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SELF-REFERENCE (STATE-DEPENDENCE) QUADRATIC INTEGRAL EQUATION OF FRACTIONAL ORDERS

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Abstract. The theory of differential equations with deviating arguments is one of the important and significant branches of nonlinear analysis with numerous applications in most fields. Usually, equations of deviating arguments with deviation depend only on the time, however, when the deviation of the arguments depends upon both the state variable x and also the time t, is incredibly important theoretically and practically. Differential equations with state-dependent delays attract the interest of specialists since they widely arise from application models, such as the two-body problem of classical Electrodynamics, which also have many applications, especially in the class of problems that have past memories.

In this paper, we study the existence of solutions of a self-reference (state dependence) quadratic integral equation of fractional orders of the form

$$x(t) = x_0 + I^{\alpha}_{0,t} f_1\big(t, x(x(\phi(t)))\big).\ I^{\beta}_{0,t} f_2\big(t, x(x(\phi(t)))\big)\ t \in [0,T],\ \alpha,\ \beta \in (0,1).$$

The uniqueness of the solution will be studied. The continuous dependence of the unique solution on the initial data and the functions f_1 and f_2 will be proved. Some examples are included. The study establishes conditions for the solution's existence and uniqueness, according to Schauder's fixed point Theorem.

1. Introduction

One of the most important branches of mathematics is the theory of differential and integral equations with deviating arguments especially when the equations have deviating arguments that depend on both the state variable x and the time t, which are called self-reference equations. This type of equations is very important in nonlinear analysis and has many applications, especially in problems that have past memory for example, in hereditary phenomena, see [19, 20, 21]. Several papers

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have appeared that are devoted to such kind of equations, see for example [1-3, 5-12, 16, 17].

Buica [3], studied the following problem

$$\begin{aligned} x'(t) &= f(t, x(x(t))),\ t \in [a,b], \\ x(0) &= x_0 \end{aligned}$$

which is equivalent to the integral equation

$$x(t) \ = \ x_0 \ + \ \int_0^t \ f(s, x(x(s))) \ ds,\ t \in [a,b],$$

the author proved the existence and uniqueness for this equation.
EL-Sayed and Ebead [9], relaxed the assumptions of Buicá and generalized their results, they studied the equation

$$x(t) \ = \ g(t, \ \int_0^t f\big(s, x(x(s))\big) \ ds),\ t \in [0,T].$$

EL-Sayed and Ebead [11], studied the self-reference quadratic integral equation

$$x(t) = a(t) + \int_0^{\phi_1(t)} f_1\big(s, x(x(s))\big) ds \int_0^{\phi_2(t)} f_2\big(s, x(x(s))\big) ds,\ t \in [0,T].$$

EL-Sayed and Hashem[13], studied the fractional order integral equation

$$x(t) = x_0 + I^\alpha f_1\big(t, x(x(\phi(t)))\big).$$

Our aim in this work is to study the existence and uniqueness of the solution $x \in C[0,T]$ of the self-reference QIE of fractional order

$$x(t) = x_0 + I_{0,t}^\alpha f_1\big(t, x(x(\phi(t)))\big).\ I_{0,t}^\beta f_2\big(t, x(x(\phi(t)))\big),\ t \in [0,T],\ \alpha,\ \beta \in (0,1), \quad (1)$$

where x_0 is a constant. Also we study the continuous dependence on the initial data x_0 and the functions $f_1,\ f_2$. To illustrate our results some examples will be given.

Let $C[a,b]$ is class of all continuous functions on the interval $I = [a,b]$, with the norm defined by

$$\|x\| = \sup_{t\ \in [a,b]} |x(t)|$$

Remark 1.

For $x(t) \in [0,T]$ *and* $\phi(t) \in [0,T]$, we can get $|x(t)| \le T,\ |x(x(t))| \le T,\ |x(x(\phi(t)))| \le T$ and

$$\|x\| = sup_{t\in[0,T]}|x(t)| = sup_{x(t)\in[0,T]}\|x(x(t))\| = sup_{x(\phi(t))\in[0,T]}|x(x(\phi(t)))|$$

2. Existence of solution

Consider (1) under the following assumptions:

(1) $f_i : [0,T] \times [0,T] \to R^+$ satisfy Carathéodory condition i.e. f_i are measurable in t for all $x \in C[0,T]$ and continuous in x for almost all $t \in [0,T]$, $i = 1, 2$.

(2) There exist two measurable bounded functions $m_1,\ m_2$ (i.e. $|m_i(t)| \le c_i$ where c_i are two positive constants, $i = 1,\ 2$) and two constants $b_1\ , b_2 > 0$ such that

$$|f_i(t,x)| \le |m_i(t)| + b_i|x|,\ i = 1, 2.$$

(3) $\phi : [0,T] \to [0,T]$ is continuous.
(4) $2L\ T_0 + x_0 \le T$ where $x_0 \ge 0$, $T_0 = \max\{T^\alpha, T^\beta\}$ and $L = 3\frac{M_1 M_2\ T_0}{\Gamma(\alpha+1)\Gamma(\beta+1)}$, where $M_1 = c_1 + b_1 T$ *and* $M_2 = c_2 + b_2 T$.

The next theorem proves the existence of a continuous solution for (1).

