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كلية التجارة – جامعة طنطا

العدد : الرابع

ديسمبر 2023  
(الجزء الثاني)

# **Bayesian GLS Identification of Autoregressive Moving Average Models**

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## Abstract

In this article, a new Bayesian approach is used to identify the autoregressive moving average models. Employing approximation error is the foundation of the suggested Bayesian methodology. We take into consideration presence of an approximation error when substituting lagged errors of the original autoregressive moving average model with suitably lagged residuals from along autoregression . The direct Bayesian identification approach is utilized for solving the Bayesian identification issue of autoregressive moving average processes employing both informative and non-informative priors. The theoretical derivations of the direct Bayesian identification approach are carried out utilizing the aforementioned priors. We compare the effectiveness of the Broemeling and Shaarawy approach with proposed Bayesian approach for determining the orders of autoregressive moving average models by utilizing an actual data set and numerous simulated experiments. The outcomes of simulations and actual data demonstrate that the suggested approach is superior to the Broemeling and Shaarawy approach for determining the orders of autoregressive moving average processes.

**MSC:** 62M10, 62C10, 37M10, 91B84.

**Keywords:** Autoregressive Moving Average Models, Generalized Least Squares (GLS) Approach, Approximate Error, Prior Distribution, Posterior Distribution, Bayesian Identification.

## 1.Introduction

Time series models play a significant position in the modeling of time series data in several fields. The literature on time series analysis can be classified into two categories: non-Bayesian (classical) and Bayesian approaches. Box –Jenkins [1976] is the most well-known non-Bayesian approach for analyzing time series data. It is composed of four stages: identification, estimation, diagnostic checking, and forecasting. In contrast, the foundation of Bayesian time series analysis is Bayes' theorem. This theorem integrates the likelihood function containing observable sample information (data) with the prior parameter distribution to acquire the

posterior distribution. Identification is the first step of time series analysis and is an essential position in time series analysis, and the accuracy of the subsequent steps depends on it. Identifying an autoregressive moving average processes involves determining the orders  $p$  and  $q$  of an autoregressive moving average models. There is no ultimate optimum identification approach. Therefore, this article focuses solely on the Bayesian identification of ARMA models applying the proposed Bayesian methodology, denoted by the Bayesian Generalized Least Squares (BGLS) approach, utilizing the approximation error's exact stochastic structure.

In the literature on the Bayesian identification approach, there are well-known direct and indirect techniques. Diaz and Farah [1981] introduced the direct technique for autoregressive (AR) models, which considers the time series model's orders are random variables that have a known maximum and are unknown, and the issue with identification is to determine the posterior mass function of these orders. Afterward, the posterior probabilities are computed to determine the model's order as a point estimate by selecting the order with the highest probability. The direct approach has been expanded to seasonal autoregressive (SAR) processes by Shaarawy and Ali [2003]. The direct technique has been expanded to ARMA processes by Ali [2003]. The direct approach was expanded to include MA models by Shaarawy et al. [2007]. The direct approach has been extended to multivariate AR models by Shaarawy and Ali [2008]. Ali (2009) extended the technique proposed by Shaarawy et al. (2007) in order to identify the mixed ARMA ( $p, q$ ) processes. Moreover, the direct approach has been extended to seasonal multivariate AR processes by Shaarawy and Ali [2015]. The direct method for determining the ordering of vector MA models with seasonality has been expanded by Shaarawy et al. [2021]. The direct technique has been extended to MA models by Al Bassam et al. [2022]. The indirect method suggested by Shaarawy and Broemeling [1987], which views the model's parameter number as an unknown constant with a known maximum. Ismail et al. [2016] have extended the indirect approach to moving average (MA) processes utilizing a suggested Bayesian methodology to identify moving average (MA) processes. In this article, we utilize the direct Bayesian approach to determine the autoregressive moving average (ARMA) models' orders. Broemeling and Shaarawy [1988] introduced an approximation approach to eliminate the nonlinearity of errors in the model. This approximation method is based on calculating the residuals recursively using nonlinear least squares estimates NLSE's and then replacing the lagged errors of the model with their

corresponding lagged residuals. This method disregards the estimation error that occurs when true errors are substituted by their estimates.

Ismail et al. [2015] have expanded the direct approach to moving average (MA) processes utilizing a proposed Bayesian methodology to identify moving average models. The technique's foundation is the substitution of suitably lagged residuals from a long autoregressive model for the lagged errors of the original MA model. In contrast to Broemeling and Shaarawy [1988], the precise structure of the approximation error when substituting genuine errors with matching residuals is obtained and utilized in the derivation of the posterior probability mass function of the model order. This article's primary goal is to establish Bayesian GLS identification for the ARMA model using the derived exact stochastic structure of approximation (estimation) error.

In numerous fields, including business, economics, engineering, and the natural sciences, an autoregressive moving average processes, often known as ARMA (p, q), is widely utilized for modeling time series data. The main issue with Bayesian time series identification of ARMA models is that the model errors are nonlinear functions in the model coefficients, and the likelihood function is complex and analytically intractable. Consequently, numerical integration is required for Bayesian identification.

Ismail (2009, 2012) proposed the Bayesian Generalized Least Squares (BGLS) method to estimate the moving average processes. Ismail and Abd El-Aziz (2010) have extended the proposed methodology to estimate autoregressive moving average models. The innovation substitution (IS) method, which was introduced by Koreisha and Pukkila (1989), is the foundation of this methodology. It is a quick and simple way to estimate errors by employing the ordinary least squares (OLS) approach rather than the costly nonlinear least squares estimates (NLS'E) used in Broemeling and Saharawy's method. Ismail et al. [2015] have expanded the direct approach to moving average (MA) processes utilizing the proposed Bayesian methodology BGLS to determine moving average processes. In contrast to Broemeling and Shaarawy [1988], the posterior probability mass function of the model order is derived using the precise structure of the approximation error when actual errors are substituted with corresponding residuals.

This article's primary goal is to propose a new Bayesian approach BGLS for the identification of ARMA processes utilizing the derived exact stochastic structure of approximation (estimation) error. This proposed approach depends on substituting lagged errors of the original autoregressive moving average processes with lagged residuals from a long autoregression. To demonstrate and evaluate the effectiveness of the suggested technique, it is compared to the Broemeling and Shaarawy methods using a real data set and many simulation studies.

