Computational Journal of Mathematical and Statistical Sciences
3(1), 57–84
DOI:10.21608/CJMSS.2023.243845.1023
https://cjmss.journals.ekb.eg/

Exponentiated Generalized Weibull Exponential Distribution: Properties, Estimation and Applications

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Abstract: Real-life sciences rely heavily on statistical modeling because new applications and phenomena pop up constantly, increasing the demand for new distributions. In this article, the exponentiated generalized Weibull exponential (EGWE) distribution is proposed and studied. The density can exhibit decreasing, increasing, right-skewed, and left-skewed shapes. The hazard rate function shows decreasing, J-shaped, bathtub, and upside-down bathtub shapes. Statistical properties such as asymptotic behavior, quantile function, moment and incomplete moments, mean and median deviations, inequality measures, moment generating function, and order statistics are studied. The estimation of the parameters of the EGWE distribution using six frequentist estimation methods, namely maximum likelihood, least squares, maximum product of spacing, weighted least squares, Anderson-Darling, and Cramér-von Mises are discussed. Monte Carlo simulation study to ascertain the behavior of the estimators in terms of average absolute biases and mean square error is carried out. All the estimators performed very well since the average absolute biases and mean square errors decrease as the sample size increases. The usefulness of the EGWE distribution is illustrated with two datasets. The results show that the EGWE distribution provides better parametric fit compared with the competing distributions.

Keywords: Weibull distribution, estimation methods, bathtub, exponentiated generalized, simulations.
Mathematics Subject Classification: 60E05, 62N02, 37M05.
Received: 21 October 2023; Revised: 14 November 2023; Accepted: 2 December 2023; Online: 5 December 2023.

1. Introduction

In fitting data to actual phenomena, statistical distributions are essential. They are frequently used to model and analyze data across variety of fields, including engineering, biology, economics, finance, and the life sciences. Despite the fact that numerous distributions have been developed and studied, there is always room to develop or propose distributions that are either more flexible or that better fit particular real-world phenomena. However, some data may display complex pattern which may not be adequately modeled using the classical and traditional distributions. This complexity in data patterns has led to the need to develop statistical distributions that are more flexible, practical, and accurate in modeling them in the literature.

One of the numerous uses for the well-known continuous probability model known as the exponential distribution is life testing. The beta exponential distribution (Nadarajah and Kotz, [1]), exponentiated exponential distribution (Gupta and Kundu, [2]), generalized exponential distribution (Gupta and Kundu, [3]), Kumaraswamy exponential distribution (Cordeiro and Castro, [4]), and inverse exponential distribution (Keller et al. [5]) are all examples of attempts to increase the flexibility of the exponential distribution. Also, the idea of exponentiated distributions were utilized to create new distributions. Cordeiro and Castro [4] extended many known distributions as normal, Weibull, gamma, Gumbel, and inverse Gaussian distributions. They expressed the ordinary moments of these new family of generalized distributions as linear functions of probability weighted moments of the parent distribution.

Also, one of the life-time distributions that is most frequently utilized in reliability and lifetime data analysis is the Weibull distribution. The associated hazard rate function can be increasing, constant, or decreasing, hence not adaptable in modeling non-monotonic failure time data. However, hazard rate function can have a bathtub form in many applications in reliability and survival analysis. See Lai and Xie [6] and Bebbington et al. [7] for more information on how the hazard rate function is essential to the work of reliability engineers.

Cordeiro et al. [8] proposed the exponentiated generalized (EG) class of distributions with cumulative distribution function (CDF) and probability density function (PDF) given by Equations (1.1) and (1.2)

$$F(x) = [1 - \{1 - G(x)\}^{\alpha}]^{\beta}, \alpha > 0, \beta > 0, x \in \mathbb{R}. \tag{1.1}$$

$$f(x) = \alpha\beta g(x) \{1 - G(x)\}^{\alpha-1} [1 - \{1 - G(x)\}^{\alpha}]^{\beta}, x \in \mathbb{R}, \tag{1.2}$$

where $\alpha > 0$ and $\beta > 0$ are shape parameters. They proposed the exponentiated generalized Fréchet (EGF), exponentiated generalized normal (EGN), exponentiated generalized Gamma (EGGa), and exponentiated generalized Gumbel (EGGu) distributions as special cases. According to Cordeiro et al. [8], even if the baseline PDF, $g(x)$ is a symmetric distribution, the resulting distribution in Equation (1.2) will not be a symmetric distribution since the two shape parameters can control the tail weights and possibly add entropy to the center of the exponentiated generalized class of distributions.

Other extensions of this class of distributions can be found in Elbatal and Muhammed [9] and Oguntunde et al. [10] proposing the exponentiated generalized inverse Weibull (EGIW) and exponentiated generalized inverted exponential (EGIE) distributions respectively. Aslo, Oguntunde et al. [11] introduce the exponentiated generalized Weibull (EGW) distribution. Reyad et al. [12] used the EG class of distributions by Cordeiro et al. [8] in extending the Topp Leone-G by Al-Shomrani et al. [13]. They proposed the exponentiated generalized Topp-Leone-G family (EGTL-G). Elsherpienyi and Almetwally [15] proposed and studied the exponentiated generalized alpha power exponential (EGAPEx) distribution. The hazard rate exhibits L-shaped, increasing, decreasing, and upside-down bathtub shaped. The EGAPEx includes the exponential, alpha power exponential, alpha power generalized exponential, generalized exponential, standardized exponential, and exponentiated generalized

exponential distributions as special cases. The parameter(s) estimation of the Weibull generalized exponential distribution (WGED) based on the adaptive Type-II progressive (ATIIP) censored sample was investigated by Almongy et al. [14]. It was evident that the Bayesian estimation was better and more efficient than the maximum likelihood estimation (MLE) and maximum product spacing (MPS) estimation according to the mean square error (MSE). Also, El-Morshedy et al. [16] introduced a new 4-parameter exponentiated generalized inverse flexible Weibull (EGIFW) distribution. They estimated the model parameters via several methods namely; maximum likelihood, maximum product of spacing, and Bayesian. For more recently papers see [17, 18].
Moreover, Bilal et al. [19] introduced a new Weibull class of distributions with CDF and PDF given by Equation (1.3) and Equation (1.4)

$$G(x) = 1 - e^{-\left[\frac{-\log[1-K(x)]^a}{c}\right]^d} a > 0, c > 0, d > 0, x \in \mathbb{R}. \tag{1.3}$$

$$g(x) = a\left(\frac{d}{c}\right)\left[\frac{-\log[1-K(x)]^a}{c}\right]^{d-1} \frac{k(x)}{1-K(x)} e^{-\left[\frac{-\log[1-K(x)]^a}{c}\right]^d}, x \in \mathbb{R}. \tag{1.4}$$

They proposed the Weibull exponential (WE) distribution as a special case. The CDF and PDF of the WE distribution is given by

$$G(x) = 1 - e^{-\left(\frac{abx}{c}\right)^d}, x > 0, a, b, c, d > 0 \tag{1.5}$$

and

$$g(x) = d\left(\frac{ab}{c}\right)^d x^{d-1} e^{-\left(\frac{abx}{c}\right)^d}, x > 0 \tag{1.6}$$

respectively.
From these concepts, this paper combine the works of Cordeiro et al. [8] and Bilal et al. [19] to a proposed new distribution known as exponentiated generalized Weibull exponential (EGWE) distribution. This is a generalization of the WE distribution by Bilal et al. [19]. To the best of our knowledge this is an attempt to generalized WE distribution by Bilal et al. [19]. We demonstrate the usefulness and flexibility of the EGWE distribution in comparison to other distributions.
The rest of the paper is organized as follows: Section 2 presents the EGWE distribution. The statistical properties of the distribution is presented in Section 3. In Section 4, six estimation methods are presented. Monte Carlo simulations are carried out in Section 5. In Section 6, the applications of the EGWE distribution is illustrated using two real life datasets and the conclusion is presented in Section 7.

2. The Exponentiated Generalized Weibull Exponential Distribution

In this section, we propose the exponentiated generalized Weibull exponential (EGWE) distribution. The CDF of the proposed distribution is obtained by substituting Equation (1.5) into Equation (1.1). Therefore, a random variable X is said to follow the EGWE distribution if the CDF is given by

$$F(x) = \left[1 - e^{-\alpha\left(\frac{abx}{c}\right)^d}\right]^\beta, x > 0, a, b, c, d, \alpha, \beta > 0. \tag{2.1}$$

The related PDF is given by

$$f(x) = \alpha\beta d\left(\frac{ab}{c}\right)^{d} x^{d-1} e^{-\alpha\left(\frac{abx}{c}\right)^{d}} \left[1 - e^{-\alpha\left(\frac{abx}{c}\right)^{d}}\right]^{\beta-1}, x > 0. \tag{2.2}$$

The hazard rate function is given by

$$h(x) = \frac{\alpha\beta d\left(\frac{ab}{c}\right)^{d} x^{d-1} e^{-\alpha\left(\frac{abx}{c}\right)^{d}} \left[1 - e^{-\alpha\left(\frac{abx}{c}\right)^{d}}\right]^{\beta-1}}{1 - \left[1 - e^{-\alpha\left(\frac{abx}{c}\right)^{d}}\right]^{\beta}}. \tag{2.3}$$

The density plots of the EGWE in Figure 1 show decreasing, increasing, right skewed, and left skewed shapes.