Theorem 2.1. *Let the assumptions* $(1)-(4)$ *be satisfied, then QIE (1) has at least one solution* $x \in C[0,T]$.

Proof. Define the set S_L by

$$S_L = \big\{x \in C[0,T] : |x(t_2) - x(t_1)| \le L\ \big[|t_2 - t_1|^\alpha + |t_2 - t_1|^\beta\big] \subset C[0,T]\big\}.$$

Now define the operator F associated with equation (1) by

$$Fx(t) \ = \ x_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1\big(s, x(x(\phi(s)))\big)ds \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2\big(s, x(x(\phi(s)))\big)ds.$$

Let $x \in C[0,T]$, then for $t \in [0,T]$, we get

$$\begin{aligned}
|Fx(t)| \ &\le \ x_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f_1(s, x(x(\phi(s))))| ds \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} |f_2(s, x(x(\phi(s))))| ds \\
&\le \ x_0 + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t (t-s)^{\alpha-1}\{m_1(s) + b_1|x(x(\phi(s)))|\} ds \\
&\quad\ \int_0^t (t-s)^{\beta-1}\{m_2(s) + b_2|x(x(\phi(s)))|\} ds \\
&\le \ x_0 + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t (t-s)^{\alpha-1}\{c_1 + b_1 T\} ds\ \int_0^t (t-s)^{\beta-1}\{c_2 + b_2 T\} ds \\
&= \ x_0 + \frac{M_1\ M_2}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t (t-s)^{\alpha-1} ds\ \ \int_0^t (t-s)^{\beta-1} ds \\
&= \ x_0 + \frac{M_1\ M_2}{\Gamma(\alpha)\Gamma(\beta)} \Big[\frac{-(t-s)^\alpha}{\alpha}\Big]_0^t \Big[\frac{-(t-s)^\beta}{\beta}\Big]_0^t \\
&= \ x_0 + \frac{M_1\ M_2}{\Gamma(\alpha+1)\Gamma(\beta+1)} \Big[t^\alpha\ t^\beta\Big] \\
&\le \ x_0 + \frac{M_1\ M_2}{\Gamma(\alpha+1)\Gamma(\beta+1)} \Big[T^\alpha\ T^\beta\Big] \\
&\le \ x_0 + \frac{M_1\ M_2}{\Gamma(\alpha+1)\Gamma(\beta+1)}\ T_0^2 \\
&\le \ x_0 + 2LT_0 \le \ T.
\end{aligned}$$

This proves that the class $\{Fx\}$ is uniformly bounded on S_L. Now let $x \in S_L$ and $t_1,\ t_2 \in [0,T]$ with $t_1 < t_2$ such that $|t_2 - t_1| < \delta$, then