The article's remaining sections are sorted as follows: In Section 2, the suggested method is presented, the autoregressive moving average processes are discussed, and the approximation (estimation) error is driven. The approximate conditional likelihood function is shown in Section 3, while Section 4 shows the direct Bayesian identification method. The simulation study is presented in Section 5. The details about real data used for Bayesian identification and results derived from this study are clarified in Section 6. Finally, some conclusions are displayed in Section 7.

## 2. Autoregressive Moving Average Models and The Approximation Error

An autoregressive moving average of order  $p$  and  $q$  for a time series  $(y_t)$ , labeled as ARMA  $(p, q)$ , able to represent in the compact form shown below [see Box and Jenkins (1976)]:

$$\Phi_p(B)y_t = \Theta_q(B)\varepsilon_t \quad (1)$$

Where  $B$  is the backshift operator, denoted as  $B^r y_t = y_{t-r}$ , the autoregressive polynomial of order  $p$  is  $\Phi_p(B) = (1 - \phi_1 B - \phi_2 B^2 - \dots - \phi_p B^p)$ , the moving polynomial of order  $q$  is  $\Theta_q(B) = (1 - \theta_1 B - \theta_2 B^2 - \dots - \theta_q B^q)$ , the errors  $\varepsilon_t$ 's are assumed to i.i.d normally distributed variable with zero mean and variance  $\tau^{-1}$ , where  $\tau = 1/\sigma^2 > 0$  is the precision parameter,  $y_t$ 's are observations. If  $\Phi_q(B)$  has roots that are not inside the circle of unit, then the ARMA  $(p, q)$  model is stationary lay outside of the circle of unit. If  $\Theta_q(B)$  has roots that are not inside the circle of unit, then the model is invertible. Using  $n$  observations  $\underline{Y}^T = (y_1, y_2, \dots, y_n)$  to calculate values the orders of the ARMA model  $p$  and  $q$ , which are unknown.

Following Ismail's approach [2009, 2010,2015] depends on replacing the original model's lagged errors with the suitable lagged residual from a long autoregression of order L as :

$$y_t = \sum_{i=1}^L \hat{\pi}_i y_{t-i} + \hat{\varepsilon}_t, \quad (2)$$

Where  $\hat{\pi}_1, \hat{\pi}_2, \dots, \hat{\pi}_L$  are the estimated parameters of the autoregression of order L,  $\hat{\varepsilon}_t$  is estimate of  $\varepsilon_t$  and the choice of L is determined as  $\sqrt{n}$  has been validated in Koreisha and Pukkila [1990a, 1990b] utilized extensive simulation studies.

The approach of Ismail does not disregard the estimation (approximation) error, denoted by  $a_t$  for short, that arise when replacing the error with their corresponding lagged residuals. It uses the innovations substitution estimation (IS) suggested by Koresiha and Pukkila (1989), which is a quick and easily implemented estimation approach for estimating the errors, as opposed to the expensive nonlinear least squares estimates utilized by Broemeling and Saharawy's approach.

Define  $a_t$  as

$$a_t = \varepsilon_t - \hat{\varepsilon}_t \quad (3)$$

Using Eq. (2),  $\hat{\varepsilon}_t$  can be expressed as

$$\begin{aligned} \hat{\varepsilon}_t &= y_t - \sum_{i=1}^L \hat{\pi}_i y_{t-i}, \\ \hat{\varepsilon}_t &= y_t - (\hat{\pi}_1 y_{t-1} + \hat{\pi}_2 y_{t-2} + \dots + \hat{\pi}_L y_{t-L}) \\ \hat{\varepsilon}_t &= \hat{\Pi}_L(B) y_t \end{aligned} \quad (4)$$

Where  $\hat{\Pi}_L(B) = [1 - \pi_1 B - \pi_2 B^2 - \dots - \pi_L B^L]$  is a polynomial function of order L. By utilizing (1) and replacing  $y_t$  in (4), we obtain

$$\hat{\varepsilon}_t = \hat{\Pi}_L(B) \frac{\Theta_q(B)}{\Phi_p(B)} \varepsilon_t \quad (5)$$

Using (3) and (5), the approximation error can be represented as a function in  $\varepsilon_t$  as follows:

$$\begin{aligned} a_t &= \varepsilon_t - \hat{\Pi}_L(B) \frac{\Theta_q(B)}{\Phi_p(B)} \varepsilon_t \\ a_t &= \left[ 1 - \hat{\Pi}_L(B) \frac{\Theta_q(B)}{\Phi_p(B)} \right] \varepsilon_t \end{aligned} \quad (6)$$

### 3. An Approximate Conditional Likelihood Function

In this section, the approximate conditional likelihood function for ARMA processes is derived, replacing for  $\varepsilon_t$  in (1) using (3), (5) and (6) results in the model

$$\begin{aligned} \Phi_p(B)y_t &= \Theta_q(B)\varepsilon_t \\ \Phi_p(B)y_t &= \Theta_q(B)[\hat{\varepsilon}_t + a_t] \\ \Phi_p(B)y_t &= \Theta_q(B)\hat{\varepsilon}_t + \Theta_q(B)a_t \end{aligned} \quad (7)$$

