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ON THE SOLVABILITY OF A DELAY TEMPERED-FRACTAL DIFFERENTIAL EQUATION

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Abstract. In this paper we define the tempered-fractal derivative

$$e^{-\lambda t}\frac{d}{dt^{\beta}}(f(t)\ e^{\lambda t})$$

and study the initial-value problem of the delay tempered-fractal differential equation

$$e^{-\lambda t}\frac{d}{dt^{\beta}}(x(t)\ e^{\lambda t})\ =\ f(t,x(\phi(t))),\ a.e.,\ t\in(0,T],\ x(0)\ =\ x_o.$$

We discuss the existence of at least one solution $x\in C[0,T]$. The Uniqueness of the solution will be proved. The continuous dependence on the initial data x_0, the delay function ϕ and on the function f is proved. The Hyers - Ulam stability of the problem itself will be established. Finally, some examples will be given to verify our results.

1. Introduction and preliminaries

It is well known that the tempered-fractal differential equations create an important new branch of non-linear analysis and have applications in describing of miscellaneous real world problems. For papers studying such kind of problems (see [1, 2, 9, 11]). This paper will focus itself on a tempered-fractal differential equation. Let $I=[0,T]$, $f:\ I\times R\rightarrow R$ be continuous, $\beta\in(0,1)$ and $\lambda>0$.

Definition 1.1. *The fractal derivative of the function f at $t_o\in(0,T)$ is defined by* [7, 10]

$$D_{\beta}f(t)\ =\ \frac{df(t)}{dt^{\beta}}\Big|_{t=t_0}=\lim_{t\rightarrow t_0}\frac{f(t)-f(t_0)}{t^{\beta}-t_0^{\beta}} \tag{1}$$

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and generally we can get

$$D_\beta f(t) = \frac{d}{dt^\beta} f(t) = \frac{t^{1-\beta}}{\beta}\ \frac{df(t)}{dt},\ \ t \in (0, T]. \tag{2}$$

Example Let $f(t) = \sqrt{t}$. Differentiating f with respect to t, we obtain

$$\frac{d\sqrt{t}}{dt} = \frac{1}{2\sqrt{t}}$$

which proves that f is not differentiable at zero. But the fractal derivative $D_\beta f(t),\ \beta = 1/2$ exists and

$$\frac{d\sqrt{t}}{d\sqrt{t}} = \lim_{t \to t_0} \frac{\sqrt{t} - \sqrt{t_0}}{\sqrt{t} - \sqrt{t_0}} = 1$$

exists.

Definition 1.2. *The tempered derivative of the function f is defined by* [12]

$${}^T D f(t) \ = \ e^{-\lambda t} \frac{d}{dt} (f(t)\ e^{\lambda t}). \tag{3}$$

Definition 1.3. *The tempered-fractal derivative of the function f is defined by*

$${}^T D_\beta f(t) \ = \ e^{-\lambda t} \frac{d}{dt^\beta} (f(t)\ e^{\lambda t}). \tag{4}$$

1.1. **General properties.** Here, we give some general properties of the tempered-fractal differential operators (2), (3) and (4).

(i) Linearity

$$\begin{aligned} D_\beta\ (af_1(t) + bf_2(t)) \ &= \ \frac{t^{1-\beta}}{\beta} \frac{d}{dt} (af_1(t) + bf_2(t)) \\ &= \ a\ \frac{t^{1-\beta}}{\beta}\ \frac{df_1(t)}{dt} + b\ \frac{t^{1-\beta}}{\beta}\ \frac{df_2(t)}{dt} \\ &= \ a\ D_\beta f_1(t) + b\ D_\beta f_2(t), \end{aligned}$$

and

$$\begin{aligned} {}^T D\ (af_1(t) + bf_2(t)) \ &= \ e^{-\lambda t} \frac{d}{dt} (\ (af_1(t) + bf_2(t))\ e^{\lambda t}\ ) \\ &= \ a\ e^{-\lambda t} \frac{d}{dt} (f_1(t)\ e^{\lambda t}) + b\ e^{-\lambda t} \frac{d}{dt} (f_2(t)\ e^{\lambda t}) \\ &= \ a\ {}^T D\ f_1(t) + b\ {}^T D\ f_2(t). \end{aligned}$$

So, we can prove that the tempered-fractal derivative (4) is linear.