$$
\begin{aligned}
|Fx(t_2) - Fx(t_1)| = \Big| & \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\, f_1\big(s, x(x(\phi(s)))\big)ds \int_0^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}\, f_2\big(s, x(x(\phi(s)))\big)ds \\
- & \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\, f_1\big(s, x(x(\phi(s)))\big)ds \int_0^{t_1} \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}\, f_2\big(s, x(x(\phi(s)))\big)ds\Big| \\
= & \Big|\Big(\int_0^{t_1} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\, f_1\big(s, x(x(\phi(s)))\big)ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\, f_1\big(s, x(x(\phi(s)))\big)ds\Big) \\
& \Big(\int_0^{t_1} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}\, f_2\big(s, x(x(\phi(s)))\big)ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}\, f_2\big(s, x(x(\phi(s)))\big)ds\Big) \\
- & \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\, f_1\big(s, x(x(\phi(s)))\big)ds \int_0^{t_1} \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}\, f_2\big(s, x(x(\phi(s)))\big)ds\Big| \\
= & \Big|\int_0^{t_1} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\, f_1\big(s, x(x(\phi(s)))\big)ds \int_0^{t_1} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}\, f_2\big(s, x(x(\phi(s)))\big)ds \\
+ & \int_0^{t_1} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\, f_1\big(s, x(x(\phi(s)))\big)ds \int_{t_1}^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}\, f_2\big(s, x(x(\phi(s)))\big)ds \\
+ & \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\, f_1\big(s, x(x(\phi(s)))\big)ds \int_0^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}\, f_2\big(s, x(x(\phi(s)))\big)ds \\
- & \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\, f_1\big(s, x(x(\phi(s)))\big)ds \int_0^{t_1} \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}\, f_2\big(s, x(x(\phi(s)))\big)ds \\
+ & \int_0^{t_1} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\, f_1\big(s, x(x(\phi(s)))\big)ds \int_0^{t_1} \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}\, f_2\big(s, x(x(\phi(s)))\big)ds \\
- & \int_0^{t_1} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\, f_1\big(s, x(x(\phi(s)))\big)ds \int_0^{t_1} \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}\, f_2\big(s, x(x(\phi(s)))\big)ds\Big| \\
\le & \int_0^{t_1} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\, |f_1\big(s, x(x(\phi(s)))\big)|ds \int_{t_1}^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}\, |f_2\big(s, x(x(\phi(s)))\big)|ds \\
+ & \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\, |f_1\big(s, x(x(\phi(s)))\big)|ds \int_0^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}\, |f_2\big(s, x(x(\phi(s)))\big)|ds \\
+ & \int_0^{t_1} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\, |f_1\big(s, x(x(\phi(s)))\big)|ds \Big[\int_0^{t_1} \Big|\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} - \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}\Big|\, |f_2\big(s, x(x(\phi(s)))\big)|ds\Big] \\
+ & \int_0^{t_1} \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}\, |f_2\big(s, x(x(\phi(s)))\big)\ ds \Big[\int_0^{t_1} \Big|\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\Big|\, |f_1\big(s, x(x(\phi(s)))\big)|ds\Big] \\
\le & \int_0^{t_1} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\, \{m_1(s) + b_1|x(x(\phi(s)))|\}ds \int_{t_1}^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}\, \{m_2(s) + b_2|x(x(\phi(s)))|\}ds \\
+ & \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\, \{m_1(s) + b_1|x(x(\phi(s)))|\}ds \int_0^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}\, \{m_2(s) + b_2|x(x(\phi(s)))|\}ds \\
+ & \int_0^{t_1} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\, \{m_1(s) + b_1|x(x(\phi(s)))|\}ds \\
& \int_0^{t_1} \Big(\Big|\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} - \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}\Big|\, \{m_2(s) + b_2|x(x(\phi(s)))|\}\Big)ds \\
+ & \int_0^{t_1} \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}\, \{m_2(s) + b_2|x(x(\phi(s)))|\}ds \\
& \int_0^{t_1} \Big(\Big|\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\Big|\, \{m_1(s) + b_1|x(x(\phi(s)))|\}\Big)ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{M_1M_2}{\Gamma(\alpha)\Gamma(\beta)}\int_0^{t_1} (t_2-s)^{\alpha-1}ds \int_{t_1}^{t_2} (t_2-s)^{\beta-1}ds\\
&+ \frac{M_1M_2}{\Gamma(\alpha)\Gamma(\beta)}\int_{t_1}^{t_2} (t_2-s)^{\alpha-1}ds \int_0^{t_2} (t_2-s)^{\beta-1}ds\\
&+ \frac{M_1M_2}{\Gamma(\alpha)\Gamma(\beta)}\int_0^{t_1} (t_2-s)^{\alpha-1}ds \int_0^{t_1}\{(t_1-s)^{\beta-1}-(t_2-s)^{\beta-1}\}ds\\
&+ \frac{M_1M_2}{\Gamma(\alpha)\Gamma(\beta)}\int_0^{t_1} (t_1-s)^{\beta-1}ds \int_0^{t_1}\{(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\}ds\\
&= \frac{M_1M_2}{\Gamma(\alpha+1)\Gamma(\beta+1)}\big[t_2^\alpha-(t_2-t_1)^\alpha\big]\big[(t_2-t_1)^\beta\big]+\frac{M_1M_2}{\Gamma(\alpha+1)\Gamma(\beta+1)}\big[(t_2-t_1)^\alpha\big][t_2^\beta]\\
&+ \frac{M_1M_2}{\Gamma(\alpha+1)\Gamma(\beta+1)}\big[t_2^\alpha-(t_2-t_1)^\alpha\big]\big[(t_2-t_1)^\beta+t_1^\beta-t_2^\beta\big]\\
&+ \frac{M_1M_2}{\Gamma(\alpha+1)\Gamma(\beta+1)}[t_1^\beta]\big[(t_2-t_1)^\alpha+t_1^\alpha-t_2^\alpha\big]\\
&\leq \frac{M_1M_2}{\Gamma(\alpha+1)\Gamma(\beta+1)}\Big[t_2^\alpha\ (t_2-t_1)^\beta+t_2^\beta\ (t_2-t_1)^\alpha+t_2^\alpha\ (t_2-t_1)^\beta\\
&+ t_2^\alpha\ (t_1^\beta-t_2^\beta)-(t_2-t_1)^\alpha(t_2-t_1)^\beta+(t_2-t_1)^\alpha\ (t_2^\beta-t_1^\beta)\ +t_1^\beta\ (t_2-t_1)^\alpha+t_1^\beta\ (t_1^\alpha-t_2^\alpha)\Big]\\
&\leq \frac{M_1M_2}{\Gamma(\alpha+1)\Gamma(\beta+1)}\Big[t_2^\alpha\ (t_2-t_1)^\beta+t_2^\beta\ (t_2-t_1)^\alpha+t_2^\alpha\ (t_2-t_1)^\beta+(t_2-t_1)^\alpha\ t_2^\beta+t_1^\beta\ (t_2-t_1)^\alpha\Big]\\
&\leq \frac{M_1M_2}{\Gamma(\alpha+1)\Gamma(\beta+1)}\Big[t_2^\alpha\ |t_2-t_1|^\beta+t_2^\beta\ |t_2-t_1|^\alpha+t_2^\alpha\ |t_2-t_1|^\beta+t_2^\beta\ |t_2-t_1|^\alpha+t_1^\beta\ |t_2-t_1|^\alpha)\Big]\\
&\leq 3\frac{M_1M_2}{\Gamma(\alpha+1)\Gamma(\beta+1)}\big[T^\alpha|t_2-t_1|^\beta+T^\beta\ |t_2-t_1|^\alpha\big]\\
&\leq 3\frac{M_1M_2\ T_0}{\Gamma(\alpha+1)\Gamma(\beta+1)}\big[|t_2-t_1|^\beta+|t_2-t_1|^\alpha\big]=L\ \big[|t_2-t_1|^\beta+|t_2-t_1|^\alpha\big].
\end{aligned}
$$