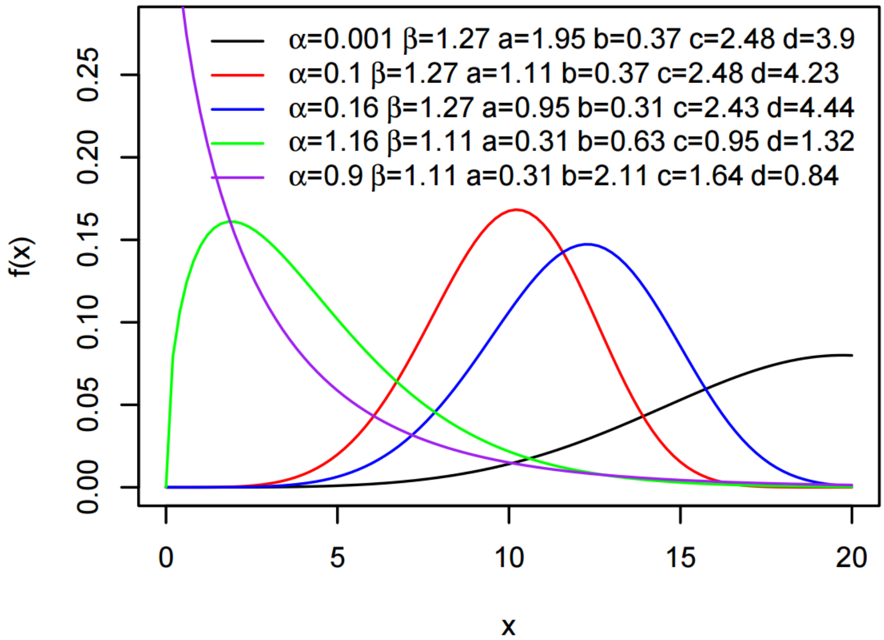


Figure 1. Plots of the density of the EGWE distribution

Figure 2 shows the hazard rate plots of the EGWE. From the plots, there is decreasing, J-shaped, bathtub and upside-down bathtub shapes.

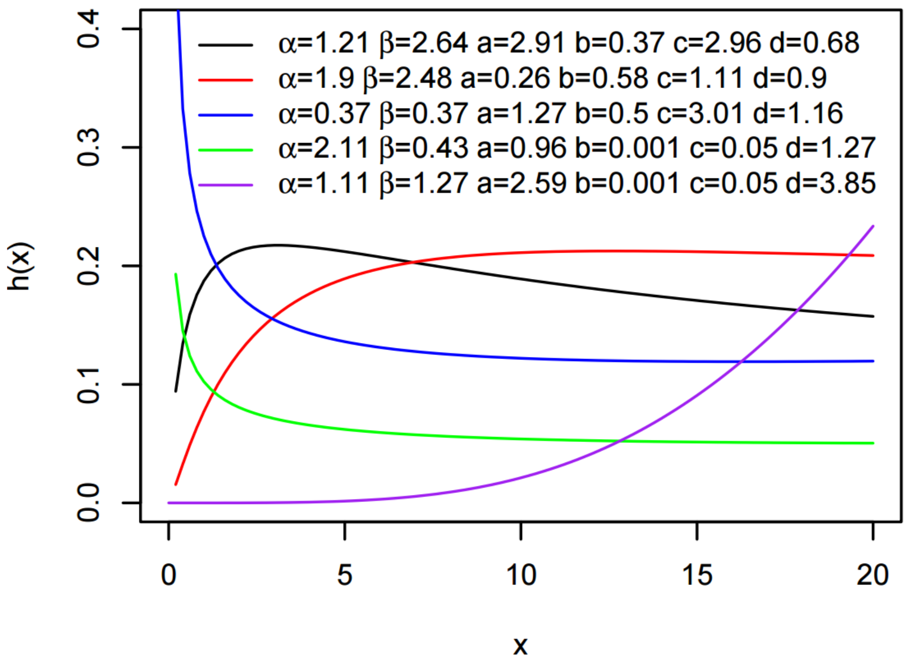


Figure 2. Plots of the hazard rate function of the EGWE distribution

2.1. Sub-models

The EGWE distribution consists of a number of vital sub-models that are widely used in modeling. These includes: Weibull exponential (WE) distribution (Bilal et al. [19]), exponential Weibull (EW) distribution (Pal et al. [20]), generalized Weibull (GW) distribution (Mudholkar and Srivastava, [21]), exponential (E) distribution, Weibull (W) distribution, Rayleigh distribution, exponentiated exponential Weibull (EEWD) distribution (Al-Sulami, [22]), and exponentiated generalized Weibull (EGWeibull) distribution (Oguntunde et al. [11]) . These are displayed in Table 1.

Table 1. Sub-models from the EGWE distribution

Distribution	α	β	a	b	c	d
Rayleigh	1	1	a	b	c	2
EEWD	α	β	1	1	c	d
EGWeibull	1	β	1	1	c	d
WE	1	1	a	b	c	d
EW	1	β	a	1	1	d
GW	1	β	1	b	1	d
W	1	1	1	1	c	d
E	1	1	1	1	c	1

2.2. Useful Expansions

In deriving the statistical properties of the EGWE distribution, it is essential that the expansion of the density be obtained.
Using the generalized binomial expansion of the form; $(1-t)^n = \sum_{j=0}^{\infty}(-1)^n\binom{n}{j}t^j, |t| \leq 1,$

$$\left[1-e^{-\alpha\left(\frac{abx}{c}\right)^d}\right]^{\beta-1} = \sum_{j=0}^{\infty}(-1)^j\binom{\beta-1}{j}e^{-j\alpha\left(\frac{abx}{c}\right)^d}. \tag{2.4}$$

Equation (2.4) can be rewritten as

$$\left[1-e^{-\alpha\left(\frac{abx}{c}\right)^d}\right]^{\beta-1} = \sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(\beta)}{j!\Gamma(\beta-j)}e^{-j\alpha\left(\frac{abx}{c}\right)^d}. \tag{2.5}$$

Substituting Equation (2.5) into Equation (2.2), we have

$$f(x) = \alpha\beta d\left(\frac{ab}{c}\right)^d\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(\beta)}{j!\Gamma(\beta-j)}x^{d-1}e^{-\alpha\left(\frac{ab}{c}\right)^d(1+j)x^d}$$

letting $\Psi_j = \frac{(-1)^j\Gamma(\beta)}{j!\Gamma(\beta-j)}$, we obtain the expansion of the PDF of the EGWE distribution as

$$f(x) = \alpha\beta d\left(\frac{ab}{c}\right)^d\sum_{j=0}^{\infty}\Psi_j x^{d-1}e^{-\alpha\left(\frac{ab}{c}\right)^d(1+j)x^d}. \tag{2.6}$$

3. Statistical Properties

In this section, some statistical properties of the EGWE distribution are studied.

3.1. Asymptotic Behavior

The behavior of the CDF of the EGWE distribution is investigated as $x \to 0$ and as $x \to \infty$.
As $x \to 0$;

$$\lim_{x\to 0}F(x) = \lim_{x\to 0}\left[\left[1-e^{-\alpha\left(\frac{abx}{c}\right)^d}\right]^{\beta}\right] = 0.$$

As $x \to \infty$;

$$\lim_{x\to\infty}F(x) = \lim_{x\to\infty}\left[\left[1-e^{-\alpha\left(\frac{abx}{c}\right)^d}\right]^{\beta}\right] = 1.$$

In like manner, the behavior of the PDF of the EGWE distribution is investigated as $x \to 0$ and as $x \to \infty$.
As $x \to 0$;

$$\lim_{x\to 0}f(x) = \lim_{x\to 0}\left[\alpha\beta d\left(\frac{ab}{c}\right)^d x^{d-1}e^{-\alpha\left(\frac{abx}{c}\right)^d}\left[1-e^{-\alpha\left(\frac{abx}{c}\right)^d}\right]^{\beta-1}\right] = 0.$$

As $x \to \infty$;

$$\lim_{x\to\infty}f(x) = \lim_{x\to\infty}\left[\alpha\beta d\left(\frac{ab}{c}\right)^d x^{d-1}e^{-\alpha\left(\frac{abx}{c}\right)^d}\left[1-e^{-\alpha\left(\frac{abx}{c}\right)^d}\right]^{\beta-1}\right] = 0.$$

3.2. Quantile Function

To obtain the quantile function of the EGWE distribution, one has to determine the inverse of the CDF of the EGWE distribution, which is used in obtaining randomly generated datasets from the EGWE distribution. Thus, the quantile function of the EGWE distribution for $u \in (0, 1)$ is given by

$$x_u = \frac{c}{\alpha ab}\left[-\log\left(1 - u^{1/\beta}\right)\right]^{1/d}. \tag{3.1}$$

3.3. Moments and Incomplete moments

The moments of a distribution is important in estimating measures of variation like the variance, standard deviation, coefficient of variation, mean deviation, median deviation, kurtosis, skewness amongst others.
The r^{th} non-central moment by defintion is given by

$$\mu_r^{'} = \int_0^\infty x^r f(x)dx.$$

This implies that,

$$\mu_r^{'} = \alpha\beta d\left(\frac{ab}{c}\right)^d \sum_{j=0}^{\infty} \Psi_j \int_0^\infty x^r x^{d-1} e^{-\alpha\left(\frac{ab}{c}\right)^d (1+j)x^d} dx.$$

Letting $y = \alpha\left(\frac{ab}{c}\right)^d (1 + j)x^d$, implies that, if $x \to 0$, $y \to 0$, and if $x \to \infty$, $y \to \infty$. Also, $dx = \frac{dy}{\alpha d\left(\frac{ab}{c}\right)^d (i+j)x^{d-1}}$ and $x = \left[\frac{y}{\alpha\left(\frac{ab}{c}\right)^d (1+j)}\right]^{1/d}$. Using the identity; $\Gamma(s) = \int_0^\infty y^{s-1}e^{-y}dy$ and after some algebra, we get

$$\mu_r^{'} = \frac{\beta}{\alpha^{r/d}(ab/c)^r} \sum_{j=0}^{\infty} \Psi_j (1 + j)^{-(r/d+1)} \Gamma(1 + r/d), \quad r > d, \tag{3.2}$$

where $r = 1, 2, ...$
The incomplete moments are vital when estimating measures of inequality like the Bonferroni and Lorenz curves, and measures of deviation such as mean and median deviations.
By definition, the r^{th} incomplete moment is given by

$$M_r(x) = \int_0^x y^r f(y)dy.$$

Using the concept in proofing the moments and the lower incomplete gamma function; $\Gamma(w, s) = \int_0^s y^{w-1}e^{-y}dy$, we get

$$M_r(x) = \frac{\beta}{\alpha^{r/d}(ab/c)^r} \sum_{j=0}^{\infty} \Psi_j (1 + j)^{-(r/d+1)} \Gamma(1 + r/d, \alpha(ab/c)^d (1 + j)x^d), \quad r > d, c \neq 0. \tag{3.3}$$

3.4. Mean and Median Deviations

The totality of the deviations from the mean and median can be used to estimate the variation in a population with some certainty. By definition, the mean deviation is given as $\nu_1 = 2\mu F(\mu) - 2\int_0^{\mu} xf(x)dx$, where $\int_0^{\mu} xf(x)dx$ is simplified using the first incomplete moment. Therefore, the mean deviation of the EGWE distribution is given by

$$\nu_1 = 2\mu F(\mu) - \frac{2\beta}{\alpha^{1/d}(ab/c)} \sum_{j=0}^{\infty} \Psi_j (1+j)^{-(1/d+1)} \Gamma(1+1/d, \alpha(ab/c)^d (1+j)x^d), \;\; d > 1, c \neq 0, \tag{3.4}$$

where $\mu = \mu_1$ is the mean of X.
Also, the median deviation is defined as $\nu_2 = \mu - 2\int_0^{M} xf(x)dx$. Thus, the median deviation of the EGWE distribution is given by

$$\nu_2 = \mu - \frac{2\beta}{\alpha^{1/d}(ab/c)} \sum_{j=0}^{\infty} \Psi_j (1+j)^{-(1/d+1)} \Gamma(1+1/d, \phi), \;\; d > 1, \tag{3.5}$$

where $\phi = \alpha(ab/c)^d (1+j)M^d$.