$$\begin{aligned} \Phi_p(B)y_t &= -\sum_{j=1}^q \theta_j \hat{\varepsilon}_{t-j} + \hat{\varepsilon}_t + \Theta_q(B) \left[ 1 - \hat{\Pi}_L(B) \frac{\Theta_q(B)}{\Phi_p(B)} \right] \varepsilon_t \\ \Phi_p(B)y_t &= -\sum_{j=1}^q \theta_j \hat{\varepsilon}_{t-j} + \hat{\Pi}_L(B) \frac{\Theta_q(B)}{\Phi_p(B)} \varepsilon_t + \Theta_q(B) \left[ 1 - \hat{\Pi}_L(B) \frac{\Theta_q(B)}{\Phi_p(B)} \right] \varepsilon_t \\ \Phi_p(B)y_t &= -\sum_{j=1}^q \theta_j \hat{\varepsilon}_{t-j} + \left[ \frac{\hat{\Pi}_L(B) \Theta_q(B)}{\Phi_p(B)} + \Theta_q(B) \left[ 1 - \frac{\hat{\Pi}_L(B) \Theta_q(B)}{\Phi_p(B)} \right] \right] \varepsilon_t \\ \Phi_p(B)y_t &= -\sum_{j=1}^q \theta_j \hat{\varepsilon}_{t-j} + \left[ \frac{\hat{\Pi}_L(B) \Theta_q(B)}{\Phi_p(B)} + \Theta_q(B) - \Theta_q(B) \left( \frac{\hat{\Pi}_L(B) \Theta_q(B)}{\Phi_p(B)} \right) \right] \varepsilon_t \\ \Phi_p(B)y_t &= -\sum_{j=1}^q \theta_j \hat{\varepsilon}_{t-j} + \left[ \Theta_q(B) + (1 - \Theta_q(B)) \left( \frac{\hat{\Pi}_L(B) \Theta_q(B)}{\Phi_p(B)} \right) \right] \varepsilon_t \\ \Phi_p(B)y_t &= -\sum_{j=1}^q \theta_j \hat{\varepsilon}_{t-j} + \eta_t \end{aligned}$$

$$\begin{aligned} (1 - \phi_1 B - \phi_2 B^2 - \dots - \phi_p B^p)y_t &= -\sum_{j=1}^q \theta_j \hat{\varepsilon}_{t-j} + \eta_t \\ y_t - \sum_{i=1}^p \phi_i y_{t-i} &= -\sum_{j=1}^q \theta_j \hat{\varepsilon}_{t-j} + \eta_t \end{aligned}$$

Substituting for  $\Phi(B), \Theta_q(B), a_t$  and  $\hat{\varepsilon}_t$  in (7), we get

$$y_t = \sum_{i=1}^p \phi_i y_{t-i} - \sum_{j=1}^q \theta_j \hat{\varepsilon}_{t-j} + \eta_t \quad (8)$$

Where

$$\begin{aligned} \eta_t &= \left\{ \Theta_q(B) + (1 - \Theta_q(B)) \left( \frac{\hat{\Pi}_L(B) \Theta_q(B)}{\Phi_p(B)} \right) \right\} \varepsilon_t \\ \eta_t &= \left\{ \Theta_q(B) \frac{\Phi_p(B)}{\Phi_p(B)} + (1 - \Theta_q(B)) \left( \frac{\hat{\Pi}_L(B) \Theta_q(B)}{\Phi_p(B)} \right) \right\} \varepsilon_t \end{aligned} \quad (9)$$

$$\eta_t = \left\{ \theta_q(B) \Phi_p(B) + (1 - \theta_q(B)) \left( \hat{\Pi}_L(B) \theta_q(B) \right) \right\} \frac{\varepsilon_t}{\Phi_p(B)}$$

$$\eta_t \Phi_p(B) = \left\{ \theta_q(B) \Phi_p(B) + (1 - \theta_q(B)) \left( \hat{\Pi}_L(B) \theta_q(B) \right) \right\} \varepsilon_t$$

$$\eta_t \Phi_p(B) = \Psi_{q^*}(B) \varepsilon_t$$

$$\eta_t = \sum_{i=1}^p \phi_i \eta_{t-i} + \Psi_{q^*}(B) \varepsilon_t \tag{10}$$

$$\eta_t = \sum_{i=1}^p \phi_i \eta_{t-i} + \sum_{j=1}^{q^*} \psi_j \varepsilon_{t-j} + \psi_0 \varepsilon_t \tag{11}$$

Where

$$\Psi_{q^*}(B) = \theta_q(B) \Phi_p(B) + (1 - \theta_q(B)) \left( \hat{\Pi}_L(B) \theta_q(B) \right) \tag{12}$$

$q^* = \max[(p + q), (2q + 1)]$ ,  $\psi_0 = 1$ , and  $\psi'_j$ s are determined by contrasting the coefficients in (11)

Letting the initial residuals be zeros, i.e.,  $\varepsilon_0 = \varepsilon_{-1} = \dots = \varepsilon_{p-2q^*-L} = 0$ , where  $q^* > p + 1$ , the model (1) can be represented in a matrix with the following form

$$\underline{Y} = X\underline{\gamma} + \underline{\eta} \tag{13}$$

Where  $\underline{Y}^T = (y_{p+1}, y_{p+2}, \dots, y_{n-1}, y_n)$ ,  $\underline{\gamma}^T = (\phi_1, \phi_2, \dots, \phi_p, \theta_1, \theta_2, \dots, \theta_q)$  is the parameter vector,  $\underline{\eta}^T = (\eta_{p+1}, \eta_{p+2}, \dots, \eta_{n-1}, \eta_n)$ ,  $X$  is  $(n - p) \times (p + q)$  matrix with  $t^{th}$  row  $(y_{t-1}, y_{t-2}, \dots, y_{t-p}, -\varepsilon_{t-1}, -\varepsilon_{t-2}, \dots, -\varepsilon_{t-q})$ ,  $t = p + 1, p + 2, \dots, n$ . The error vector  $\underline{\eta}$  has a multivariate normal distribution with zero mean vector and matrix of variance-covariance  $[\Sigma = \tau^{-1}\Omega]$ , the matrix  $\Omega$  is a symmetric positive definite Toeplitz matrix and the covariance structure depends on the orders of ARMA models [for details Wei (2006)]. Using model (13), the likelihood function can be written as:

$$L(\underline{\gamma}(p, q), p, q, \tau | \underline{Y}) \propto \tau^{\frac{(n-p)}{2}} \exp \left[ -\frac{\tau}{2} (\underline{Y} - X\underline{\gamma}(p, q))^T \Omega^{-1} (\underline{Y} - X\underline{\gamma}(p, q)) \right] \tag{14}$$

where  $\underline{\gamma}(p, q) = (\phi_{p1}, \phi_{p2}, \dots, \phi_{pp}, \theta_{q1}, \theta_{q2}, \dots, \theta_{qq}) \in R^{p+q}$ ,  $p = 1, 2, \dots, K_1$ ,  $q = 1, 2, \dots, K_2$ ,  $\tau > 0$ , where  $K_1, K_2$  are the maximum potential value of  $p, q$  in the case of the direct identification technique. Because the components

of  $\Omega$  are nonlinear functions in the model coefficients  $(\phi_{p1}, \phi_{p2}, \dots, \phi_{pp}, \theta_{q1}, \theta_{q2}, \dots, \theta_{qq})$ , the likelihood function (14) is a complicated function in  $\underline{\gamma}(p, q)$ . Consequently, in order to do Bayesian estimate for  $\underline{\gamma}(p, q)$ , numerical integration is required. This problem can be resolved by obtaining an estimated matrix  $\hat{\Omega}$  by substituting the elements of  $\underline{\gamma}(p, q)$  in the matrix  $\Omega$  with by their estimates  $\hat{\underline{\gamma}}(p, q)$  obtained by IS approach. This is because the IS estimates for ARMA parameters are produced by applying the OLS method to the ARMA model after substituting lagged errors with corresponding lagged residuals from along autoregressive.