This proves that $Fx(t)\in S_L$, hence $F:S_L\ \to S_L$ and the class of functions $\{Fx\}$ is equi-continuous. Using Arzela-Ascoli Theorem, [16] we find that F is compact. Now we will show that F is continuous, let $\{x_n\}\subset S_L$ such that $x_n\to x_0$ uniformly on $[0,T]$ (i.e. $|x_n(\phi(t))-x_0(\phi(t))|\leq\epsilon_1$) this implies that $|x_n(x_0(\phi(t)))-x_0(x_0(\phi(t)))|\leq\epsilon_2$ for arbitrary $\epsilon_1,\epsilon_2\geq 0$, then

$$
\begin{aligned}
|x_n(x_n(\phi(t)))-x_0(x_0(\phi(t)))| &= |x_n(x_n(\phi(t)))-x_n(x_0(\phi(t)))+x_n(x_0(\phi(t)))-x_0(x_0(\phi(t)))|\\
&\leq |x_n(x_n(\phi(t)))-x_n(x_0(\phi(t)))|+|x_n(x_0(\phi(t)))-x_0(x_0(\phi(t)))|\\
&\leq L|x_n(\phi(t))-x_0(\phi(t))|+|x_n(x_0(\phi(t)))-x_0(x_0(\phi(t)))|,\\
&\leq L\epsilon_1+\epsilon_2.
\end{aligned}
$$

This implies that

$$x_n(x_n(\phi(t)))\to x_0(x_0(\phi(t))).$$

Now $f_i,\ i=1,\ 2$ continuous in the second argument, then

$$f_i\big(t,x_n(x_n(\phi(t)))\big)\to f_i\big(t,x_0(x_0(\phi(t)))\big),\ i=1,\ 2.$$

Thus by using assumption (2) and Lebesgue dominated convergence [4] theorem, we have

$$\begin{aligned}\lim_{n\to\infty} Fx_n(t) &= x_0 + \lim_{n\to\infty}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, f_1\big(s, x_n(x_n(\phi(s)))\big)ds \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\, f_2\big(s, x_n(x_n(\phi(s)))\big)ds\\ &= x_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, f_1\big(s, x_0(x_0(\phi(s)))\big)ds \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\, f_2\big(s, x_0(x_0(\phi(s)))\big)ds\\ &= Fx_0(t)\end{aligned}$$

Then F is continuous. Now all conditions of Schauder's fixed point Theorem [15] are satisfied, then the operator F has at least one fixed point $x \in S$. Consequently, there exists at least one solution $x \in C[0,T]$ of equation (1) which completes the proof. □

3. Uniqueness of the solution

Here, we prove the uniqueness of the solution $x \in C[0,T]$ of (1). For this aim, we assume that

$(1')$ $|f_i(t,x) - f_i(t,y)| \leq b_i\ |x-y|\ i = 1,\ 2.$

$(2')$ $|f_i(t,0)| \leq c_i,$

where $b_i,\ c_i$ are positive constants, $i = 1,\ 2.$

For the uniqueness of the solution of (1), we introduce the next theorem.

Theorem 3.1. *Let the assumptions* $(1), (3), (4),\ (1')$ *and* $(2')$ *be satisfied, if*

$$\frac{(L+1)T_0^2(M_1 b_2 + M_2 b_1)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \leq 1,$$

then equation (1) has a unique solution $x \in C[0,T]$.

Proof. Assumption (2) can be deduced from assumption $(1')$ *and* $(2')$ if we put $y = 0$ in $(1')$, we get

$$\begin{aligned}|f_i(t,x)| &\leq b_i\ |x| + |f_i(t,0)|\\ &\leq b_i\ |x| + c_i\ i = 1,2,\end{aligned} \tag{2}$$

hence all assumptions of theorem (2.1) are satisfied. Then the solution of (1) exists. Now let $x,\ y$ be two solutions of (1), then

$$\begin{aligned}
|x(t)-y(t)| &= \Big| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1\big(s, x(x(\phi(s)))\big)ds \ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2\big(s, x(x(\phi(s)))\big)ds \\
&- \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1\big(s, y(y(\phi(s)))\big)ds \ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2\big(s, y(y(\phi(s)))\big)ds\Big| \\
&= \Big| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1\big(s, x(x(s))\big)\ ds \ \Big[\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \big\{ f_2\big(s, x(x(\phi(s)))\big) - f_2\big(s, y(y(\phi(s)))\big)\big\}\ ds \ \Big] \\
&+ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2\big(s, y(y(\phi(s)))\big)\ ds \ \Big[\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \big\{ f_1\big(s, x(x(\phi(s)))\big) - f_1\big(s, y(y(\phi(s)))\big)\big\}\ ds \ \Big]\Big| \\
&\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f_1\big(s, x(x(s))\big)|\ ds \ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} |f_2\big(s, x(x(\phi(s)))\big) - f_2\big(s, y(y(\phi(s)))\big)|\ ds \\
&+ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} |f_2\big(s, y(y(\phi(s)))\big)|\ ds \ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f_1\big(s, x(x(\phi(s)))\big) - f_1\big(s, y(y(\phi(s)))\big)|\ ds \\
&\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f_1\big(s, x(x(\phi(s)))\big)|\ ds\ b_2 \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} |x\big(x(\phi(s))\big) - y\big(y(\phi(s))\big)|ds \\
&+ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} |f_2\big(s, y(y(\phi(s)))\big)|\ ds\ b_1 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |x\big(x(\phi(s))\big) - y\big(y(\phi(s))\big)|ds.
\end{aligned} \tag{3}$$