3.5. Inequality Measures

The Lorenz and Bonferroni curves are frequently used to assess the level of economic inequality in a population. The Bonferroni curve, $B_F(x)$, is the scaled conditional mean curve, which is the ratio of the group mean income of the population. The Lorenz curve, $L_F(x)$, indicates the proportion of total income volume accumulated by those units with income lower than or equal to volume x.
The Bonferroni curve, $B_F(x)$ and the Lorenz curve, $L_F(x)$ of the EGWE distribution is by Equations (3.6) and (3.7) respectively.

$$B_F(x) = \frac{\beta}{(ab/c)\alpha^{1/d}\mu F(x)} \sum_{j=0}^{\infty} \Psi_j (1+j)^{-(1/d+1)} \Gamma(1+1/d, \alpha(ab/c)^d (1+j)x^d). \tag{3.6}$$

$$L_F(x) = \frac{\beta}{(ab/c)\alpha^{1/d}\mu} \sum_{j=0}^{\infty} \Psi_j (1+j)^{-(1/d+1)} \Gamma(1+1/d, \alpha(ab/c)^d (1+j)x^d). \tag{3.7}$$

3.6. Moment Generating Function

The moment generating function (MGF) of a random variable X by definition is given as, $M_x(z) = \mathbb{E}(e^{zx})$, if it exists. Using Taylor series; $M_x(z) = \sum_{r=0}^{\infty} \frac{z^r}{r!}\mu_r'$. Therefore, the MGF of the EGWE distribution is given by

$$M_x(z) = \sum_{r=0}^{\infty} \frac{z^r \beta}{r!\alpha^{r/d}(ab/c)^r} \sum_{j=0}^{\infty} \Psi_j (1+j)^{-(r/d+1)} \Gamma(1+r/d) \tag{3.8}$$

3.7. *Order Statistics*

Let $X_1, X_2, \ldots, X_n$ be a sample of size n from the EGWE distribution and $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ denote the order statistics of the sample.
The PDF of the first-order statistics is defined as

$$f_{1:n}(x) = n[1 - F(x)]^{n-1} f(x). \tag{3.9}$$

Substituting the CDF and PDF of the EGWE distribution into Equation (3.9), we get

$$f_{1:n}(x) = n\alpha\beta d\left(\frac{ab}{c}\right)^d \left[1 - \left[1 - e^{-\alpha\left(\frac{abx}{c}\right)^d}\right]^\beta\right]^{n-1} x^{d-1} e^{-\alpha\left(\frac{abx}{c}\right)^d} \left[1 - e^{-\alpha\left(\frac{abx}{c}\right)^d}\right]^{\beta-1}. \tag{3.10}$$

Also, the PDF of the n^{th} order statistics is defined as

$$f_{n:n}(x) = n[F(x)]^{n-1} f(x). \tag{3.11}$$

Therefore, substituting the CDF and PDF of the EGWE distribution in Equation (3.11), we get the PDF of the n^{th} order statistics as

$$f_{n:n}(x) = n\alpha\beta d\left(\frac{ab}{c}\right)^d \left[\left[1 - e^{-\alpha\left(\frac{abx}{c}\right)^d}\right]^\beta\right]^{n-1} x^{d-1} e^{-\alpha\left(\frac{abx}{c}\right)^d} \left[1 - e^{-\alpha\left(\frac{abx}{c}\right)^d}\right]^{\beta-1}. \tag{3.12}$$

4. Estimation Methods

This section discusses the estimation of the EGWE parameters via six estimation approaches. These are the maximum likelihood, maximum product of space, least squares, weighted least squares, Anderson-Darling, and Cramér-von Mises methods.

4.1. *Maximum Likelihood Estimation*

The maximum likelihood estimators (MLEs) of the EGWE parameters are discussed. Let $X_1, X_2, ..., X_n$ be n random sample from the EGWE distribution and $\Theta = (\alpha, \beta, a, b, c, d)^T$, then the log-likelihood function $\ell = \ell(\Theta)$ is given by

$$\begin{aligned} \ell =& n\log\left(\alpha\beta d(ab/c)^d\right) + (d-1)\sum_{i=1}^{n}\log(x_i) - \alpha(ab/c)^d \sum_{i=1}^{n} x_i^d \\ &+ (\beta - 1)\sum_{i=1}^{n}\log\left(1 - e^{-\alpha\left(\frac{abx_i}{c}\right)^d}\right). \end{aligned} \tag{4.1}$$

By maximizing the total likelihood function with respect to the parameters $\hat{\alpha}$, $\hat{\beta}$, $\hat{a}$, $\hat{b}$, $\hat{c}$, and $\hat{d}$, the ML estimates of the parameters can be obtained. Nevertheless, when the log-likelihood function in Equation (4.1) is differentiated with respect to each parameter the score functions are obtained as

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} - \left(\frac{ab}{c}\right)^d \sum_{i=1}^{n} x_i^d + (\beta - 1)\sum_{i=1}^{n} \frac{e^{\alpha\left(\frac{abx_i}{c}\right)^d}\left[\left(\frac{abx_i}{c}\right)^d - e^{-\alpha\left(\frac{abx_i}{c}\right)^d}\left(e^{\alpha\left(\frac{abx_i}{c}\right)^d} - 1\right)\left(\frac{abx_i}{c}\right)^d\right]}{\left(e^{\alpha\left(\frac{abx_i}{c}\right)^d} - 1\right)}, \tag{4.2}$$

$$\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \log\left[e^{-\alpha\left(\frac{abx_i}{c}\right)^d}\left(e^{\alpha\left(\frac{abx_i}{c}\right)^d} - 1\right)\right], \tag{4.3}$$

$$\begin{aligned}\frac{\partial \ell}{\partial a} =& \frac{nd}{a} - \frac{\alpha bd\left(\frac{ab}{c}\right)^{d-1}}{c}\sum_{i=1}^{n} x_i^d + (\beta - 1)\sum_{i=1}^{n} e^{\alpha\left(\frac{abx_i}{c}\right)^d} \\ &\times\left[\frac{bdx_i(abx_i/c)^d - bde^{-\alpha\left(\frac{abx_i}{c}\right)^d}\left(e^{\alpha\left(\frac{abx_i}{c}\right)^d} - 1\right)\alpha x_i(abxi/c)^{d-1}}{c\left(e^{\alpha\left(\frac{abx_i}{c}\right)^d} - 1\right)}\right],\end{aligned} \tag{4.4}$$

$$\begin{aligned}\frac{\partial \ell}{\partial b} =& \frac{nd}{b} - \frac{\alpha ad\left(\frac{ab}{c}\right)^{d-1}}{c}\sum_{i=1}^{n} x_i^d + (\beta - 1)\sum_{i=1}^{n} e^{\alpha\left(\frac{abx_i}{c}\right)^d} \\ &\times\left[\frac{\alpha adx_i(abx_i/c)^d - ade^{-\alpha\left(\frac{abx_i}{c}\right)^d}\left(e^{\alpha\left(\frac{abx_i}{c}\right)^d} - 1\right)\alpha x_i(abxi/c)^{d-1}}{c\left(e^{\alpha\left(\frac{abx_i}{c}\right)^d} - 1\right)}\right],\end{aligned} \tag{4.5}$$

$$\begin{aligned}\frac{\partial \ell}{\partial c} =& \frac{nd}{c} - \frac{\alpha ad\left(\frac{ab}{c}\right)^{d-1}}{c}\sum_{i=1}^{n} x_i^d + (\beta - 1)\sum_{i=1}^{n} e^{\alpha\left(\frac{abx_i}{c}\right)^d} \\ &\times\left[\frac{abde^{-\alpha\left(\frac{abx_i}{c}\right)^d}\left(e^{\alpha\left(\frac{abx_i}{c}\right)^d} - 1\right)\alpha x_i(abxi/c)^{d-1} - \alpha abdx_i(abx_i/c)^d}{c^2\left(e^{\alpha\left(\frac{abx_i}{c}\right)^d} - 1\right)}\right],\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}\frac{\partial \ell}{\partial d} =& \frac{n(ab/c)^{-d}\left[\alpha\beta(ab/c)^d + (ab/c)^d d\alpha\beta\log(ab/c)\right]}{d\alpha\beta} - \alpha(ab/c)^d\log(ab/c)\sum_{i=1}^{n} x_i^d \\ &+ \sum_{i=1}^{n}\log(x_i) - \alpha(ab/c)^d\sum_{i=1}^{n} x_i^d\log(x_i) + (\beta - 1)\sum_{i=1}^{n} e^{\alpha\left(\frac{abx_i}{c}\right)^d} \\ &\times\left[\frac{\alpha(abx_i/c)^d\log(abx_i/c) - e^{-\alpha\left(\frac{abx_i}{c}\right)^d}\left(e^{\alpha\left(\frac{abx_i}{c}\right)^d} - 1\right)(abx_i/c)^d\alpha(abxi/c)^{d-1}}{c^2\left(e^{\alpha\left(\frac{abx_i}{c}\right)^d} - 1\right)}\right].\end{aligned} \tag{4.7}$$

Equating the score functions to zero and solving the resulting system of equations numerically, the estimates $\hat{\alpha}$, $\hat{\beta}$, $\hat{a}$, $\hat{b}$, $\hat{c}$, and $\hat{d}$ are obtained.