Substituting the matrix  $\Omega$  by its estimate  $\hat{\Omega}$  in (14) yields an approximate conditional likelihood function that like the following:

$$L^*(\underline{\gamma}(p, q), p, q, \tau | \underline{Y}) \propto \tau^{\frac{(n-p)}{2}} \exp \left[ -\frac{\tau}{2} (\underline{Y} - X\underline{\gamma}(p, q))^T \hat{\Omega}^{-1} (\underline{Y} - X\underline{\gamma}(p, q)) \right] \quad (15)$$

The direct application of (15) is challenging since it requires the Toeplitz matrix  $\hat{\Omega}$  to be inverted, which can be computationally demanding, especially for high n. Nevertheless, the precise transformation matrix R, such as that  $R^T R = \Omega^{-1}$  was determined by Galbraith and Zinde-Walsh [1992]. It is feasible to create an estimated transformation matrix  $\hat{R}$  using the IS parameters estimates, such that  $\hat{\Omega}^{-1} = \hat{R}^T \hat{R}$ . Consequently, the following is an approximate conditional probability function:

$$L^*(\underline{\gamma}(p, q), p, q, \tau | \underline{Y}) \propto \tau^{\frac{(n-p)}{2}} \exp \left[ -\frac{\tau}{2} (\underline{Y} - X\underline{\gamma}(p, q))^T (\hat{R}^T \hat{R}) (\underline{Y} - X\underline{\gamma}(p, q)) \right] \quad (16)$$

$$L^*(\underline{\gamma}(p, q), p, q, \tau | \underline{Y}) \propto \tau^{\frac{(n-p)}{2}} \exp \left[ -\frac{\tau}{2} (\underline{Y}^* - X^* \underline{\gamma}(p, q))^T (\underline{Y}^* - X^* \underline{\gamma}(p, q)) \right] \quad (17)$$

Where  $\underline{Y}^* = \hat{R}\underline{Y}$  and  $X^* = \hat{R}X$ .

The approximation conditional likelihood function employed by Broemeling and Shaarawy [1988] is a simplification of (16) when  $\hat{R} = \hat{\Omega}^{-1} = I_{n-p}$  where  $I_{n-p}$  is the unit matrix of order n-p.

#### 4. Direct Bayesian Identification

This technique considers the orders  $p$  and  $q$  of the mixed ARMA models are unidentified random variables with maximum values that are known. The challenge is to determine the posterior probabilities over all conceivable orders by determining the joint posterior probability function of orders  $p$  and  $q$ . subsequently the model order is chosen, corresponding to the highest posterior probability as the identified orders. In other words, the identified model selects the value of  $(p, q)$  with the highest probability. In contrast to Broemeling and Shaarawy [1988], this section presents the direct Bayesian identification procedure for the mixed ARMA processes utilizing the suggested approach, it is dependent on the innovation substitution method (IS) and takes into consideration estimate (approximation) error into. The IS estimates for the ARMA parameters are obtained via the ordinary least squares (OLS) approach after appropriate lagged residuals from a long autoregressive are substituted for lagged errors in the ARMA model.

Using both normal-gamma and Jefferys' priors, the posterior distribution of an autoregressive moving average model is calculated. The rationale for using normal gamma prior to the approximate conditional likelihood function (15) is that a function in the parameters is a normal gamma density (see Broemeling [1985]). The following is an appropriate choice for proper prior distribution to conjugate prior distribution:

Consider the following prior assumptions:

- The conditional prior density of  $\underline{\gamma}(p, q)$  given  $p, q$ , and  $\tau$  has a multivariate normal prior distribution with a vector of mean  $\underline{M}(p, q)$  and precision matrix  $\tau V(p, q)$  (i.e. matrix of variance-covariance  $\Sigma = 1/\text{precision matrix} = \tau^{-1}V^{-1}(p, q)$ ), denoted by  $\xi_1(\underline{\gamma}(p, q)|p, q, \tau) \sim N(\underline{M}(p, q), \tau^{-1}V^{-1}(p, q))$ , where  $\tau > 0$ ,  $V(p, q)$  is a square positive definite matrix of order  $(p+q)$  as the following form:

$$\xi_1(\underline{\gamma}(p, q)|p, q, \tau) = \frac{\tau^{\frac{p+q}{2}} |V(p, q)|^{\frac{1}{2}}}{(2\pi)^{\frac{p+q}{2}}} \exp\left\{-\frac{\tau}{2} (\underline{\gamma}(p, q) - \underline{M}(p, q))^T V(p, q) (\underline{\gamma}(p, q) - \underline{M}(p, q))\right\} \quad (18)$$

- Let us assume the independence of  $p$ ,  $q$ , and  $\tau$ . Consequently, the marginal prior density of  $\tau$  has a gamma density with parameters  $\alpha$  and  $\beta$  as the following:

$$\xi_2(\tau) \sim \text{Gamma}(\alpha, \beta)$$

$$\xi_2(\tau) \propto \tau^{\alpha-1} e^{-\tau\beta}, \tau > 0, \alpha > 0 \text{ and } \beta > 0 \quad (19)$$

- The marginal prior probability mass function density of  $p$  and  $q$  is uniform.

$$\xi_3(p, q) = K_1^{-1} \times K_2^{-1} \quad (20)$$

$$p = 1, 2, \dots, K_1, q = 1, 2, \dots, K_2$$

The direct technique assigns a probability to each pair of values of  $(p, q)$  or, more specifically, to each of the  $K_1 \times K_2$  of the ARMA process.