But we have

$$\begin{aligned}
|x(x(\phi(s))) - y(y(\phi(s)))| &= |x(x(\phi(s))) - y(y(\phi(s))) + x(y(\phi(s))) - x(y(\phi(s)))| \\
&\le |x(x(\phi(s))) - x(y(\phi(s)))| + |x(y(\phi(s))) - y(y(s\phi(s)))| \\
&\le L|x(\phi(s))) - y(\phi(s))| + |x(y(\phi(s))) - y(y(\phi(s)))| \\
&\le L\|x-y\| \ + \ \|x-y\| = (L+1)\|x-y\|.
\end{aligned} \tag{4}$$

Also, we have

$$\begin{aligned}
|f_i\big(s, x(x(s))\big)| &\le b_i|x(x(\phi(s)))| + c_i \\
&\le b_i\ T + c_i = M_i,\ i = 1,\ 2.
\end{aligned} \tag{5}$$

Substituting by (4) and (5) in (3), we get

$$\begin{aligned}
|x(t)-y(t)| &\le \frac{M_1\ b_2\ (L+1)\ t^\alpha t^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)}\ \|x-y\| \ + \frac{M_2\ b_1\ (L+1)\ t^\alpha t^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)}\ \|x-y\| \\
&\le \frac{(L+1)\ T^\alpha\ T^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)}(M_1\ b_2 + M_2\ b_1)\|x-y\| \\
&\le \frac{(L+1)\ T_0^2}{\Gamma(\alpha+1)\Gamma(\beta+1)}(M_1\ b_2 + M_2\ b_1)\|x-y\|.
\end{aligned}$$

Thus we have

$$\Big(1 - \frac{(L+1)\ T_0^2\ (M_1\ b_2 + M_2\ b_1)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\Big)\ \|x-y\| \le 0.$$

Since

$$\frac{(L+1)\ T_0^2\ (M_1\ b_2 + M_2\ b_1)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \le 1,$$

then we get $x = y$ and hence the solution of (1) is unique solution. □

4. Continuous dependence

4.1. Continuous dependence on the functions f_1 and f_2. Here we prove that the solution of (1) depends continuously on the function f_1.

Definition 4.1. *The solution of (1) depends continuously on the function f_1 if $\forall\, \epsilon_1 > 0\ \exists\ \delta_1(\epsilon_1) > 0$ such that*

$$|f_1(t,x) - f_1^*(t,x)| \leq \delta_1 \Rightarrow \|x - x^*\| \leq \epsilon_1$$

where x^* is the unique solution of

$$x^*(t) = x_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1^*\big(s, x^*(x^*(\phi(s)))\big)ds \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2\big(s, x^*(x^*(\phi(s)))\big)ds,\ t \in [0,T]. \tag{6}$$

For the continuous dependence of the unique solution on the function f_1, we have the following theorem.

Theorem 4.1. *Let the assumptions of Theorem 3.1 be satisfied, assume that $|f_1(t,x) - f_1^*(t,x)| \leq \delta_1$, then the solution of (1) depends continuously on the function f_1.*

Proof. Let $x(t)$ *and* $x^*(t)$ be the two solutions of (1) and (6) respectively, then