(ii) Multiplication

$$\begin{aligned} D_\beta\ (f_1(t)\ f_2(t)) \ &= \ \frac{t^{1-\beta}}{\beta} \frac{d}{dt} (f_1(t)\ f_2(t)) \\ &= \ \frac{t^{1-\beta}}{\beta}\ f_2(t)\ \frac{d}{dt} f_1(t) + \frac{t^{1-\beta}}{\beta}\ f_1(t)\ \frac{d}{dt} f_2(t) \\ &= \ f_2(t)\ \frac{t^{1-\beta}}{\beta}\ \frac{d}{dt} f_1(t) + f_1(t)\ \frac{t^{1-\beta}}{\beta}\ \frac{d}{dt} f_2(t) \\ &= \ f_2(t)\ D_\beta f_1(t) + f_1(t)\ D_\beta f_2(t), \end{aligned}$$

but

$$\begin{aligned} {}^{T}D\ (f_1(t)\ f_2(t)) &= e^{-\lambda t}\frac{d}{dt}(\ f_1(t)\ f_2(t)\ e^{\lambda t}\ ) \\ &= e^{-\lambda t}\ f_2(t)\ e^{\lambda t}\ \frac{d}{dt}f_1(t) + e^{-\lambda t}\ f_1(t)\ \frac{d}{dt}(f_2(t)\ e^{\lambda t}) \\ &= f_2(t)\ \frac{df_1(t)}{dt} + f_1(t)\ {}^{T}D\ f_2(t) \\ &\neq f_2(t)\ {}^{T}D\ f_1(t) + f_1(t)\ {}^{T}D\ f_2(t). \end{aligned}$$

So, we can prove that the tempered-fractal derivative (4) does not satisfy the rule of derivative of multiplication of two functions.

(iii) Division

$$\begin{aligned} D_\beta\ (\frac{f_1(t)}{f_2(t)}) &= \frac{dt}{dt^\beta}\frac{d}{dt}(\frac{f_1(t)}{f_2(t)}) \\ &= \frac{dt}{dt^\beta}\ \frac{f_2(t)\ \frac{df_1(t)}{dt} - x(t)\ \frac{df_2(t)}{dt}}{(f_2(t))^2} \\ &= \frac{f_2(t)\ \frac{dt}{dt^\beta}\ \frac{df_1(t)}{dt} - f_1(t)\ \frac{dt}{dt^\beta}\ \frac{df_2(t)}{dt}}{(f_2(t))^2} \\ &= \frac{f_2(t)\ D_\beta\ f_1(t) - f_1(t)\ D_\beta\ f_2(t)}{(f_2(t))^2}, \end{aligned}$$

but

$${}^{T}D\ (\frac{f_1(t)}{f_2(t)}) \neq \frac{f_2(t)\ {}^{T}D\ f_1(t) - f_1(t)\ {}^{T}D\ f_2(t)}{(f_2(t))^2}.$$

So, we can prove that the tempered-fractal derivative (4) does not satisfy the rule of derivative of division of two functions.

(iv) Chain rule

We can obtain

$$D_\beta\ f(g(t)) = \frac{t^{1-\beta}}{\beta}\frac{d}{dt}f(g(t)) = \frac{t^{1-\beta}}{\beta}\ \frac{df}{dg}\ \frac{dg}{dt} = \frac{df}{dg}\ \frac{dg}{dt^\beta}.$$

Now, consider the initial value problem

$$\frac{d}{dt}x(t)\ =\ f(t, x(t)),\ t \in (0, T],\ x(0)\ =\ x_o. \tag{5}$$

Definition 1.4. *Let the solution of the initial value problem (5) be exists, then the problem (5) is Hyers-Ulam stable* [5] *if* $\forall \epsilon > 0, \exists \delta > 0,$ *such that for any* $\delta-$*approximate solution* x_s *of the initial value problem (5) satisfying*

$$|\frac{d}{dt}x_s(t) - f(t, x_s(t))| < \delta,$$

then

$$||x - x_s|| < \epsilon.$$

Now, let $\phi : I \to I,\ \phi(t) \leq t$ be continuous. $\beta \in (0,1)$ and $\lambda > 0$. Consider the initial-value problem of the tempered fractal differential equation with delay

$$ {}^{T}D_{\beta}x(t) \ = \ e^{-\lambda t}\frac{d}{dt^{\beta}}(x(t)\ e^{\lambda t}) \ = \ f(t, x(\phi(t))),\ a.e.,\ t \in (0, T],\ x(0) \ = \ x_o. \quad (6) $$

Here, we study the existence of solutions $x \in C(I)$ of (6). The sufficient condition of the uniqueness will be given and the continuous dependence of the unique solution on the initial value x_o, the delay function ϕ and the function f will be proved. The Hyers-Ulam stability is proved.