Using these quantities, we assert the subsequent theorem.

The approximate marginal posterior probability mass function of the autoregressive moving average (ARMA) model orders  $p$  and  $q$ , which can be obtained by integrating out  $\underline{\gamma}(p, q)$  and  $\tau$ , respectively, as shown by the subsequent theorem.

**Theorem:** Given  $n$  observation  $\underline{Y}^T = (y_1, y_2, \dots, y_n)$  from autoregressive moving average (ARMA) model given by (1), an approximate conditional likelihood function in (17) and the joint prior density given by (21), then the approximate marginal posterior probability mass function of the autoregressive moving average orders  $p$  and  $q$  is

$$\xi^*(p, q | \underline{Y}) \propto \frac{|A|^{\frac{-1}{2}} \Gamma^{\frac{(2\alpha+n-p)}{2}}}{|V(p, q)|^{\frac{-1}{2}} (\pi)^{\frac{n-p}{2}}} [C - B^T A^{-1} B]^{-\frac{(2\alpha+n-p)}{2}} \quad (21)$$

$$p = 1, 2, \dots, K_1, q = 1, 2, \dots, K_2$$

Where

$$A = [X^{*T} X^* + V(p, q)], B = [X^{*T} \underline{Y}^* + V(p, q) \underline{M}(p, q)], \text{ and } C = \underline{Y}^{*T} \underline{Y}^* + \underline{M}(p, q)^T V(p, q) \underline{M}(p, q) + 2\beta.$$

$$\because \underline{Y}^* = \hat{R} \underline{Y}, X^* = \hat{R} X, \text{ and } \hat{\Omega}^{-1} = \hat{R}^T \hat{R}.$$

$$\because \underline{Y}^{*T} \underline{Y}^* = (\hat{R} \underline{Y})^T \hat{R} \underline{Y} = Y^T \hat{R}^T \hat{R} \underline{Y} = Y^T \hat{\Omega}^{-1} \underline{Y}$$

$$X^{*T} X^* = (\hat{R} X)^T \hat{R} X = X^T \hat{R}^T \hat{R} X = X^T \hat{\Omega}^{-1} X$$

$$X^{*T} \underline{Y}^* = (\hat{R} X)^T \hat{R} \underline{Y} = X^T \hat{R}^T \hat{R} \underline{Y} = X^T \hat{\Omega}^{-1} \underline{Y}$$

**Proof:** The following procedures can be used to prove the theorem

- Multiplying Eq. (18) by both Eq. (19) and Eq. (20), The joint prior distribution of the model parameters  $\underline{\gamma}(p, q), p, q$  and  $\tau$  is as follows:

$$\xi(\underline{\gamma}(p, q), p, q, \tau) \propto K_1^{-1} K_2^{-1} \tau^{\frac{p+q+2\alpha}{2}-1} |V(p, q)|^{\frac{1}{2}} \exp\left\{-\frac{\tau}{2}(\underline{\gamma}(p, q) - \underline{M}(p, q))^T V(p, q) (\underline{\gamma}(p, q) - \underline{M}(p, q)) + 2\beta\right\} \quad (22)$$

where  $\underline{\gamma}(p, q) = (\phi_{p1}, \phi_{p2}, \dots, \phi_{pp}, \theta_{q1}, \theta_{q2}, \dots, \theta_{qq}) \in R^{p+q}, p = 1, 2, \dots, K_1, q = 1, 2, \dots, K_2,$  where  $K_1, K_2$  are the biggest possible orders of  $p, q,$  respectively.

- Combining the approximate likelihood function  $L^*(\underline{\gamma}(p, q), p, q, \tau | \underline{Y})$  in Eq. (17), with the joint prior density  $\xi(\underline{\gamma}(p, q), p, q, \tau)$  in Eq. (22) through Bayes theorem yields an approximate joint posterior distribution of the parameters  $\underline{\gamma}(p, q), p, q,$  and  $\tau$  as follows:

$$\xi^*(\underline{\gamma}(p, q), p, q, \tau | \underline{Y}) \propto L^*(\underline{\gamma}(p, q), p, q, \tau | \underline{Y}) \xi(\underline{\gamma}(p, q), p, q, \tau) \exp\left\{-\frac{\tau}{2}(\underline{\gamma}(p, q) - \underline{M}(p, q))^T V(p, q) (\underline{\gamma}(p, q) - \underline{M}(p, q)) + 2\beta + (\underline{Y}^* - X^* \underline{\gamma}(p, q))^T (\underline{Y}^* - X^* \underline{\gamma}(p, q))\right\} \quad (23)$$

$$\xi^*(\underline{\gamma}(p, q), p, q, \tau | \underline{Y}) \propto \frac{\tau^{\frac{2\alpha+q+n}{2}-1}}{(2\pi)^{\frac{n+q}{2}}} |V(p, q)|^{\frac{1}{2}}$$

The above is an approximate joint posterior distribution of the parameters  $\underline{\gamma}(p, q), p, q,$  and  $\tau$  has also the normal gamma distribution.

- The identification of the terms in the exponent of the approximate joint posterior distribution and reformulation in a quadratic form is as follows:

$$1- (\underline{Y}^* - X^* \underline{\gamma}(p, q))^T (\underline{Y}^* - X^* \underline{\gamma}(p, q)) = \underline{Y}^{*T} \underline{Y}^* - 2 \underline{\gamma}(p, q)^T X^{*T} \underline{Y}^* + \underline{\gamma}(p, q)^T (X^{*T} X^*) \underline{\gamma}(p, q)$$

$$2- (\underline{\gamma}(p, q) - \underline{M}(p, q))^T V(p, q) (\underline{\gamma}(p, q) - \underline{M}(p, q))$$

$$= \underline{\gamma}(p, q)^T V(p, q) \underline{\gamma}(p, q) - 2 \underline{\gamma}(p, q)^T V(p, q) \underline{M}(p, q) + \underline{M}(p, q)^T V(p, q) \underline{M}(p, q)$$

$$3 - 2\beta$$

When adding 1, 2, and 3, we obtain:

$$\xi^*(\underline{\gamma}(p, q), p, q, \tau | \underline{Y}) \propto \frac{\tau^{\frac{2\alpha+q+n}{2}-1}}{(2\pi)^{\frac{n+q}{2}}} |V(p, q)|^{\frac{1}{2}} \exp\left\{-\frac{\tau}{2}(C - 2 \underline{\gamma}(p, q)^T B + \underline{\gamma}(p, q)^T A \underline{\gamma}(p, q))\right\} \quad (24)$$