4.2. Maximum Product of Spacing Estimation

Consider the order statistics of a random sample from the EGWE distribution, denoted by $x_{(1:n)}, x_{(2:n)}, ..., x_{(n:n)}$, and consider the uniform spacings for the random sample:

$$D_i(\alpha,\beta,a,b,c,d) = F(x_{(i)}|\alpha,\beta,a,b,c,d) - F(x_{(i-1)}|\alpha,\beta,a,b,c,d), i = 1,2,...,n+1,$$

where $F(x_{(0)}|\alpha,\beta,a,b,c,d) = 0$, $F(x_{(n+1)}|\alpha,\beta,a,b,c,d) = 1$, $\sum_{i=1}^{n+1} D_i(\alpha,\beta,a,b,c,d) = 1$, $F(x_{(i)}|\alpha,\beta,a,b,c,d) = \left[1 - e^{-\alpha\left(\frac{abx_i}{c}\right)^d}\right]^\beta$, $F(x_{(i-1)}|\alpha,\beta,a,b,c,d) = \left[1 - e^{-\alpha\left(\frac{abx_{i-1}}{c}\right)^d}\right]^\beta$. Therefore, the maximum product spacing estimators (MPEs) of $\hat{\alpha}_{MPSE}$, $\hat{\beta}_{MPSE}$, $\hat{a}_{MPSE}$, $\hat{b}_{MPSE}$, $\hat{c}_{MPSE}$, and $\hat{d}_{MPSE}$ follow by maximizing either the geometric mean spacings or the logarithm of the sample geometric mean spacings which are defined by

$$MS(\alpha,\beta,a,b,a,d) = \left[\prod_{i=1}^{n+1} D_i(\alpha,\beta,a,b,c,d)\right]^{\frac{1}{n+1}}$$

and

$$LM(\alpha,\beta,a,b,a,d) = \frac{1}{n+1}\sum_{i=1}^{n+1} \log[D_i(\alpha,\beta,a,b,c,d)],$$

with respect to α, β, a, b, c, and d.
The MPSEs of the EGWE parameters can also be obtained by solving the following non-linear equations:

$$\frac{1}{n+1}\sum_{i=1}^{n+1} \frac{1}{D_i(\alpha,\beta,a,b,c,d)} \left[\Lambda_r(x_{(i)}|\alpha,\beta,a,b,c,d) - \Lambda_r(x_{(i-1)}|\alpha,\beta,a,b,c,d)\right] = 0, r = 1,2,...,6,$$

where

$\Lambda_1(x_{(i)}|\alpha,\beta,a,b,c,d) = \frac{\partial}{\partial\alpha}F(x_{(i)}|\alpha,\beta,a,b,c,d)$, $\Lambda_2(x_{(i)}|\alpha,\beta,a,b,c,d) = \frac{\partial}{\partial\beta}F(x_{(i)}|\alpha,\beta,a,b,c,d)$, $\Lambda_3(x_{(i)}|\alpha,\beta,a,b,c,d) = \frac{\partial}{\partial a}F(x_{(i)}|\alpha,\beta,a,b,c,d)$, $\Lambda_4(x_{(i)}|\alpha,\beta,a,b,c,d) = \frac{\partial}{\partial b}F(x_{(i)}|\alpha,\beta,a,b,c,d)$, $\Lambda_5(x_{(i)}|\alpha,\beta,a,b,c,d) = \frac{\partial}{\partial c}F(x_{(i)}|\alpha,\beta,a,b,c,d)$, and $\Lambda_6(x_{(i)}|\alpha,\beta,a,b,c,d) = \frac{\partial}{\partial d}F(x_{(i)}|\alpha,\beta,a,b,c,d)$.

The Λ_r for $r = 1,2,...,6$. can be solved numerically.

4.3. Least Squares and Weighted Least Squares Estimation

Consider the order statistics of a random sample from the EGWE distribution denoted by $x_{(1:n)}, x_{(2:n)}, ..., x_{(n:n)}$. The least squares estimators (LSEs) of the EGWE parameters $\hat{\alpha}_{LSE}$, $\hat{\beta}_{LSE}$, $\hat{a}_{LSE}$, $\hat{b}_{LSE}$, $\hat{c}_{LSE}$, and $\hat{d}_{LSE}$ follow by minimizing

$$L(\alpha,\beta,a,b,c,d) = \sum_{i=1}^{n} \left[F(x_{(i:n)}|\alpha,\beta,a,b,c,d) - \frac{i}{n+1}\right]^2,$$

with respect to α, β, a, b, c, and d. Alternatively, the LSEs are obtained by solving the non-linear

$$\sum_{i=1}^{n}\left[F(x_{(i:n)}|\alpha,\beta,a,b,c,d)-\frac{i}{n+1}\right]\Lambda_r(x_{(i:n)}|\alpha,\beta,a,b,c,d)=0, r=1,2,...,6,$$

where $\Lambda_1(.|\alpha,\beta,a,b,c,d)$, $\Lambda_2(.|\alpha,\beta,a,b,c,d)$, $\Lambda_3(.|\alpha,\beta,a,b,c,d)$, $\Lambda_4(.|\alpha,\beta,a,b,c,d)$, $\Lambda_5(.|\alpha,\beta,a,b,c,d)$, and $\Lambda_6(.|\alpha,\beta,a,b,c,d)$ are as defined earlier.
The weighted least squares estimators (WLSEs) of the EGWE parameters $\hat{\alpha}_{WLSE}, \hat{\beta}_{LWSE}, \hat{a}_{WLSE}, \hat{b}_{WLSE}$, $\hat{c}_{WLSE}$, and $\hat{d}_{WLSE}$, can be obtained by minimizing the Equation (4.8) with respect to the parameters

$$W(\alpha,\beta,a,b,c,d)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_{(i:n)}|\alpha,\beta,a,b,c,d)-\frac{i}{n+1}\right]^2. \tag{4.8}$$

Also, the WLSEs can be obtained by solving the non-linear equation in Equation (4.9)

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_{(i:n)}|\alpha,\beta,a,b,c,d)-\frac{i}{n+1}\right]\Lambda_r(.|\alpha,\beta,a,b,c,d)=0, r=1,2,...,6. \tag{4.9}$$

4.4. Cramér-von Mises Estimation

The Cramér-von Mises estimators (CVMEs) of the EGWE parameters can be estimated by minimizing Equation (4.10) with respect to α, β, a, b, c, and d

$$C(\alpha,\beta,a,b,c,d)=\frac{1}{12n}+\sum_{i=1}^{n}\left[F(x_{(i:n)}|\alpha,\beta,a,b,c,d)-\frac{2i-1}{2n}\right]^2. \tag{4.10}$$

Alternatively, the CVMEs can be obtained by solving Equation (4.11) numerically.

$$\sum_{i=1}^{n}\left[F(x_{(i:n)}|\alpha,\beta,a,b,c,d)-\frac{2i-1}{2n}\right]\Lambda_r(.|\alpha,\beta,a,b,c,d)=0, r=1,2,...,6. \tag{4.11}$$

4.5. Anderson-Darling Estimation

The Anderson-Darling estimators (ANDEs) are considered a type of minimum distance estimators. The ANDEs of the EGWE parameters can be estimated by minimizing Equation (4.12)

$$A(\alpha,\beta,a,b,c,d)=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left\{\log\left[F(x_{(i:n)}|\alpha,\beta,a,b,c,d)\right]+\log\left[\bar{F}(x_{(i:n)}|\alpha,\beta,a,b,c,d)\right]\right\} \tag{4.12}$$

with respect to α, β, a, b, c, and d. These estimates can also be obtained by solving Equation (4.13) numerically.

$$\sum_{i=1}^{n}(2i-1)\left[\frac{\Lambda_r(x_{(i:n)}|\alpha,\beta,a,b,c,d)}{F(x_{(i:n)}|\alpha,\beta,a,b,c,d)}-\frac{\Lambda_j(x_{(n+1-i:n)}|\alpha,\beta,a,b,c,d)}{\bar{F}(x_{(n+1-i:n)}|\alpha,\beta,a,b,c,d)}\right]=0, r,j=1,2,...,6, \tag{4.13}$$

where $\bar{F}(x_{(n+1-i:n)}|\alpha,\beta,a,b,c,d)=1-\left[1-e^{-\alpha\left(\frac{abx_{n+1-i}}{c}\right)^d}\right]^\beta$.

5. Monte Carlo Simulation

In this section, simulation study to assess the performance of the six different estimators of the EGWE parameters. We generated 2000 samples from the EGWE distribution for sample sizes, $n = 40, 70, 100, 200, 400$ and for $\alpha = 1.02$, $\beta = 0.32$, $a = 1$, $b = 0.1$, $c = 0.14$, and $d = 0.2$ and $\alpha = 1.25$, $\beta = 0.41$, $a = 0.21$, $b = 0.17$, $c = 0.61$, and $d = 1$. The properties of the estimators are investigated by computing average absolute biases (AVBs) and mean square errors (MSEs) for each of the parameters. Simulation results of the six estimation methods are presented in Tables 3 and 5. For all the estimation methods, the AVBs approaches zero as the sample size increases, evident that these estimates behave as asymptotically unbiased estimators. Also, the MSEs for all the estimation methods decrease for all parameters combinations as the sample size increases, an indication that the estimators are consistent. Again, all the estimates of the EGWE parameters obtained from the six estimation methods are fairly good, providing credible MSEs and small AVBs. Therefore, the results show that all the estimation methods perform well in estimating the parameters of the EGWE distribution.

