is continuous in $D = \mathbb{C}\diagdown\left\{\frac{1-\lambda+\lambda\alpha}{\lambda(\alpha-1)}\right\}\times\mathbb{C}$,

ii) $(1,0) \in D$ and $\operatorname{Re}\{\psi(1,0)\} = 1-\eta$,

iii) $\operatorname{Re}\{\psi(iu_2,\upsilon_1)\} \le 0$ for all $(iu_2,\upsilon_1) \in D$ and such that $\upsilon_1 \le -\frac{(1+u_2^2)}{2}$,

$$\operatorname{Re}\{\psi(iu_2,\upsilon_1)\} = \operatorname{Re}\left\{\frac{\lambda(1-\eta)\left[1-\lambda+\lambda\alpha+\lambda(1-\alpha)\,h_1(x,y)\right]\upsilon_1}{\left[1-\lambda+\lambda\alpha+\lambda(1-\alpha)\,h_1(x,y)\right]^2+\left[\lambda(1-\alpha)\,h_2(x,y)\right]^2}\right\} < 0,$$

for $\upsilon_1 < 0$, and $\hat{H}(z) = h_1(x,y) + ih_2(x,y)$, $h_1(x,y)$ and $h_2(x,y)$ being functions of $x,\ y$ and $\operatorname{Re}\left\{\hat{H}(z)\right\} = h_1(x,y) > 0$. Thus we have $\operatorname{Re}\{h(z)\} > 0$, that is, $f \in C_{\lambda,\mu}(n+1,\delta,\eta,\alpha)$.

Theorem 2.4. For $f(z) \in S$, we have

$$C^*_{\lambda,\mu}(n,\delta,\eta,\alpha) \subset C^*_{\lambda,\mu}(n+1,\delta,\eta,\alpha) \ \ (\eta \geq 0,\ 0 \leq \alpha < 1).$$

Proof. Just as we derived Theorem 2.2 as a consequence of Theorem 2.1 by means of the equivalence (1.18), we can prove Theorem 2.4 by applying analogously to Theorem 2.3 and using the equivalence (19).

Now, we consider the generalized Libera integral operator F_c (see [11]), defined by

$$\begin{aligned} F_c f(z) &= \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)\,dt \\ &= z + \sum_{k=2}^{\infty} \frac{c+1}{c+k} a_k z^k \ (c > -1). \end{aligned} \tag{30}$$

From (30), we have

$$z\left(I^{n,\delta}_{\lambda,\mu}F_c(f)(z)\right)' = (c+1)\, I^{n,\delta}_{\lambda,\mu} f(z) - c I^{n,\delta}_{\lambda,\mu} F_c(f)(z). \tag{31}$$

Theorem 2.5. Let $c+1 > 0$ and $f \in S^*_{\lambda,\mu}(n,\ \delta;\alpha)$, then $F_c(f)(z) \in S^*_{\lambda,\mu}(n,\ \delta;\alpha)$.

Proof. Let $f \in S^*_{\lambda,\mu}(n,\ \delta;\alpha)$ and put

$$\frac{z\left(I^{n,\delta}_{\lambda,\mu}F_c(f)(z)\right)'}{I^{n,\delta}_{\lambda,\mu}F_c(f)(z)} = \alpha + (1-\alpha)\, h(z), \tag{32}$$

where $h(z)$ given by (20). Applying (31) in (32), we have

$$\frac{I^{n,\delta}_{\lambda,\mu}F_c f(z)}{I^{n,\delta}_{\lambda,\mu}F_c(f)(z)} = \frac{1}{c+1}\left\{c + \alpha + (1-\alpha)\, h(z)\right\}. \tag{33}$$