2. Existence of solutions

Let $C(I) = C[0, T]$ be the class of all continuous functions with norm [3]

$$ ||x|| = \sup_{t\in(0,T]} |x(t)|. $$

Definition 2.5. *By a solution of (6) we mean the function $x \in C(I)$ which satisfies the problem (6).*
We consider the problem (6) under the following assumptions
(i) $f : I \times R \to R$ is measurable in t for any $x \in R$ and continuous in $x \in R$ for all t.
(ii) There exists a bounded measurable function $a \in L_1(I)$ and a positive constant b such that

$$ |f(t, x)| \leq a(t) + b|x|. $$

(iii) $bT^{\beta} < 1$.
Now we have the following equivalent lemma.

Lemma 2.1. *Let the solution of the problem (6) be exists, then it can be given by*

$$ x(t) = e^{-\lambda t}\ x_0 + \beta \int_0^t s^{\beta-1}\ e^{-\lambda(t-s)}\ f(s, x(\phi(s)))ds. \quad (7) $$

Proof. Let x be a solution of the problem (6), then

$$ \begin{aligned} e^{-\lambda t}\frac{d}{dt^{\beta}}(e^{\lambda t}\ x(t)) &= f(t, x(\phi(t))), \\ \frac{d}{dt^{\beta}}(e^{\lambda t}\ x(t)) &= e^{\lambda t}\ f(t, x(\phi(t))), \\ \frac{dt}{dt^{\beta}}\ \frac{d}{dt}(e^{\lambda t}\ x(t)) &= e^{\lambda t}\ f(t, x(\phi(t))), \\ \frac{1}{\beta\ t^{\beta-1}}\ \frac{d}{dt}(e^{\lambda t}\ x(t)) &= e^{\lambda t}\ f(t, x(\phi(t))) \end{aligned} $$

and

$$ \frac{d}{dt}(e^{\lambda t}\ x(t)) \ = \ \beta\ t^{\beta-1}\ e^{\lambda t}\ f(t, x(\phi(t))). $$

Integrating both side from 0 to t, we obtain

$$ \begin{aligned} e^{\lambda t}\ x(t) - e^{0}\ x(0) &= \int_0^t \beta\ s^{\beta-1}\ e^{\lambda s}\ f(s, x(\phi(s)))\ ds, \\ e^{\lambda t}\ x(t) &= x_0 + \int_0^t \beta\ s^{\beta-1}\ e^{\lambda s}\ f(s, x(\phi(s)))\ ds \end{aligned} $$

and

$$x(t) = e^{-\lambda t}\ x_0 + \int_0^t \beta\ s^{\beta-1}\ e^{-\lambda(t-s)}\ f(s, x(\phi(s)))\ ds.$$

Conversely, let $x(t)$ be a solution of (7); we want to obtain (6) multiplying both side by $e^{\lambda t}$, then

$$\begin{aligned}
e^{\lambda t}\ x(t) &= x_0 + \beta \int_0^t s^{\beta-1}\ e^{\lambda s}\ f(s, x(\phi(s)))\ ds, \\
\frac{d}{dt^\beta}\ (e^{\lambda t}\ x(t)) &= \frac{dx_0}{dt^\beta} + \beta\ \frac{d}{dt^\beta} \int_0^t s^{\beta-1}\ e^{\lambda s}\ f(s, x(\phi(s)))\ ds \\
&= \frac{1}{\beta t^{\beta-1}} \frac{dx_0}{dt} + \beta\ \frac{dt}{dt^\beta}\ \frac{d}{dt} \int_0^t s^{\beta-1}\ e^{\lambda s}\ f(s, x(\phi(s)))\ ds \\
&= \beta\ \frac{dt}{dt^\beta}\ t^{\beta-1}\ e^{\lambda t}\ f(t, x(\phi(t))) \\
&= \beta\ \frac{1}{\beta\ t^{\beta-1}}\ t^{\beta-1}\ e^{\lambda t}\ f(t, x(\phi(t))) \\
&= e^{\lambda t}\ f(t, x(\phi(t)))
\end{aligned}$$

and

$$e^{-\lambda t} \frac{d}{dt^\beta}(e^{\lambda t}\ x(t)) = f(t, x(\phi(t)))\ a.e.,\ t \in (0, T].$$