6. Applications

This section illustrates the usefulness and flexibility of the EGWE distribution using real life datasets.
The performance of the EGWE distribution is compared with other distributions. The performance of the distributions about providing proper parametric fit to the dataset is compared using the AIC, BIC, Cramér-von Misses (W^*), Anderson-Darling (A^*) and K-S statistics. The distribution with the least of these measures provides a reasonable fit to the dataset. The competitive distributions of the EGWE distribution as listed in Table 2.

Table 2. Competing distributions of the EGWE distribution

Competing Distribution	Abbreviation	Author(s)
Weibull Exponential	WE	Bilal et al. [19]
Exponentiated Exponential Inverse Weibull	EEIW	Badr and Sobahi [23]
Exponentiated Generalized Fréchet	EGF	Cordeiro et al. [8]
Marshall-Olkin Power Lomax	MOPLX	ul Haq et al. [25]
Weibull	W	-
Exponential	E	-

6.1. Blood Cancer

The second dataset consists of lifetime (in years) of 40 blood cancer (leukemia) patients from one military hospital in Saudi Arabia. This data was also used by Klakattawa [26]. It is available in R package DataSetsUni by Imran et al. [24]. Figure 7 shows the TTT plot of the hazard rate of the blood cancer dataset, there is evidence of increasing hazard rate function. The box plot, violin plot, histogram and kernel density plot of the blood cancer dataset is as displayed in Figure 8.

Table 3. Simulation results of several estimation methods for $\alpha = 1.02$, $\beta = 0.32$, $a = 1$, $b = 0.1$, $c = 0.14$, $d = 0.2$

Parameter	n	MLEs	ANDEs	CVMEs	MPEs	LSEs	WLEs
				AVBs			
	40	0.1229	0.2199	0.3925	0.1129	0.3620	0.1788
	70	0.0646	0.1423	0.2465	0.0590	0.2473	0.1098
α	100	0.0466	0.1066	0.1911	0.0434	0.2104	0.0820
	200	0.0242	0.0564	0.1237	0.0253	0.1171	0.0474
	400	0.0195	0.0347	0.0771	0.0190	0.0770	0.0304
	40	0.2808	0.3276	1.9227	0.2399	2.1162	0.3840
	70	0.1630	0.1802	0.2784	0.1480	0.2827	0.1933
β	100	0.1205	0.1445	0.1921	0.1217	0.2225	0.1446
	200	0.0750	0.0941	0.1281	0.0754	0.1216	0.0988
	400	0.0530	0.0625	0.0851	0.0534	0.0862	0.0654
	40	0.5702	0.5550	0.9806	0.4126	1.3987	1.5624
	70	0.5518	0.5469	0.5125	0.3732	0.5266	0.7347
a	100	0.5111	0.5113	0.4757	0.3249	0.4965	0.6099
	200	0.4958	0.5088	0.4739	0.3127	0.4884	0.6026
	400	0.4864	0.5019	0.4674	0.2754	0.4881	0.5781
	40	0.9303	1.0175	1.4871	0.9199	1.8661	1.9410
	70	0.9167	0.9483	0.9836	0.9107	0.9821	1.1405
b	100	0.9124	0.9382	0.9562	0.9078	0.9714	1.0085
	200	0.9072	0.9205	0.9318	0.9034	0.9186	0.9315
	400	0.9062	0.9119	0.9176	0.9018	0.9186	0.9315
	40	0.4859	0.6207	0.8387	1.2055	1.0841	1.5213
	70	0.4280	0.5576	0.7471	1.1065	0.7670	0.4771
c	100	0.4102	0.5116	0.7190	0.9889	0.7400	0.4536
	200	0.3614	0.4505	0.5938	0.9756	0.6235	0.4163
	400	0.3286	0.4167	0.5238	0.8868	0.5248	0.3665
	40	0.1883	0.0917	0.1349	1.0894	0.1331	0.1048
	70	0.1109	0.0722	0.1041	1.0628	0.1029	0.0797
d	100	0.0677	0.0617	0.0895	0.0504	0.0920	0.0680
	200	0.0396	0.0435	0.0658	0.0349	0.0630	0.0489
	400	0.0265	0.0306	0.0440	0.0240	0.0447	0.0329

Table 4. Cont.:Simulation results of several estimation methods for $\alpha = 1.02$, $\beta = 0.32$, $a = 1$, $b = 0.1$, $c = 0.14$, $d = 0.2$

Parameter	n	MLEs	ANDEs	CVMEs	MPEs	LSEs	WLEs
				MSEs			
	40	0.1046	0.1838	0.5916	0.0877	0.4789	0.1621
	70	0.0234	0.0659	0.1825	0.0186	0.1841	0.0568
α	100	0.0126	0.0408	0.1088	0.0099	0.1306	0.0322
	200	0.0021	0.0122	0.0523	0.0105	0.0458	0.0097
	400	0.0008	0.0040	0.0214	0.0005	0.0216	0.0029
	40	0.4703	3.6679	8.9208	0.2796	5.8799	2.0992
	70	0.0588	0.0872	0.5010	0.0522	0.5127	0.1200
β	100	0.0313	0.0528	0.1108	0.0330	0.4198	0.0488
	200	0.0102	0.0176	0.0318	0.0102	0.0275	0.0176
	400	0.0053	0.0067	0.0127	0.0050	0.0131	0.0070
	40	0.4490	1.7992	2.4970	0.3070	3.4764	9.0713
	70	0.3875	0.4087	0.4928	0.2532	0.4647	2.4993
a	100	0.3872	0.3932	0.3728	0.2009	0.5931	0.5560
	200	0.3870	0.3745	0.3693	0.1656	0.3854	0.4469
	400	0.3789	0.3654	0.3489	0.1184	0.3651	0.4415
	40	0.8910	2.9257	1.7057	0.8483	8.0646	2.4440
	70	0.8430	0.9201	1.1090	0.8299	1.0765	2.9913
b	100	0.8336	0.8946	0.9307	0.8243	1.3080	1.1517
	200	0.8234	0.8514	0.8728	0.8163	0.8661	0.9531
	400	0.8218	0.8337	0.8434	0.8133	0.8462	0.8752
	40	0.4912	0.7773	1.5146	2.6497	5.3615	0.6009
	70	0.3251	0.5953	1.0958	2.1419	1.1381	0.5056
c	100	0.2850	0.4606	0.9429	1.7472	1.0174	0.4018
	200	0.2024	0.3076	0.6302	1.5527	0.6734	0.3225
	400	0.1438	0.2297	0.4344	1.1183	0.4233	0.2241
	40	0.1032	0.0138	0.0307	0.0139	0.0293	0.0183
	70	0.0471	0.0086	0.0191	1.0066	0.0184	0.0105
d	100	0.0170	0.0064	0.0144	0.0041	0.0149	0.0081
	200	0.0046	0.0031	0.0079	0.0020	0.0073	0.0042
	400	0.0018	0.0015	0.0034	0.0009	0.0036	0.0018

Table 5. Simulation results of several estimation methods for $\alpha = 1.25, \beta = 0.41, a = 0.21, b = 0.17, c = 0.61, d = 1$

Parameter	n	MLEs	ANDEs	CVMEs	MPEs	LSEs	WLEs
				AVBs			
	40	0.2466	0.1966	0.2185	0.2392	0.2176	0.4189
	70	0.2187	0.1191	0.1681	0.1853	0.1766	0.3158
α	100	0.2128	0.0983	0.1606	0.1806	0.1491	0.2415
	200	0.2131	0.0650	0.1317	0.1607	0.1328	0.1572
	400	0.2051	0.0360	0.1028	0.1295	0.1075	0.1125
	40	0.3430	0.3571	1.9030	0.3700	9.1583	0.7496
	70	0.2000	0.2264	0.3799	0.1715	0.3718	0.2450
β	100	0.1436	0.1706	0.2525	0.1434	0.2545	0.1926
	200	0.0976	0.1032	0.1630	0.0822	0.1582	0.1197
	400	0.0682	0.0554	0.1055	0.0455	0.1083	0.0813
	40	0.1097	0.1427	1.7774	0.1031	4.3931	0.2077
	70	0.0969	0.1461	0.1490	0.0911	0.1530	0.1021
a	100	0.0924	0.1457	0.1432	0.0824	0.1471	0.3987
	200	0.0889	0.1375	0.1482	0.0660	0.1490	0.1656
	400	0.0883	0.1204	0.1246	0.0556	0.1550	0.1683
	40	0.9798	1.0167	3.0707	0.9367	9.0912	1.5213
	70	0.9291	0.9219	0.9337	0.8815	0.9152	1.0605
b	100	0.9027	0.8889	0.8910	0.8741	0.8947	0.9925
	200	0.8880	0.8577	0.8684	0.8599	0.8622	0.9185
	400	0.0883	0.8439	0.8540	0.8495	0.8537	0.8964
	40	1.4299	0.7872	2.1427	1.1917	0.6178	8.7772
	70	1.2699	0.2602	0.3391	1.1771	0.3270	7.4854
c	100	1.2504	0.1767	0.3106	1.1210	0.2893	3.9764
	200	1.2333	0.0926	0.2472	1.1045	0.2459	0.2170
	400	1.2525	0.0563	0.2055	1.0279	0.2008	0.1551
	40	0.5231	0.4857	0.7384	0.4665	0.6788	0.5474
	70	0.3668	0.3463	0.5195	0.3055	0.5545	0.4194
d	100	0.2764	0.2923	0.4453	0.2573	0.4330	0.3325
	200	0.1828	0.1905	0.3056	0.1489	0.3002	0.2196
	400	0.1252	0.1038	0.2022	0.0832	0.2079	0.1526

Table 6. Cont. :Simulation results of several estimation methods for $\alpha = 1.25$, $\beta = 0.41$, $a = 0.21$, $b = 0.17$, $c = 0.61$, $d = 1$