Applying logarithm differentiating, we have

$$\frac{z\left(I^{n,\delta}_{\lambda,\mu} f(z)\right)'}{I^{n,\delta}_{\lambda,\mu} f(z)} = \frac{z\left(I^{n,\delta}_{\lambda,\mu}F_c(f)(z)\right)'}{I^{n,\delta}_{\lambda,\mu}F_c(f)(z)} + \frac{(1-\alpha)\, z h'(z)}{c+\alpha+(1-\alpha)\, h(z)},$$

where, in view of (32), leads to

$$\frac{z\left(I^{n,\delta}_{\lambda,\mu} f(z)\right)'}{I^{n,\delta}_{\lambda,\mu} f(z)} = \alpha + (1-\alpha)\, h(z) + \frac{(1-\alpha)\, z h'(z)}{c+\alpha+(1-\alpha)\, h(z)}. \tag{34}$$

Let

$$\varphi(u,v) = (1-\alpha)\, u + \frac{(1-\alpha)\, v}{c+\alpha+(1-\alpha)\, u}, \tag{35}$$

with $h(z) = u = u_1 + iu_2,\ zh'(z) = v = v_1 + iv_2$. Then

i) $\varphi(u,v)$ is continuous in $D = \mathbb{C} \diagdown \left\{\frac{c+\alpha}{\alpha-1}\right\} \times \mathbb{C}$,

ii) $(1,0) \in D$ and $\operatorname{Re}\{\varphi(1,0)\} = 1-\alpha$,

iii) $\operatorname{Re}\{\varphi(iu_2, v_1)\} \leq 0$ for all $(iu_2, v_1) \in D$ and such that $v_1 \leq -\frac{(1+u_2^2)}{2}$,

$$\begin{aligned}\operatorname{Re}\{\varphi(iu_2,\upsilon_1)\} &= \operatorname{Re}\left\{\frac{(1-\alpha)\upsilon_1}{c+\alpha+(1-\alpha)iu_2}\right\}\\ &= \frac{(c+\alpha)(1-\alpha)\upsilon_1}{(c+\alpha)^2+(1-\alpha)^2u_2^2}\\ &< 0,\end{aligned}$$

for $\upsilon_1 < 0$. Therefore, the function $\varphi(u,v)$ satisfies the conditions in Lemma 2.1. Thus we have $\operatorname{Re}\{h(z)\} > 0$ $(z \in \mathbb{U})$, that is, $F_c(f)(z) \in S^*_{\lambda,\mu}(n,\delta,\alpha)$.

Theorem 2.6. Let $c+1 > 0$ and $f \in K_{\lambda,\mu}(n,\delta,\alpha)$, then $F_c(f)(z) \in K_{\lambda,\mu}(n,\delta,\alpha)$.

Proof. Applying Theorem 2.5 and (1.18), we have

$$\begin{aligned} f(z) &\in K_{\lambda,\mu}(n,\delta,\alpha) \Longleftrightarrow zf'(z) \in S^*_{\lambda,\mu}(n,\delta,\alpha)\\ &\Longrightarrow F_c\left(zf'(z)\right)(z) \in S^*_{\lambda,\mu}(n,\delta,\alpha) \Longleftrightarrow z\left(F_c(f)(z)\right)' \in S^*_{\lambda,\mu}(n,\delta,\alpha)\\ &\Longleftrightarrow F_c(f)(z) \in K_{\lambda,\mu}(n,\delta,\alpha).\end{aligned}$$

Theorem 2.7. Let $c+1 > 0$ and $f \in C_{\lambda,\mu}(n,\delta,\eta,\alpha)$, then $F_c(f)(z) \in C_{\lambda,\mu}(n,\delta,\eta,\alpha)$.

Proof. Let $f \in C_{\lambda,\mu}(n,\delta,\eta,\alpha)$. Then , from (1.16) there exist a function $g(z) \in S^*_{\lambda,\mu}(n,\delta,\alpha)$ such that

$$\operatorname{Re}\left\{\frac{z\left(I^{n,\delta}_{\lambda,\mu}f(z)\right)'}{I^{n,\delta}_{\lambda,\mu}g(z)}\right\} > \eta \ \ (z \in \mathbb{U}).$$