Theorem 2.1. *Let the assumptions $(i) - (iii)$ be satisfied, then there exist at least one solution $x \in C(I)$ of the problem (6).*

Proof. Define the set $Q_r = \{x \in C(I) : |x(t)| < r\},\ r = \frac{x_0 + \beta A}{1 - bT^\beta},$ where $A = \int_0^T s^{\beta-1}\ |a(s)|\ ds.$
Let $x \in Q_r$, then we have

$$\begin{aligned}
|Fx(t)| &= |\ e^{-\lambda t}\ x_0 + \beta \int_0^t s^{\beta-1}\ e^{-\lambda(t-s)}\ f(s, x(\phi(s)))\ ds| \\
&\le e^{-\lambda t}\ |x_0| + \beta \int_0^t s^{\beta-1}\ e^{-\lambda(t-s)}\ |f(s, x(\phi(s)))|\ ds \\
&\le |x_0| + \beta \int_0^t s^{\beta-1}\ |f(s, x(\phi(s)))|\ ds \\
&\le |x_0| + \beta \int_0^t s^{\beta-1}\ (\ |a(s)| + b|x(\phi(s))|\ )\ ds \\
&\le |x_0| + \beta \int_0^t s^{\beta-1}\ |a(s)|\ ds + \beta\ b \int_0^t s^{\beta-1}\ |x(\phi(s))|\ ds \\
&\le |x_0| + \beta \int_0^t s^{\beta-1}\ |a(s)|\ ds + \beta\ b\ r \int_0^t s^{\beta-1}\ ds \\
&\le |x_0| + \beta\ A + \beta\ b\ r\ \frac{t^\beta}{\beta} \\
&\le |x_0| + \beta\ A + \ b\ r\ T^\beta = r
\end{aligned}$$

and $\{Fx(t)\} \subset Q_r$.

Now let $x \in Q_r$. Let $t_2,\ t_1 \in I$ be such that $t_2 > t_1 > 0$ and $|t_2 - t_1| < \delta$, then