$$\begin{aligned}
|x(t)-x^*(t)| &= \Big|x_0+\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1\big(s,x(x(\phi(s)))\big)ds \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2\big(s,x(x(\phi(s)))\big)ds \\
&- \ x_0 \ -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1^*\big(s,x^*(x^*(\phi(s)))\big)ds \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2\big(s,x^*(x^*(\phi(s)))\big)ds\Big| \\
&= \Big|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1\big(s,x(x(\phi(s)))\big)ds \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2\big(s,x(x(\phi(s)))\big)ds \\
&- \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1\big(s,x^*(x^*(\phi(s)))\big)ds \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2\big(s,x(x(\phi(s)))\big)ds \\
&+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1\big(s,x^*(x^*(\phi(s)))\big)ds \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2\big(s,x(x(\phi(s)))\big)ds \\
&- \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1^*\big(s,x^*(x^*(\phi(s)))\big)ds \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2\big(s,x^*(x^*(\phi(s)))\big)ds\Big| \\
&= \Big|\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2\big(s,x(x(\phi(s)))\big)ds \\
&\quad \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Big[f_1\big(s,x(x(\phi(s)))\big)-f_1\big(s,x^*(x^*(\phi(s)))\big)\Big]ds \\
&+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1\big(s,x^*(x^*(\phi(s)))\big)ds \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2\big(s,x(x(\phi(s)))\big)ds \\
&- \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1^*\big(s,x^*(x^*(\phi(s)))\big)ds \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2\big(s,x^*(x^*(\phi(s)))\big)ds \\
&+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1\big(s,x^*(x^*(\phi(s)))\big)ds \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2\big(s,x^*(x^*(\phi(s)))\big)ds \\
&- \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1\big(s,x^*(x^*(\phi(s)))\big)ds \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2\big(s,x^*(x^*(\phi(s)))\big)ds\Big| \\
&= \Big|\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2\big(s,x(x(\phi(s)))\big)ds \\
&\quad \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Big[f_1\big(s,x(x(\phi(s)))\big)-f_1\big(s,x^*(x^*(\phi(s)))\big)\Big]ds \\
&+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1\big(s,x^*(x^*(\phi(s)))\big)ds \\
&\quad \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \Big[f_2\big(s,x(x(\phi(s)))\big)-f_2\big(s,x^*(x^*(\phi(s)))\big)\Big]ds \\
&+ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2\big(s,x^*(x^*(\phi(s)))\big)ds \\
&\quad \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Big[f_1\big(s,x^*(x^*(\phi(s)))\big)-f_1^*\big(s,x^*(x^*(\phi(s)))\big)\Big]ds\Big|
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} |f_2\big(s, x(x(\phi(s)))\big)| ds \\
&\quad \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Big| f_1\big(s, x(x(\phi(s)))\big) - f_1\big(s, x^*(x^*(\phi(s)))\big)\Big| ds \\
&+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f_1\big(s, x^*(x^*(\phi(s)))\big)| ds \\
&\quad \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \Big| f_2\big(s, x(x(\phi(s)))\big) - f_2\big(s, x^*(x^*(\phi(s)))\big)\Big| ds \\
&+ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} |f_2\big(s, x^*(x^*(\phi(s)))\big)| ds \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Big| f_1\big(s, x^*(x^*(\phi(s)))\big) - f_1^*\big(s, x^*(x^*(\phi(s)))\big)\Big| ds \\
&\leq \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \{|f_2(s,0)| + b_2|x(x(\phi(s)))|\} ds \ b_1 \ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \big| x(x(\phi(s))) - x^*(x^*(\phi(s)))\big| ds \\
&+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \{|f_1(s,0)| + b_1|x^*(x^*(\phi(s)))|\} ds \ b_2 \ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \big| x(x(\phi(s))) - x^*(x^*(\phi(s)))\big| ds \\
&+ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \{|f_2(s,0)| + b_2|x^*(x^*(\phi(s)))|\} ds \ \delta_1 \ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds.
\end{aligned}
$$

Using (4), we get

$$
\begin{aligned}
|x(t) - x^*(t)| &\leq \{c_2 + b_2 T\} \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} ds \ b_1 \ (L+1) \ \|x - x^*\| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds \\
&+ \{c_1 + b_1 T\} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds \ b_2 \ (L+1) \ \|x - x^*\| \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} ds \\
&+ \{c_2 + b_2 T\} \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} ds \ \delta_1 \ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds \\
&\leq M_2 \ b_1 \ (L+1) \ \frac{T^\beta}{\Gamma(\beta+1)} \ \frac{T^\alpha}{\Gamma(\alpha+1)} \ \|x - x^*\| \\
&+ M_1 \ b_2 \ (L+1) \ \frac{T^\beta}{\Gamma(\beta+1)} \ \frac{T^\alpha}{\Gamma(\alpha+1)} \ \|x - x^*\| \\
&+ M_2 \ \delta_1 \ \frac{T^\beta}{\Gamma(\beta+1)} \ \frac{T^\alpha}{\Gamma(\alpha+1)} \\
&\leq \frac{T_0^2}{\Gamma(\alpha+1) \ \Gamma(\beta+1)} \ (L+1) \ (M_2 \ b_1 + M_1 \ b_2) \ \|x - x^*\| + M_2 \ \delta_1 \ \frac{T_0^2}{\Gamma(\alpha+1) \ \Gamma(\beta+1)}.
\end{aligned}
$$

Thus we have

$$
\|x - x^*\| \Big(1 - \frac{T_0^2 \ (L+1)}{\Gamma(\alpha+1) \ \Gamma(\beta+1)} \big(M_1 \ b_2 \ + \ M_2 \ b_1 \big) \Big) \leq \ \frac{T_0^2 \ M_2 \ \delta_1}{\Gamma(\alpha+1) \ \Gamma(\beta+1)}.
$$

This implies

$$
\|x - x^*\| \leq \frac{\frac{T_0^2 \ M_2 \ \delta_1}{\Gamma(\alpha+1) \ \Gamma(\beta+1)}}{\Big(1 - \frac{T_0^2 \ (L+1)}{\Gamma(\alpha+1) \ \Gamma(\beta+1)} \big(M_1 \ b_2 \ + \ M_2 \ b_1 \big) \Big)} = \epsilon_1.
$$

Since $\frac{T_0^2 \ (L+1)}{\Gamma(\alpha+1) \ \Gamma(\beta+1)} \big(M_1 \ b_2 \ + \ M_2 \ b_1 \big) \leq 1$, then the solution of (1) depends continuously on the function f_1. □

Corollary 4.0. *Let the assumptions of Theorem 3.1 be satisfied, assume that* $|f_2(t,x) - f_2^*(t,x)| \leq \delta_2$, *then the solution of (1) depends continuously on the function* f_2.