Put

$$\frac{z\left(I^{n,\delta}_{\lambda,\mu}F_c(f)(z)\right)'}{I^{n,\delta}_{\lambda,\mu}F_c(g)(z)} = \eta + (1-\eta)h(z), \tag{36}$$

where $h(z)$ given by (20). Applying (31), in (36), differentiating the resulting equation, we have

$$\begin{aligned}(c+1)z\left(I^{n,\delta}_{\lambda,\mu}(f)(z)\right)' &= \{\eta+(1-\eta)h(z)\}z\left(I^{n,\delta}_{\lambda,\mu}F_c(g)(z)\right)'\\ &+(1-\eta)zh'(z)I^{n,\delta}_{\lambda,\mu}F_c(g)(z) + cz\left(I^{n,\delta}_{\lambda,\mu}F_cf(z)\right)'. \end{aligned}\tag{37}$$

Since $g \in S^*_{p,\lambda}(n,\ \delta;\alpha)$, then by Theorem 2.5, we have $F_c(g)(z) \in S^*_{p,\lambda}(n,\ \delta;\alpha)$. Let

$$\frac{z\left(I^{n,\delta}_{\lambda,\mu}F_c(g)(z)\right)'}{I^{n,\delta}_{\lambda,\mu}F_c(g)(z)} = \alpha + (1-\alpha)\hat{H}(z), \ \ \operatorname{Re}\left\{\hat{H}(z)\right\} > 0. \tag{38}$$

Applying (31) in (38), we have

$$(c+1)\frac{I^{n,\delta}_{\lambda,\mu}(g)(z)}{I^{n,\delta}_{\lambda,\mu}F_c(g)(z)} = c+\alpha+(1-\alpha)\hat{H}(z). \tag{39}$$

From (37) and (39) , we have

$$\frac{z\left(I_{\lambda,\mu}^{n,\delta}f\left(z\right)\right)^{'}}{I_{\lambda,\mu}^{n,\delta}g\left(z\right)}-\eta=\left(1-\eta\right)h\left(z\right)+\frac{\left(1-\eta\right)zh^{'}\left(z\right)}{c+\alpha+\left(1-\alpha\right)\hat{H}\left(z\right)}.$$

Now, let

$$\varphi\left(u,\upsilon\right)=\left(1-\eta\right)u+\frac{\left(1-\eta\right)\upsilon}{c+\alpha+\left(1-\alpha\right)\hat{H}\left(z\right)},$$

it is easy to see that $\varphi\left(u,\upsilon\right)$ satisfies the conditions (i) and (ii) of Lemma 2.1 in $D=\mathbb{C}\diagdown\left\{\frac{c+\alpha}{\alpha-1}\right\}\times\mathbb{C},$. To verify the condition (iii), we proceed as follows:

$$\operatorname{Re}\left\{\varphi\left(iu_2,\upsilon_1\right)\right\}=\operatorname{Re}\left\{\frac{\left(1-\eta\right)\left[c+\alpha+\left(1-\alpha\right)h_1\left(x,y\right)\right]\upsilon_1}{\left[c+\alpha+\left(1-\alpha\right)h_1\left(x,y\right)\right]^2+\left[\left(1-\alpha\right)h_2\left(x,y\right)\right]^2}\right\}<0,$$

for $\upsilon_1<0$ and $\hat{H}\left(z\right)=h_1\left(x,y\right)+ih_2\left(x,y\right)$, $h_1\left(x,y\right)$ and $h_2\left(x,y\right)$ being functions of x, y and $\operatorname{Re}\left\{\hat{H}\left(z\right)\right\}=h_1\left(x,y\right)>0$. Thus we have $Re\left\{h\left(z\right)\right\}>0$, that is, $F_cf\left(z\right)\in C_{\lambda,\mu}\left(n,\delta,\eta,\alpha\right).$

Similary we can prove

Theorem 2.8. Let $c+1>0$ and $f\in C_{\lambda,\mu}^{*}\left(n,\delta,\eta,\alpha\right)$, then $F_cf\left(z\right)\in C_{\lambda,\mu}^{*}\left(n,\delta,\eta,\alpha\right).$

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