$$\begin{aligned}
|\ Fx(t_2) - Fx(t_1)\ | &= |\ e^{-\lambda t_2}\ x_0 + \beta \int_0^{t_2} s^{\beta-1}\ e^{-\lambda(t_2-s)}\ f(s, x(\phi(s))\ ds \\
&- \ e^{-\lambda t_1}\ x_0 - \beta \int_0^{t_1} s^{\beta-1}\ e^{-\lambda(t_1-s)}\ f(s, x(\phi(s)))\ ds\ | \\
&= |\ (e^{-\lambda t_2} - e^{-\lambda t_1}\ )\ x_0 - \beta \int_0^{t_1} s^{\beta-1}\ e^{-\lambda(t_1-s)}\ f(s, x(\phi(s)))\ ds \\
&+ \ \beta \int_0^{t_2} s^{\beta-1}\ e^{-\lambda(t_2-s)}\ f(s, x(\phi(s))\ ds\ | \\
&= |\ (e^{-\lambda t_2} - e^{-\lambda t_1}\ )\ x_0 - \beta \int_0^{t_1} s^{\beta-1}\ e^{-\lambda(t_1-s)}\ f(s, x(\phi(s)))\ ds \\
&+ \ \beta\ (\int_0^{t_1} s^{\beta-1}\ e^{-\lambda(t_2-s)}\ f(s, x(\phi(s)))\ ds \\
&+ \int_{t_1}^{t_2} s^{\beta-1}\ e^{-\lambda(t_2-s)}\ f(s, x(\phi(s)))\ ds\ )\ | \\
&= |\ (e^{-\lambda t_2} - e^{-\lambda t_1}\ )\ x_0 \\
&+ \ \beta \int_0^{t_1} s^{\beta-1}\ (e^{-\lambda(t_2-s)} - e^{-\lambda(t_1-s)})\ f(s, x(\phi(s)))\ ds \\
&+ \ \beta \int_{t_1}^{t_2} s^{\beta-1}\ e^{-\lambda(t_2-s)}\ f(s, x(\phi(s)))\ ds\ | \\
&\leq |\ e^{-\lambda t_2} - e^{-\lambda t_1}\ |\ |x_0| \\
&+ \ \beta \int_0^{t_1} s^{\beta-1}\ |\ e^{-\lambda(t_2-s)} - e^{-\lambda(t_1-s)}\ |\ |\ f(s, x(\phi(s)))\ |\ ds \\
&+ \ \beta \int_{t_1}^{t_2} s^{\beta-1}\ e^{-\lambda(t_2-s)}\ |\ f(s, x(\phi(s)))\ |\ ds \\
&\leq |\ e^{-\lambda t_2} - e^{-\lambda t_1}\ |\ |x_0| \\
&+ \ \beta \int_0^{t_1} s^{\beta-1}\ |\ e^{-\lambda(t_2-s)} - e^{-\lambda(t_1-s)}\ |\ |\ f(s, x(\phi(s)))\ |\ ds \\
&+ \ \beta \int_{t_1}^{t_2} s^{\beta-1}\ |\ f(s, x(\phi(s)))\ |\ ds \\
&\leq |\ e^{-\lambda t_2} - e^{-\lambda t_1}\ |\ |x_0| \\
&+ \ \beta \int_0^{t_1} s^{\beta-1}\ |e^{-\lambda(t_2-s)} - e^{-\lambda(t_1-s)}|\ (\ |a(s)| + b\ |x(\phi(s))|\ )\ ds \\
&+ \ \beta \int_{t_1}^{t_2} s^{\beta-1}\ (\ |a(s)| + b\ |x(\phi(s))|\ )\ ds \\
&\leq |e^{-\lambda t_2} - e^{-\lambda t_1}|\ |x_0| + \beta \int_0^{t_1} s^{\beta-1}\ |e^{-\lambda(t_2-s)} - e^{-\lambda(t_1-s)}|\ |a(s)|\ ds \\
&+ \ \beta\ b\ r \int_0^{t_1} s^{\beta-1}\ |\ e^{-\lambda(t_2-s)} - e^{-\lambda(t_1-s)}\ |\ ds \\
&+ \ \beta \int_{t_1}^{t_2} s^{\beta-1}\ |a(s)|\ ds\ + \beta\ b\ r \int_{t_1}^{t_2} s^{\beta-1}\ ds
\end{aligned}$$

$$\begin{aligned}
&\leq \; |e^{-\lambda t_2} - e^{-\lambda t_1}| \; |x_0| + \beta \int_0^T s^{\beta-1} \; |e^{-\lambda(t_2-s)} - e^{-\lambda(t_1-s)}| \; |a(s)| \; ds \\
&+ \; \beta \; b \; r \int_0^T s^{\beta-1} \; |e^{-\lambda(t_2-s)} - e^{-\lambda(t_1-s)}| \; ds \\
&+ \; \beta \int_0^T s^{\beta-1} \; |a(s)| \; ds \; + \beta \; b \; r \int_0^T s^{\beta-1} \; ds.
\end{aligned}$$

Then $\{Fx(t)\}$ is equi-continuous on I and by Arzela-Ascoli Theorem [3] $\{Fx(t)\}$ is relatively compact. Then F is compact [4].
Let $\{x_n\} \subset Q_r$ such that $x_n \to x_0$, then

$$\begin{aligned}
Fx_n(t) &= e^{-\lambda t} \; x_0 + \beta \int_0^t s^{\beta-1} \; e^{-\lambda(t-s)} \; f(s, x_n(\phi(s))) \; ds, \\
\lim_{n\to\infty} Fx_n(t) &= e^{-\lambda t} \; x_0 + \beta \lim_{n\to\infty} \int_0^t s^{\beta-1} \; e^{-\lambda(t-s)} \; f(s, x_n(\phi(s))) \; ds \\
&= e^{-\lambda t} \; x_0 + \beta \int_0^t s^{\beta-1} \; e^{-\lambda(t-s)} \lim_{n\to\infty} f(s, x_n(\phi(s))) \; ds \\
&= e^{-\lambda t} \; x_0 + \beta \int_0^t s^{\beta-1} \; e^{-\lambda(t-s)} \; f(s, \lim_{n\to\infty} x_n(\phi(s))) \; ds \\
&= e^{-\lambda t} \; x_0 + \beta \int_0^t s^{\beta-1} \; e^{-\lambda(t-s)} \; f(s, x_0) \; ds \\
&= Fx_0
\end{aligned}$$

and F is continuous.
By Schauder's fixed point Theorem [8], there exist at least one solution $x \in Q_r \subset C(I)$ of the integral equation (7).
Consequently, there exist at least one solution $x \in C(I)$ of the problem (6).