Where

$$\begin{aligned} A &= [X^{*T}X^* + V(p, q)] \\ B &= [X^{*T}\underline{Y}^* + V(p, q)\underline{M}(p, q)] \\ C &= \underline{Y}^{*T}\underline{Y}^* + \underline{M}(p, q)^T V(p, q)\underline{M}(p, q) + 2\beta \end{aligned}$$

• Completing the exponent square in (24) with regard to  $\underline{\gamma}(p, q)$  and then integrate out over  $\underline{\gamma}(p, q)$  and  $\tau$ , respectively. The joint posterior distribution of  $\underline{\gamma}(p, q), p, q$  and  $\tau$  becomes.

$$\begin{aligned} \xi^*(\underline{\gamma}(p, q), p, q, \tau | \underline{Y}) &\propto \frac{\tau^{\frac{2\alpha+q+n}{2}-1}}{(2\pi)^{\frac{n+q}{2}}} |V(p, q)|^{\frac{1}{2}} \exp\left\{-\frac{\tau}{2}(C - B^T A^{-1} B)\right\} \\ &\quad \exp\left\{-\frac{\tau}{2}(\underline{\gamma}(p, q) - A^{-1} B)^T A (\underline{\gamma}(p, q) - A^{-1} B)\right\} \end{aligned} \quad (25)$$

Following that, integration regarding  $\underline{\gamma}(p, q)$ , the marginal joint posterior distribution of  $p, q$  and  $\tau$  is

$$\xi^*(p, q, \tau | \underline{Y}) \propto \frac{\tau^{\frac{2\alpha+n-p}{2}-1} |V(p, q)|^{\frac{1}{2}}}{(2\pi)^{\frac{n-p}{2}} |A|^{\frac{1}{2}}} \exp\left\{-\frac{\tau}{2} [C - B^T A^{-1} B]\right\}$$

Afterward, eliminating  $\tau$  leads by integration regarding  $\tau$ , we get the marginal posterior mass function of  $p$  and  $q$

$$\xi^*(p, q | \underline{Y}) \propto \frac{|A|^{\frac{-1}{2}} \Gamma\left(\frac{2\alpha+n-p}{2}\right)}{|V(p, q)|^{\frac{-1}{2}} (\pi)^{\frac{n-p}{2}}} [C - B^T A^{-1} B]^{-\frac{(2\alpha+n-p)}{2}}$$

The proof has been completed.

When there is minimal knowledge regarding hyperparameters, namely  $\underline{M}(p, q)$ ,  $V(p, q)$ ,  $\alpha$  and  $\beta$ , one may use Jefferys' prior replaces Normal-Gamma prior distribution. Consequently, the Jefferys' prior distribution is

$$\xi(\underline{\gamma}(p, q), p, q, \tau) \propto \tau^{-1} \quad (26)$$

Therefore, Jefferys' prior (26) is a particular case of the joint prior distribution of the parameters  $\underline{\gamma}(p, q), p, q$ , and  $\tau$  when  $\beta = 0, V(p, q) = 0$ , and  $\alpha = -(p + q)/2$ , as demonstrated by the following corollary. The following corollary shows that.

**Corollary:** The approximate marginal posterior probability mass function of the autoregressive moving average orders  $p$  and  $q$  can be determined by combining the approximate likelihood function (17) and the non-informative prior density (26), which is expressed as follows:

$$\xi^{**}(p, q | \underline{Y}) \propto \frac{|A^*|^{-\frac{1}{2}}}{(2\pi)^{\frac{n-2p-q}{2}}} [C^* - B^{*T} A^{*-1} B^*]^{-\frac{(n-2p-q)}{2}} \Gamma\left(\frac{n-2p-q}{2}\right)$$

Where  $A = X^{*T} X^*$ ,  $B = X^{*T} \underline{Y}^*$  and  $C = \underline{Y}^{*T} \underline{Y}^*$ .

**Proof:** We can establish the corollary from the above theorem by replacing when  $\beta = 0$ ,  $V(p, q) = 0$ , and  $\alpha = -(p + q)/2$

## 5. The Numerical Analysis and Simulation

This section's objective is to evaluate and compare the effectiveness and accuracy of the direct identification technique for the ARMA process utilizing the three Bayesian identification methods. The proposed Bayesian Generalized Least Squares (BGLS) method is illustrated in section 4. Broemeling and Shaarawy's (1988) is the second one. The second method, known as BS-NLS, employs Nonlinear Least Squares (NLS) estimates to estimate the errors, which is discussed in section 3. The third method, known as BS-IS, is a modified version of Broemeling and Shaarawy's (1988) method where IS estimates are employed to estimate the errors rather than Nonlinear Least Squares (NLS) estimates. The aim is accomplished via various simulation studies. Additionally, the effectiveness of the suggested Bayesian approach (BGLS), as detailed in section (3), is the main focus of this part. The suggested method is compared to Broemeling and Shaarawy's approach, denoted as (BS-NLS), which calculates the errors using nonlinear least squares estimates. The first subsection introduces the effectiveness criterion employed in the study. The second subsection presents the simulation design and objectives. Finally, the results and the comments are explained in detail in the third section.

### 5.1 Effectiveness Criterion

The effectiveness study employed the percentage of correct model identification as the effectiveness criteria to assess and contrast the performance of the aforementioned Bayesian identification technique.

Assuming  $n$  is the number of times where we choose the correct model, then the percentage of correctly identified models as follows:

$$P = \frac{n}{N} \times 100.$$

Where  $N$  is the total number of series generated from the original autoregressive moving average (ARMA) model.

## 5.2 Simulation Design

The assessment and comparison are based on various simulation experiments. The following steps establish the simulation process

1. A time series following an autoregressive moving average ARMA process is generated. There are two phases in the generating process:
  - After that,  $(n+200)$  observations are generated .
  - subsequently that, in order to eliminate the initialization impact, the first 200 observations are removed.
2. For determining which model is best suited to the generated time series, the methods BGLS, BS-NLS, and BS-IS are utilized in the direct technique. The sample size ( $n$ ) for the time series lengths is determined to be 50, 100, 150, 200, and 300. These time series lengths were chosen to reflect the range of time series lengths, from tiny to enormous. There are a thousand realizations, and it is assumed that the maximum order, which is known, equals three and four times, respectively.
3. The first two stages are repeated 1000 times.
4. Lastly, for each time series, we determine the percentage of correct identification using each approach.