4.2. **Continuous dependence on the initial data.**

Definition 4.2. *The solution of (1) depends continuously on the initial data* x_0 *if* $\forall\ \epsilon_3 > 0\ \exists\ \delta_3(\epsilon_3) > 0$ *such that*

$$|x_0 - x_0^*| \leq \delta_3 \Rightarrow \|x - x^*\| \leq \epsilon_3,$$

where x^* *is the unique solution of*

$$x^*(t) = x_0^* + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1\big(s, x^*(x^*(\phi(s)))\big)ds \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2\big(s, x^*(x^*(\phi(s)))\big)ds,\ t \in [0,T]. \tag{7}$$

Theorem 4.2. *Let the assumptions of Theorem 3.1 be satisfied, assume that* $|x_0 - x_0^*| \leq \delta_3$, *then the solution of (1) depends continuously on* x_0.

Proof. Let $x(t)$ *and* $x^*(t)$ be the two solutions of (1) and (7) respectively, then

$$\begin{aligned}
& |x(t) - x^*(t)| \\
= & \Big| x_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1\big(s, x(x(\phi(s)))\big)ds \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2\big(s, x(x(\phi(s)))\big)ds \\
- & x_0^* - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1\big(s, x^*(x^*(\phi(s)))\big)ds \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2\big(s, x^*(x^*(\phi^*(s)))\big)ds \Big| \\
= & \Big| x_0 - x_0^* + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1\big(s, x(x(\phi(s)))\big)ds \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2\big(s, x(x(\phi(s)))\big)ds \\
- & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1\big(s, x(x(\phi(s)))\big)ds \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2\big(s, x^*(x^*(\phi(s)))\big)ds \\
+ & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1\big(s, x(x(\phi(s)))\big)ds \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2\big(s, x^*(x^*(\phi(s)))\big)ds \\
- & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1\big(s, x^*(x^*(\phi(s)))\big)ds \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2\big(s, x^*(x^*(\phi^*(s)))\big)ds \Big| \\
= & \Big| x_0 - x_0^* + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1\big(s, x(x(\phi(s)))\big)ds \\
& \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \big[f_2\big(s, x(x(\phi(s)))\big) - f_2\big(s, x^*(x^*(\phi(s)))\big)\big]ds \\
+ & \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2\big(s, x^*(x^*(\phi(s)))\big)ds \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \big[f_1\big(s, x(x(\phi(s)))\big) - f_1\big(s, x^*(x^*(\phi(s)))\big)\big]ds \Big|
\end{aligned}$$

$$
\begin{aligned}
&\leq && |x_0 - x_0^*| + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f_1\big(s, x(x(\phi(s)))\big)| ds \\
& && \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \Big| f_2\big(s, x(x(\phi(s)))\big) - f_2\big(s, x^*(x^*(\phi(s)))\big)\Big| ds \\
&+ && \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} |f_2\big(s, x^*(x^*(\phi(s)))\big)| ds \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Big| f_1\big(s, x(x(\phi(s)))\big) - f_1\big(s, x^*(x^*(\phi(s)))\big)\Big| ds \\
&\leq && \delta_3 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \ \{|f_1(t,0)| + b_1 |x(x(\phi(s)))|\} ds \ b_2 \ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \Big| x(x(\phi(s))) - x^*(x^*(\phi(s)))\Big| ds \\
&+ && \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \ \{|f_2(t,0)| + b_2 x^*(x^*(\phi(s)))\} ds \ b_1 \ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Big| x(x(\phi(s))) - x^*(x^*(\phi(s)))\Big| ds \\
&\leq && \delta_3 + \{c_1 + b_1 \ T\} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \ ds \ b_2 \ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \Big| x(x(\phi(s))) - x^*(x^*(\phi(s)))\Big| ds \\
&+ && \{c_2 + b_2 \ T\} \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} ds \ b_1 \ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Big| x(x(\phi(s))) - x^*(x^*(\phi(s)))\Big| ds.
\end{aligned}
$$

Using (4) we get

$$
\begin{aligned}
|x(t) - x^*(t)| &\leq \delta_3 + M_1 \ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \ ds \ b_2 \ (L+1) \ \|x - x^*\| \ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} ds \\
&+ M_2 \ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} ds \ b_1 \ (L+1) \ \|x - x^*\| \ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds \\
&= \delta_3 + M_1 \ \frac{t^\alpha}{\Gamma(\alpha+1)} \ b_2 \ (L+1) \ \|x - x^*\| \ \frac{t^\beta}{\Gamma(\beta+1)} \\
&+ M_2 \ \frac{t^\beta}{\Gamma(\beta+1)} \ b_1 \ (L+1) \ \|x - x^*\| \ \frac{t^\alpha}{\Gamma(\alpha+1)} \\
&\leq \delta_3 + M_1 \ b_2 \ (L+1) \ \frac{T^\alpha \ T^\beta}{\Gamma(\alpha+1) \ \Gamma(\beta+1)} \|x - x^*\| \\
&+ M_2 \ b_1 \ (L+1) \ \frac{T^\alpha \ T^\beta}{\Gamma(\alpha+1) \ \Gamma(\beta+1)} \|x - x^*\| \\
&\leq \delta_3 + \frac{T_0^2 \ (L+1)}{\Gamma(\alpha+1) \ \Gamma(\beta+1)} \big(M_1 \ b_2 \ + \ M_2 \ b_1\big) \|x - x^*\|.
\end{aligned}
$$