Parameter	n	MLEs	ANDEs	CVMEs	MPEs	LSEs	WLEs
				MSEs			
	40	0.1671	0.3875	0.1733	0.2539	0.1997	1.3612
	70	0.0999	0.0570	0.0651	0.0702	0.1004	0.9954
α	100	0.0770	0.0349	0.0598	0.0545	0.0467	0.3292
	200	0.0577	0.0145	0.0344	0.0390	0.0406	0.1327
	400	0.0486	0.0043	0.0179	0.0296	0.0198	0.0591
	40	1.5840	0.6987	5.6071	3.0888	4.7365	4.6399
	70	0.1258	0.1775	0.8772	0.0719	1.2625	0.1930
β	100	0.0416	0.0731	0.2125	0.0439	0.2151	0.0855
	200	0.0166	0.0235	0.0573	0.0146	0.0554	0.0261
	400	0.0078	0.0075	0.0196	0.0047	0.0208	0.0261
	40	0.0299	0.0239	8.7568	0.3016	9.3069	6.5420
	70	0.0119	0.0242	0.0349	0.0120	0.0432	3.7078
a	100	0.0108	0.0239	0.0236	0.0104	0.0257	1.8232
	200	0.0098	0.0229	0.0235	0.0076	0.0237	0.0288
	400	0.0093	0.0203	0.0217	0.0062	0.0219	0.0285
	40	1.1748	1.4592	7.2156	1.1932	6.4995	6.6657
	70	0.9211	0.9850	1.1196	0.7884	0.9438	1.6042
b	100	0.8355	0.8308	0.8167	0.7717	0.8471	1.2640
	200	0.8048	0.7423	0.7622	0.7430	0.7493	0.8961
	400	0.7995	0.7137	0.7319	0.7227	0.7313	0.8284
	40	4.9904	9.8948	6.9909	3.2280	8.3435	5.3675
	70	2.9155	0.9278	0.5650	2.1949	0.5268	3.6583
c	100	2.4274	0.3666	0.4249	2.1597	0.3394	2.6055
	200	2.0310	0.0612	0.1455	2.1092	0.1471	0.1781
	400	1.9411	0.0082	0.0885	2.0100	0.0829	0.0766
	40	0.5158	0.4431	0.9951	0.4189	0.8524	0.5691
	70	0.2480	0.2254	0.5166	0.2014	0.5907	0.3417
d	100	0.1385	0.1649	0.4000	0.1348	0.3671	0.2086
	200	0.0598	0.0774	0.1840	0.0474	0.1781	0.0894
	400	0.0278	0.0264	0.0735	0.0165	0.0792	0.0404

6.2. Annual Wheat Yield

This first dataset consists of annual wheat yield for the period from 1951 to 2010. The units are tons per hectares. This data was also studied by Ristić et al. [27] in fitting the generalized beta exponential distribution. Also, this dataset is available in R package DataSetsUni by Imran et al. [24]. Figure 3 shows the TTT plot of the hazard rate of the annual wheat yield dataset, there is evidence of increasing hazard rate function. The box plot, violin plot, histogram and kernel density plot of the annual wheat yield dataset is as displayed in Figure 4. The MLEs, their standard errors (SEs) and the values of AIC,

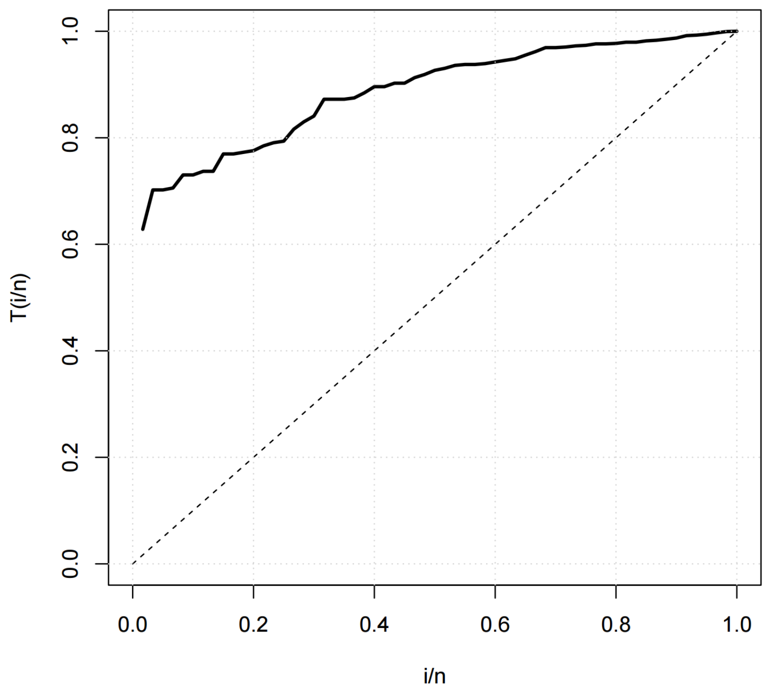


Figure 3. TTT plot of the Annual Wheat Yield dataset

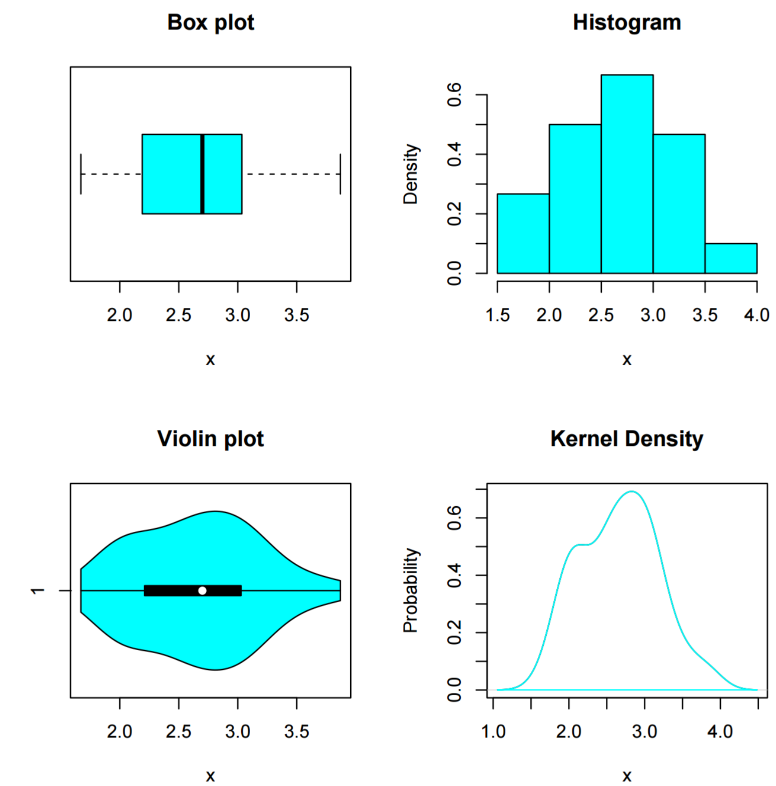


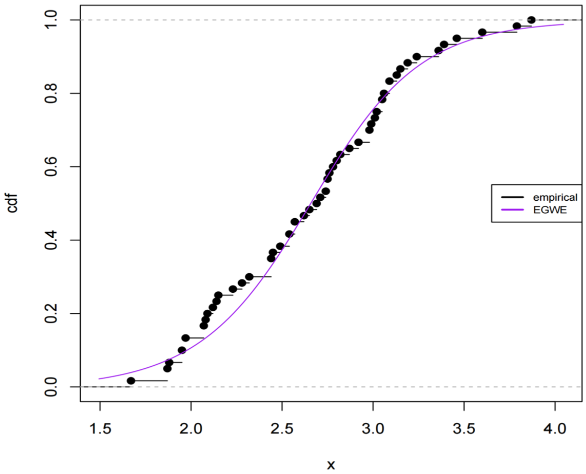
Figure 4. Box plot, violin plot, histogram and kernel density plot of the Annual Wheat Yield dataset

BIC, K-S, p-value, W*, and A* measures are shown in Table 7. From the information criteria and goodness-of-fit measures, the EGWE distribution provides the best parametric fit to the annual wheat yield dataset compared to the other competing distributions.

Table 7. MLEs, SEs, Information criteria, goodness-of-fit measures for annual wheat yield data

Model		MLEs(SEs)	AIC	BIC	W*	A*	K-S(P-value)
EGWE	$\hat{\alpha}$	1.0238(0.0484)	96.3129	100.5016	0.0467	0.4178	0.0789(0.8989)
	$\hat{\beta}$	3.8600(0.2146)					
	$\hat{a}$	1.0421(0.1430)					
	$\hat{b}$	1.1683(0.1276)					
	$\hat{c}$	2.6626(0.0560)					
	$\hat{d}$	3.0110(1.3801)					
WE	$\hat{\alpha}$	0.6854(0.0125)	99.8126	108.1900	0.0482	0.4365	0.0917(0.7138)
	$\hat{\lambda}$	1.3527(0.0063)					
	$\hat{\gamma}$	2.6616(0.0032)					
	$\hat{c}$	5.6778(0.5539)					
EEIW	$\hat{\alpha}$	1.6790(0.4788)	521.6035	527.8865	0.1098	0.6518	0.8020(<0.0002)
	$\hat{\beta}$	21.4822(22.3213)					
	$\hat{c}$	17.3836(3.5324)					
EGF	$\hat{\alpha}$	0.2092(0.1749)	107.9495	116.3269	0.2796	1.5429	0.1342(0.2297)
	$\hat{\beta}$	34.9899(49.0802)					
	$\hat{\lambda}$	24.3685(20.1387)					
	$\hat{\sigma}$	1.1696(0.2951)					
MOPLX	$\hat{\alpha}$	2.5742(2.4473)	101.1392	109.5165	0.0671	0.5177	0.9999(<0.0002)
	$\hat{\beta}$	4.8391(2.2357)					
	$\hat{\gamma}$	58.4800(137.6630)					
	$\hat{\lambda}$	15.0154(32.9476)					
W	$\hat{\alpha}$	5.2972(0.6355)	101.2384	113.8045	0.0643	0.4336	0.0917(0.7715)
	$\hat{\gamma}$	0.0039(0.0029)					
E	$\hat{\lambda}$	0.3761(0.0486)	239.3390	241.4333	0.0791	0.4958	0.4884(0.0070)

Figure 5 shows the empirical, fitted CDF and density of the EGWE distribution for the annual wheat yield. From the plot, it is evident that the EGWE distribution provides a good parametric fit to the annual wheat yield dataset.