Table (1) shows the ARMA models used in the simulation investigation with various orders and parameter values. The parameters of this model are chosen inside the invertibility domain.

**Table (1):** The ARMA (p,q) Model's Simulation Design

Model	Order	Phi1	Phi2	Theta1	Theta2
ARMA(2,2)	p=2, q=2	0	-0.2	0	0.9

In experiment, the precision parameter was set to one and Jeffery's' prior was utilized.

Various priors for the orders are used when employing the direct approach and checking its sensitivity to the prior choice, and the subsequent three priors are used

- The conditionally prior density of  $\underline{\gamma}(p, q)$  given  $p, q, \text{ and } \tau$  has a multivariate normal prior distribution with a vector of mean  $\underline{M}(p, q)$  and precision matrix  $\tau V(p, q)$  (i.e. matrix of variance-covariance  $\Sigma = 1/\text{precision matrix} = \tau^{-1}V^{-1}(p, q)$ ), denoted by  $\xi_1(\underline{\gamma}(p, q)|p, q, \tau) \sim N(\underline{M}(p, q), \tau^{-1}V^{-1}(p, q))$ , where  $\tau > 0$ ,  $V(p, q)$  is a square positive definite matrix of order  $(p+q)$  as following form:

$$\xi_1(\underline{\gamma}(p, q)|p, q, \tau) = \frac{\tau^{\frac{p+q}{2}} |V(p, q)|^{\frac{1}{2}}}{(2\pi)^{\frac{p+q}{2}}} \exp\left\{-\frac{\tau}{2} (\underline{\gamma}(p, q) - \underline{M}(p, q))^T V(p, q) (\underline{\gamma}(p, q) - \underline{M}(p, q))\right\}$$

- Let us assume the independence of  $p, q, \text{ and } \tau$ . Consequently, the marginal prior density of  $\tau$  has a gamma density with parameters  $\alpha$  and  $\beta$  as the following:

$$\xi_2(\tau) \sim \text{Gamma}(\alpha, \beta)$$

$$\xi_2(\tau) \propto \tau^{\alpha-1} e^{-\tau\beta}, \quad \tau > 0, \alpha > 0 \text{ and } \beta > 0$$

- The marginal prior probability mass function density of  $p$  and  $q$  is uniform.

$$\xi_3(p, q) = K_1^{-1} x K_2^{-1}$$

$$p = 1, 2, \dots, K_1$$

$$q = 1, 2, \dots, K_2$$

The direct technique assigns a probability to each pair of values of  $(p, q)$  or, in other words, to each of the  $K_1 \times K_2$  of the ARMA model.

The GAUSS/ARIMA library simarma procedure was employed to simulate all ARMA models, and GAUSS 10 was utilized for all computations.

### 5.3 Results of the simulation

This section summarizes and discusses the simulation studies' results conducted for ARMA sources. The direct Bayesian identification technique utilizing BGLS, BS-NLS, and BS-IS approaches is employed to determine

a model for the series after 1000 series are generated from a particular ARMA (2,2) processes.

For each series, the marginal posterior mass function of the orders of the ARMA process ( $p=1, 2, \dots, K_1$  and  $q=1, 2, \dots, K_2$ ) is computed using the direct technique, assuming that the maximum order is  $K_1=3, K_2=3$ , and  $K_1=4, K_2=4$ , with various prior functions for the order  $p, q$ . The ARMA process that has the highest probability is chosen as the identified model. For the direct technique, the ratio of correct identification  $P$  is calculated.

Table (2) includes the identification process' results conducted for the ARMA models. It includes the results of the three methods mentioned above. It is divided into five blocks corresponding to the five aforementioned time series lengths. The cells of the table include the ratio of the correct identification. The columns of the table are divided according to the method and the considered maximum order.

**Table (2):** Percentages of Correct Identification for ARMA(2,2) Models [Tau=1]

N	BGLS		BS-NLS		BS-IS	
	Max=3	Max=4	Max=3	Max=4	Max=3	Max=4
PRIOR 1						
50	73.8	32.7	59.6	47.6	51.8	48.1
100	81.4	68.1	72.8	76	76.2	43.5
150	80.7	90.9	75.6	83.1	81.2	47.6
200	80.7	94.3	74.6	83.1	78.0	46.4
300	100	100	100	100	100	100
	Max=3	Max=4	Max=3	Max=4	Max=3	Max=4
PRIOR 2						
50	75.8	36.7	66.3	47.9	62.8	76.2
100	87.3	56.3	77.9	67.6	81.5	40.5
150	87.3	88.5	81.1	78.7	87.9	62.4
200	86.6	96.9	81.2	83.6	87.5	61.2
300	100	100	100	100	100	100
	Max=3	Max=4	Max=3	Max=4	Max=3	Max=4
PRIOR 3						
50	73.8	32.7	59.6	47.6	51.8	48.1
100	81.4	68.1	72.8	76	76.2	43.5
150	80.7	90.9	75.6	83.1	81.2	47.6
200	80.7	94.3	74.6	83.1	78.0	46.4
300	100	100	100	100	100	100

Based on the previous table:

1. As  $n$  increases, the percentages of correct identification for direct techniques increase. The results with max equals 3 are superior to those with max equals 4 because it is simpler for identification technique to choose between a lower number of models.
2. Both BGLS and BS-IS provide slightly better identification for each technique at the same time series length compared to the results obtained using BS-NLS. Therefore, it can be concluded that the BGLS approach enables the identification technique to produce a superior identification for the model.

The remainder of this section demonstrates the findings of the three simulation studies in detail. Examining the results of Table 2, it is noticed that as the time series length  $n$  grows, correspondingly rises the ratio of the correct identification of both BGLS and BS-NLS increases. For each time series length, the results of the direct procedure utilizing the second prior are better than those obtained using the first prior and the third prior. The maximum order 3 results are superior to those of the maximum order 4 results since it is simpler for any identification approach to choose among a smaller number of models. Furthermore, the Broemeling and Shaarawy approach BS-NLS yields lower percentages of correct identification than the suggested method BGLS. Consequently, it can be concluded that the BGLS method achieved the identification approach to obtain a more accurate model identification.