Thus we have

$$
\|x - x^*\| \Big(1 - \frac{T_0^2 \ (L+1)}{\Gamma(\alpha+1) \ \Gamma(\beta+1)} \big(M_1 \ b_2 \ + \ M_2 \ b_1\big)\Big) \leq \delta_3.
$$

This implies

$$
\|x - x^*\| \leq \frac{\delta_3}{\Big(1 - \frac{T_0^2 \ (L+1)}{\Gamma(\alpha+1) \ \Gamma(\beta+1)} \big(M_1 \ b_2 \ + \ M_2 \ b_1\big)\Big)} = \epsilon_3.
$$

Since $\frac{T_0^2 \ (L+1)}{\Gamma(\alpha+1) \ \Gamma(\beta+1)} \big(M_1 \ b_2 \ + \ M_2 \ b_1\big) \leq 1$, then the solution of (1) depends continuously on x_0. $\square$

5. Examples

Example 5.1. *Consider the following integral equation*

$$x(t) = \frac{1}{4} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big(\frac{1}{6}s\ e^{-s^2} + \frac{1}{9}\ ln(1+|x(x(s^2))|)\Big)ds$$
$$\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\Big(\frac{1}{16}|sin(5(s+1))| + \frac{1}{7}x(x(s^2))\Big)ds \qquad (8)$$

where $t \in [0, \frac{1}{2}]$, $\alpha = \frac{1}{2}$, $\beta = 0.4$, $\phi(t) = t^2$, $T_0 = \max\{T^{0.5},\ T^{0.4}\} = (\frac{1}{2})^{0.4}$. *Here we have,* $x(0) = \frac{1}{4}$,

$$f_1\big(t, x(x(\phi(t)))\big) = \frac{1}{6}t\ e^{-t^2} + \frac{1}{9}\ ln(1+|x(x(t^2))|),$$

thus

$$|f_1\big(t, x(x(\phi(t)))\big)| \leq \frac{1}{6}t\ e^{-t^2} + \frac{1}{9}|x(x(t^2))|$$

and $m_1(t) = \frac{1}{6}t\ e^{-t^2}$, $c_1 = \frac{1}{12}$, $b_1 = \frac{1}{9}$

$$f_2\big(t, x(x(\phi(t)))\big) = \frac{1}{16}|sin(5(t+1))| + \frac{1}{7}x(x(t^2)),$$

thus

$$|f_2\big(t, x(x(\phi(t)))\big)| \leq \frac{1}{16}|sin(5(t+1))| + \frac{1}{7}|x(x(t^2))|$$

and $m_2(t) = \frac{1}{16}|sin(5(t+1))|$, $c_2 = \frac{1}{16}$, $b_2 = \frac{1}{7}$. *So we get* $M_1 \simeq 0.1389$, $M_2 \simeq 0.1339$, *and* $L \simeq 0.053$ *also we have*

$$2L\ T_0 + |x(0)| \simeq 0.33 \leq \frac{1}{2}.$$

Now clear that all assumptions of theorem 2.1 are satisfied, then equation (8) has at least one solution $x \in C[0, \frac{1}{2}]$.

Example 5.2. *Consider the following equation*

$$x(t) = \frac{1}{8} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big(\frac{1}{32}s + \frac{3}{32}x(x(\gamma s))\Big)\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\Big(\frac{1}{12}s + \frac{3}{12}x(x(\gamma s))\Big) \qquad (9)$$

where $\gamma \in (0,1]$, $t \in [0,2]$, $\phi(t) = \gamma\ t$, $\alpha = 0.1$, $\beta = \frac{1}{4}$. *Here we have* $T_0 = \max\{T^{0.1},\ T^{0.25}\} = 2^{0.25}$, $x(0) = \frac{1}{8}$,

$$f_1(t,x) = \frac{1}{32}t + \frac{3}{32}x$$

so that

$$|f_1(t,x) - f_1(t,y)| \leq \frac{3}{32}|x-y|,$$

thus $b_1 = \frac{3}{32}$,

$$f_2(t,x) = \frac{1}{12}t + \frac{3}{12}x$$

so that

$$|f_2(t,x) - f_2(t,y)| \leq \frac{3}{12}|x-y|,$$

thus $b_2 = \frac{3}{12}$. *Also, we have*

$$|f_1(t,0)| \leq \frac{1}{16} = c_1,$$

$$|f_2(t,0)| \leq \frac{1}{6} = c_2,$$

so that we have $M_1 = \frac{1}{4}, M_2 = \frac{2}{3}$ *and* $L \simeq 0.69$. *Moreover we have*

$$2L\ T_0 + |x(0)| \simeq 1.766 \leq 2$$

and

$$\frac{(L+1)\ T_0^2\ (M_1\ b_2 + M_2\ b_1)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \simeq 0.3465 < 1.$$

Clear all assumptions of theorem 3.1 are satisfied, then equation (9) has a unique solution $x \in C[0,2]$.

6. Conclusion

Here, we considered a self-reference quadratic integral equation of fractional order. The existence of solutions has been studied. The uniqueness of the solution has been proved. The continuous dependence of the unique solution on the initial data and the functions f_1 and f_2 has been analyzed. To illustrate our results, some examples have been introduced.

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