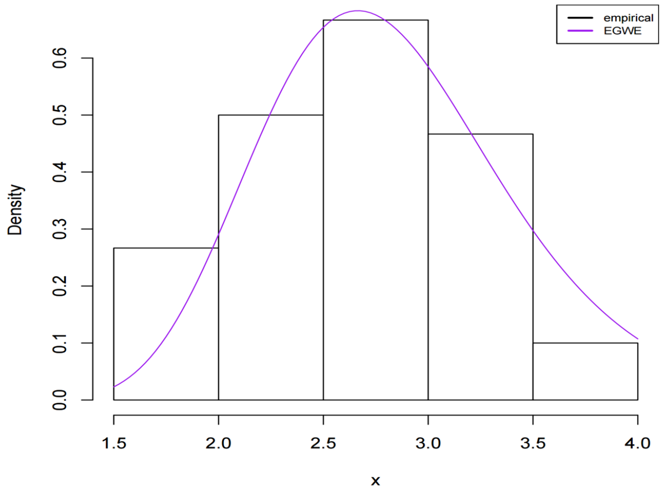


Figure 5. Empirical, Fitted CDF and density of the EGWE distribution for annual wheat yield dataset

The six estimation methods are used to estimate the EGWE parameters from the annual wheat yield dataset. This is reported in Table 8. From the K-S and p-value, the CVMEs is recommended for estimating the EGWE parameters for annual wheat yield dataset. Nevertheless, it can be concluded that all the six estimation methods performed well. This is supported by the comparison of the histogram of the annual wheat yield dataset with the fitted PDFs of the six estimation methods as shown in Figure 6.

Table 8. Estimates of EGWE parameters using six estimation methods for annual wheat yield

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$	$\hat{c}$	$\hat{d}$	**K-S**	**P-value**
MLEs	0.3852	3.8599	0.6501	0.8978	0.9228	3.0111	0.0902	0.7138
ANDEs	1.5796	1.8365	0.3401	0.8436	0.8240	4.0255	0.0735	0.9107
CVMEs	0.0626	0.9963	3.3527	3.0881	18.0682	5.6361	0.0724	0.9118
MPSEs	1.7274	4.0874	0.4521	0.4028	0.4679	2.7817	0.0773	0.8662
LSEs	0.0242	1.0013	4.1076	2.7669	16.4589	5.4822	0.0767	0.8723
WLEs	0.2982	2.4192	0.7513	0.9878	1.2685	3.4945	0.0757	0.8818

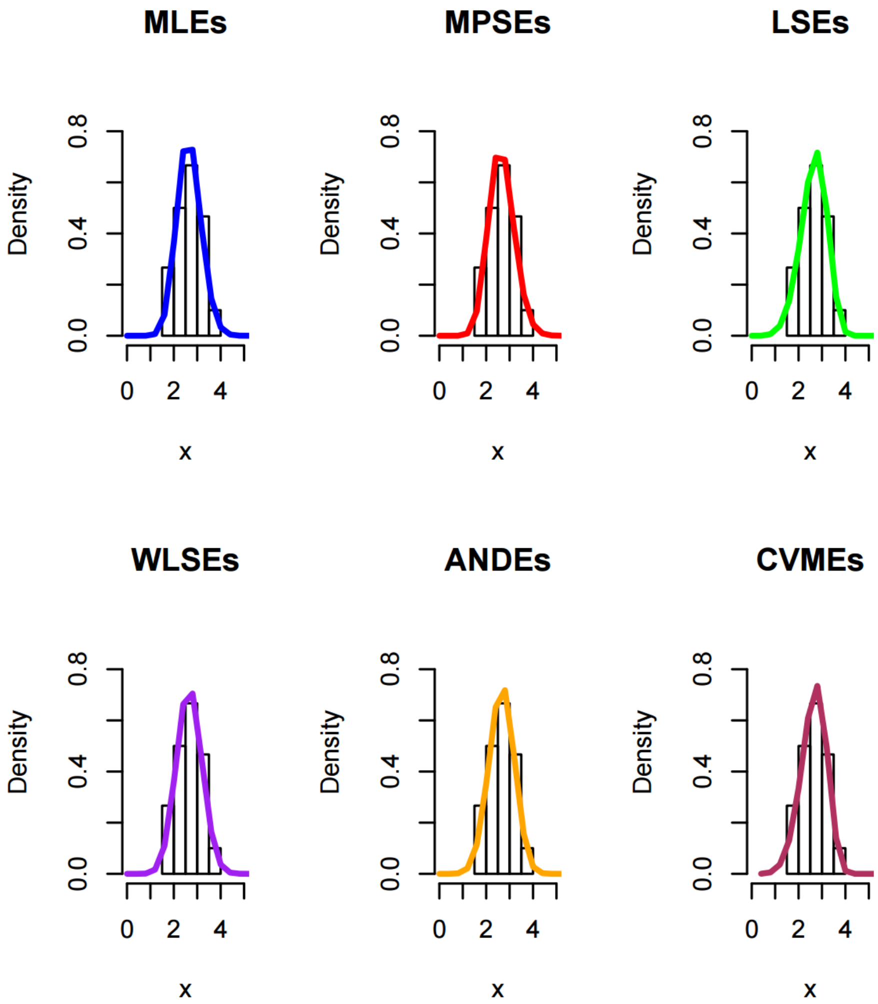


Figure 6. Histogram of the annual wheat yield and the fitted EGWE densities of the six estimation methods

The MLEs, their standard errors (SEs) and the values of AIC, BIC, K-S, p-value, W*, and A* measures are shown in Table 9. From the information criteria and goodness-of-fit measures, the EGWE distribution provides the best parametric fit to the blood cancer dataset compared to the other competing distributions.

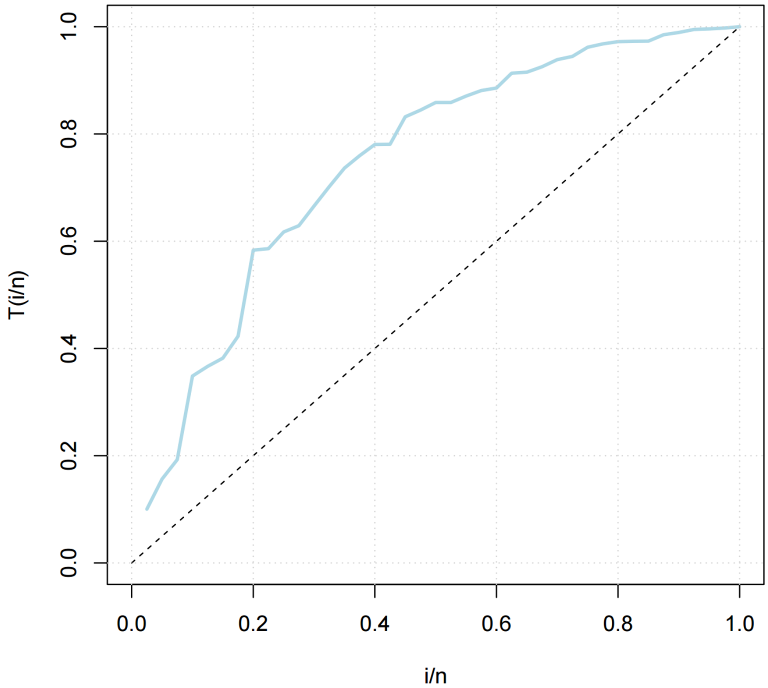


Figure 7. TTT plot of the Blood Cancer dataset

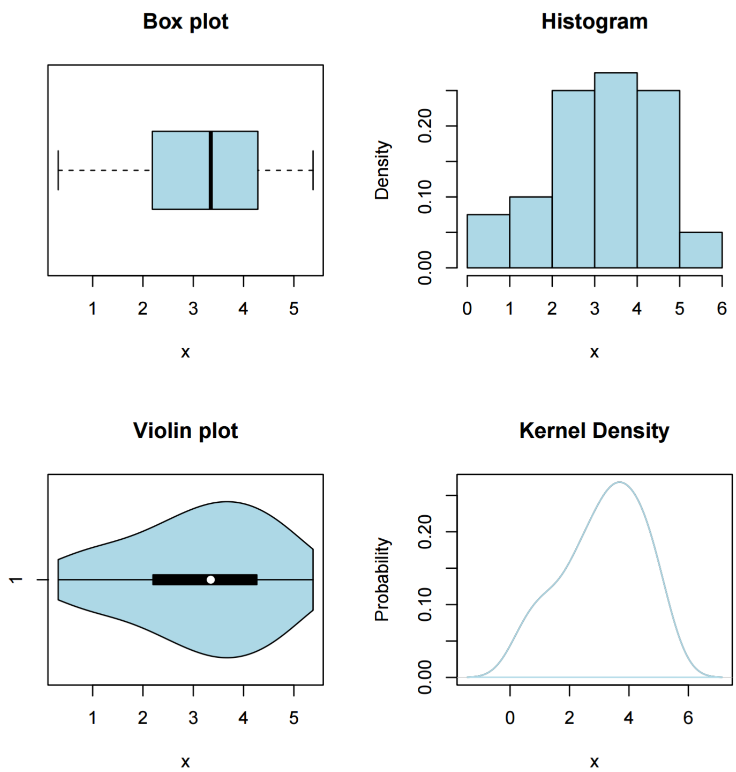


Figure 8. Box plot, violin plot, histogram and kernel density plot of the Blood Cancer dataset

Table 9. MLEs, SEs, Information criteria, goodness-of-fit measures for blood cancer data