## 6.Data Analysis

We utilize an actual dataset from the chemical field to illustrate the superior performance of the direct Bayesian identification technique employing the suggested method over Broemeling and Shaarawy approach.

### **Dataset:** Chemical Process Concentration Readings

This dataset shows the chemical process concentration values every two hours, 197 observations, as published by Box et al. (1994). The observations are as follows:

17.0	16.6	16.3	16.1	17.1	16.9	16.8	17.4	17.1	17.0	16.7	17.4	17.2
17.4	17.4	17.0	17.3	17.2	17.4	16.8	17.1	17.4	17.4	17.5	17.4	17.6
17.4	17.3	17.0	17.8	17.5	18.1	17.5	17.4	17.4	17.1	17.6	17.7	17.4
17.8	17.6	17.5	16.5	17.8	17.3	17.3	17.1	17.4	16.9	17.3	17.6	16.9
16.7	16.8	16.8	17.2	16.8	17.6	17.2	16.6	17.1	16.9	16.6	18.0	17.2

17.3	17.0	16.9	17.3	16.8	17.3	17.4	17.7	16.8	16.9	17.0	16.9	17.0
16.6	16.7	16.8	16.7	16.4	16.5	16.4	16.6	16.5	16.7	16.4	16.4	16.2
16.4	16.3	16.4	17.0	16.9	17.1	17.1	16.7	16.9	16.5	17.2	16.4	17.0
17.0	16.7	16.2	16.6	16.9	16.5	16.6	16.6	17.0	17.1	17.1	16.7	16.8
16.3	16.6	16.8	16.9	17.1	16.8	17.0	17.2	17.3	17.2	17.3	17.2	17.2
17.5	16.9	16.9	16.9	17.0	16.5	16.7	16.8	16.7	16.7	16.6	16.5	17.0
16.7	16.7	16.9	17.4	17.1	17.0	16.8	17.2	17.2	17.4	17.2	16.9	16.8
17.0	17.4	17.2	17.2	17.1	17.1	17.1	17.4	17.2	16.9	16.9	17.0	16.7
16.9	17.3	17.8	17.8	17.6	17.5	17.0	16.9	17.1	17.2	17.4	17.5	17.9
17.0	17.0	17.0	17.2	17.3	17.4	17.4	17.0	18.0	18.2	17.6	17.8	17.7
17.0	17.4											

They have determined an ARMA (p=1,q=1) model for this dataset utilizing the autocorrelation function (ACF) and partial autocorrelation function (PACF). A time plot for the series A is displayed in Figure 1.

Figure 1: Time series of the Chemical Process Concentration Reading

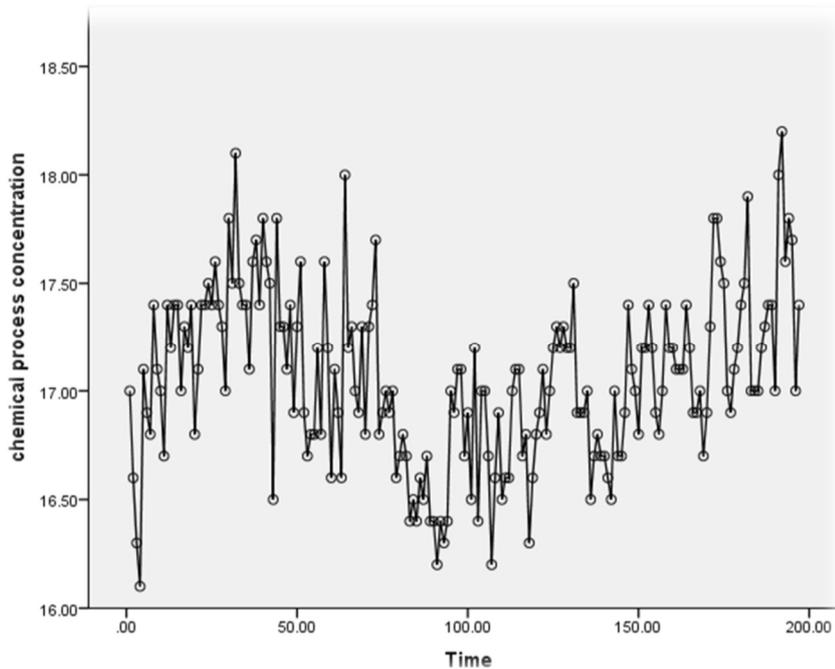


Table displays the outcomes of the Box-Jenkins methodology, the Broemeling and Shaarawy approach, and the proposed approach for each time series. Through examination of Table 3 results, we are able to notice

that the non-Bayesian (Box-Jenkins) approach and the Bayesian (BGLS and (BS-NLS) techniques both concur that ARMA is the choice model [p=1,q=1].

**Table 3:** Identified models for this dataset with various approaches.

		Series					
Max		3			4		
Technique	Prior	BGLS	BS-NLS	BS-IS	BGLS	BS-NLS	BS-IS
Box-Jenkins		ARMA (1,1)					
Direct	Prior1	ARMA (1,1)					
	Prior2	ARMA (1,1)					
	Prior3	ARMA (1,1)					

## 7. Conclusion

This article suggested a novel Bayesian technique for identifying autoregressive moving average (ARMA) processes. In contrast to Broemeling and Shaarawy, where the estimation errors that result from substituting the errors with their estimates are ignored, we suggested the use innovation substitution method for estimating the errors and exploiting the stochastic structure of the approximation error in establishing Bayesian identification for ARMA processes. Using the suggested approach, the direct Bayesian identification methodology has been created. Through simulation experiments, the effectiveness of the suggested technique has been verified and contrasted with the Broemeling and Shaarawy approach. The results of the simulation demonstrate the superiority of the suggested approach over Bromeling and Shaarawy's approach. We have verified the accuracy of our simulation studies by comparing our outcomes for Bromeling and Shaarawy with those found in the published literature.

## Acknowledgement:

The author extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through General Research Project under grant number (GRP/206/44).

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