Model		MLEs(SEs)	AIC	BIC	W*	A*	K-S(P-value)
	$\hat{\alpha}$	0.1730(0.0503)					
	$\hat{\beta}$	0.1466(0.0536)					
EGWE	$\hat{a}$	2.9710(0.2103)	142.0847	152.2179	0.0196	0.1398	0.0655(0.9955)
	$\hat{b}$	0.4010(0.0407)					
	$\hat{c}$	3.5027(2.2504)					
	$\hat{d}$	11.1345(3.2894)					
	$\hat{\alpha}$	1.7345(0.1138)					
WE	$\hat{\lambda}$	0.1314(0.0086)	147.1159	153.8714	0.1187	0.7730	0.1184(0.6288)
	$\hat{\gamma}$	0.8018(25.1382)					
	$\hat{c}$	2.4993(0.3370)					
	$\hat{\alpha}$	0.4340(0.0713)					
EEIW	$\hat{\beta}$	60.8941(22.3213)	158.1529	163.2195	0.3174	1.9084	0.1708(0.1708)
	$\hat{c}$	6.9759(0.9052)					
	$\hat{\alpha}$	80.7544(56.9057)					
EGF	$\hat{\beta}$	0.3831(0.2078)	156.7574	163.5129	0.2683	1.6423	0.1856(0.1270)
	$\hat{\lambda}$	0.6252(0.1485)					
	$\hat{\sigma}$	49.1291(28.9483)					
	$\hat{\alpha}$	16.0181(14.0614)					
MOPLX	$\hat{\beta}$	1.5232(0.5849)	145.0175	151.7731	0.4712	1.8031	0.1572(<0.0002)
	$\hat{\gamma}$	13.0698(21.4573)					
	$\hat{\lambda}$	33.0423(49.2064)					
W	$\hat{\alpha}$	2.4997(0.3372)	143.1159	146.4937	0.1187	0.7729	0.1184(0.6290)
	$\hat{\gamma}$	0.0431(0.0214)					
E	$\hat{\lambda}$	0.3184(0.0503)	173.5563	175.2452	0.2434	1.4964	0.3002(0.0015)

Figure 9 shows the empirical, fitted CDF and density of the EGWE distribution for the annual wheat yield. From the plot, it is evident that the EGWE distribution provides a good parametric fit to the annual wheat yield dataset.

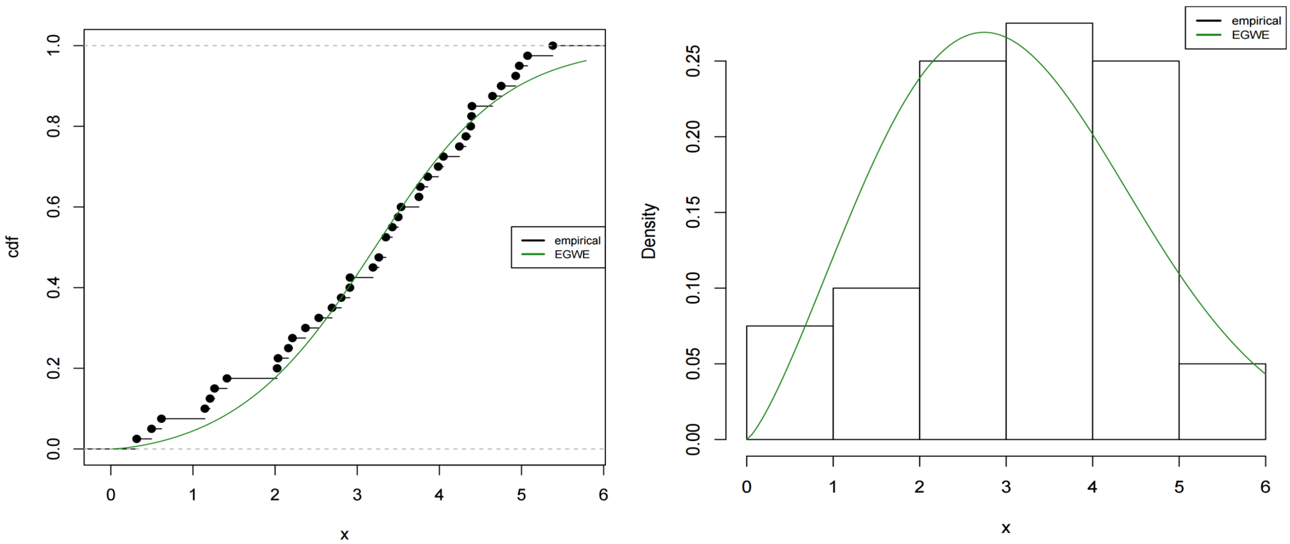


Figure 9. Empirical, Fitted CDF and density of the EGWE distribution for blood cancer dataset

The six estimation methods are used to estimate the EGWE parameters from the blood cancer dataset. This is reported in Table 10. From K-S and p-value, the ANDEs is recommended for estimating the EGWE parameters for blood cancer dataset. Nevertheless, it can be concluded that all the six estimation methods performed well. This is supported by the comparison of the histogram of the blood cancer dataset with the fitted PDFs of the six estimation methods as shown in Figure 10.

Table 10. Estimates of EGWE parameters using six estimation methods for blood cancer data

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$	$\hat{c}$	$\hat{d}$	**K-S**	**P-value**
MLEs	0.0311	0.1733	1.4770	1.1095	5.5713	9.7917	0.0667	0.9942
ANDEs	0.1172	0.1684	1.0422	0.5691	2.3633	9.8751	0.0565	0.9995
CVMEs	0.0061	0.1587	3.7185	0.8182	9.3756	10.7655	0.4400	<0.0003
MPSEs	0.1959	0.1355	0.5956	0.4212	1.0988	10.9972	0.0789	0.9642
LSEs	0.0098	0.1811	2.6596	0.7379	5.9192	9.2133	0.0601	0.9987
WLEs	0.0344	2.1649	0.4042	1.8039	2.5939	9.8185	0.4221	<0.0001

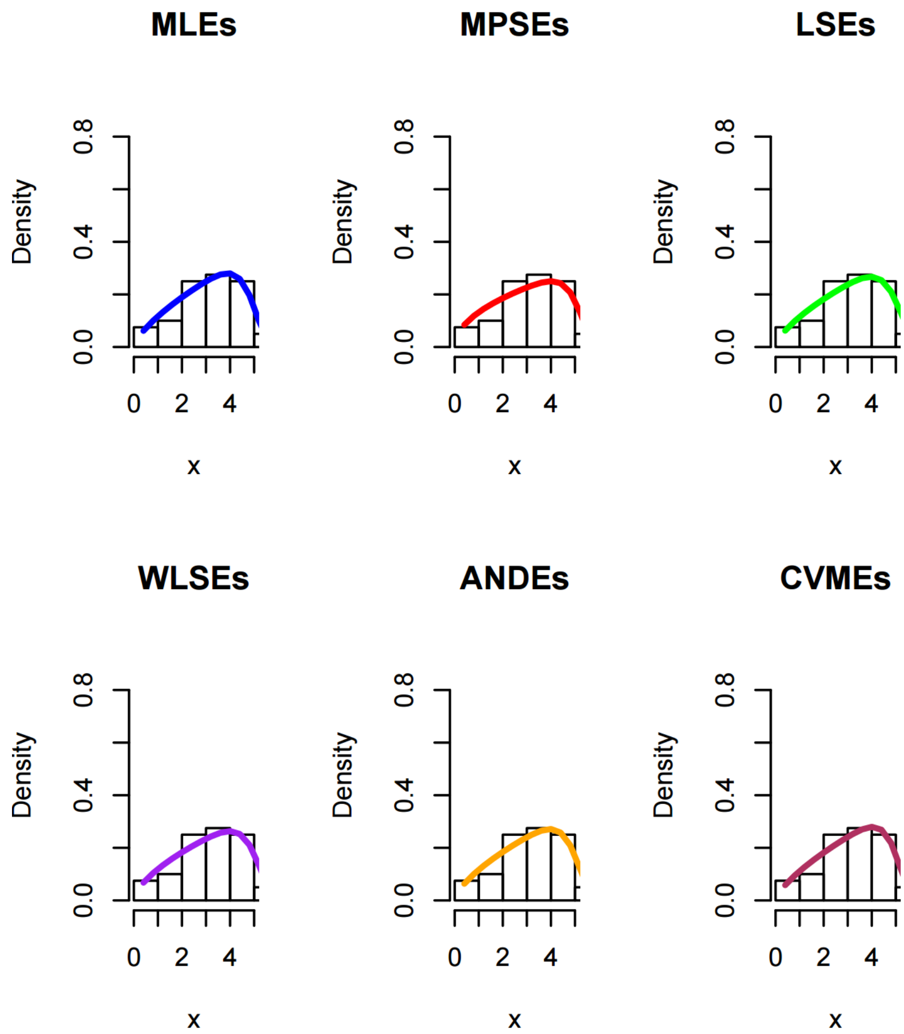


Figure 10. Histogram of the blood cancer and the fitted EGWE densities of the six estimation methods

7. Conclusion

In this article, a new distribution called exponentiated generalized Weibull exponential (EGWE) distribution is proposed and studied. The density can exhibit decreasing, increasing, right-skewed, and left-skewed shapes. The hazard rate function shows decreasing, J-shaped, bathtub, and upside-down bathtub shapes. Eight sub-models, namely Rayleigh, exponential, Weibull, exponentiated exponential Weibull distribution, Weibull exponential, exponentiated Weibull, and generalized Weibull distributions are identified. Statistical properties such as asymptotic behaviors, quantile function, moment and incomplete moments, mean and median deviations, inequality measures, moment generating function, and order statistics are studied. The estimation of the parameters of the EGWE distribution using six frequentist estimation methods, namely maximum likelihood, least squares, maximum product of spacing, weighted least squares, Anderson-Darling, and Cramér-von Mises are discussed. A detailed simulation study to ascertain the behavior of the estimators in terms of average absolute biases and mean square error was carried out. The results showed that all estimators performed well since the average absolute biases and mean square errors decrease as the sample size increases. The usefulness and flexibility of the EGWE distribution is illustrated with two real-life data, namely the blood cancer and annual wheat yield datasets. From the two datasets, the EGWE distribution provides better parametric fit compared with the competing distributions. In estimating the parameters of the EGWE distribution from the six estimation methods, the CVMEs is most appropriate for estimating the EGWE parameters from the annual wheat yield data whereas the ANDEs is the most appropriate for estimating parameters from the blood cancer data. Nevertheless, the performance of the six estimators is good in the case of the two datasets. It is our hope that this model will receive much attention in economics, finance, reliability, medicine, and other related fields.

Acknowledgments: The authors want to thank the editor and the anonymous reviewer(s) for their insightful comments and suggestions in improving this paper.

Conflict of interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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