3. Uniqueness of the solution

(i)* $f : I \times R \to R$ is measurable in $t \in I$ for every $x \in R$ and satisfies the Lipschitz condition

$$|f(t,x) - f(t,y)| \leq b|x-y|, \; b > 0 \tag{8}$$

with respect to $x \in R$ for every $t \in I$ and $f(t,0) = a(t)$ is bounded.

Remark. From (8) we can obtain

$$\begin{aligned}
|f(t,x)| - |f(t,0)| &\leq |f(t,x) - f(t,0)| \leq b|x| \\
|f(t,x)| &\leq |a(t)| + b|x|, \; a(t) = f(t,0),
\end{aligned}$$

this show that the assumptions of Theorem 2.1 are satisfied.

Theorem 3.2. *Let the assumption (i)* be satisfied. If $bT^\beta < 1$, then the solution of the problem (6) is unique.*

Proof. Let x, y be two solutions of (6), then

$$\begin{aligned}
\mid x(t)-y(t)\mid &= \mid e^{-\lambda t}\, x_0+\beta \int_0^t s^{\beta-1}\, e^{-\lambda(t-s)}\, f(s,x(\phi(s)))\, ds \\
&- \; e^{-\lambda t}\, x_0-\beta \int_0^t s^{\beta-1}\, e^{-\lambda(t-s)}\, f(s,y(\phi(s)))\, ds \mid \\
&= \mid \beta \int_0^t s^{\beta-1}\, e^{-\lambda(t-s)}\, (f(s,x(\phi(s)))-f(s,y(\phi(s))))\, ds| \\
&\leq \beta \int_0^t s^{\beta-1}\, e^{-\lambda(t-s)}\, |f(s,x(\phi(s)))-f(s,y(\phi(s)))|\, ds \\
&\leq \beta \int_0^t s^{\beta-1}\, |f(s,x(\phi(s)))-f(s,y(\phi(s)))|\, ds \\
&\leq \beta \int_0^t s^{\beta-1}\, b\, |x(\phi(s)-y(\phi(s)|\, ds \\
&\leq \beta \int_0^t s^{\beta-1}\, b\, ||x-y||\, ds \\
&\leq \beta\, b\, ||x-y|| \int_0^t s^{\beta-1}\, ds \\
&\leq \beta\, b\, ||x-y||\, \frac{t^\beta}{\beta} \\
\sup_t \mid x(t)-y(t)\mid &\leq b\, T^\beta\, ||x-y|| \\
||\, x-y\, || &\leq b\, T^\beta\, ||x-y|| \\
(1-b\, T^\beta)\, ||\, x-y\, || &\leq 0.
\end{aligned}$$

Since $bT^\beta < 1$, then $||\, x-y\, || = 0$ and this implies that $x = y$ and the solution $x \in C(I)$ of (6) is unique.

4. Continuous Dependence

Definition 4.6. [6] *The solution $x \in C(I)$ of (6) depends continuously on initial data x_0 , if $\forall \epsilon > 0,\ \exists \delta > 0$ s.t. $|x_0 - x_0^*| < \delta \ \Rightarrow\ ||x - x^*|| < \epsilon$.*

Theorem 4.3. *Let the assumption of Theorem 3.2 be satisfied. Then the solution of the problem (6) depends continuously on x_0.*

Proof. Let $|x_0 - x_0^*| < \delta$, then

$$\begin{aligned}
|x(t)-x^*(t)| &= \mid e^{-\lambda t}\, x_0+\beta \int_0^t s^{\beta-1}\, e^{-\lambda(t-s)}\, f(s,x(\phi(s))\, ds \\
&- \; e^{-\lambda t}\, x_0^*-\beta \int_0^t s^{\beta-1}\, e^{-\lambda(t-s)}\, f(s,x^*(\phi(s))\, ds \mid \\
&= \mid (x_0-x_0^*)\, e^{-\lambda t}+\beta \int_0^t s^{\beta-1}\, e^{-\lambda(t-s)}\, (f(s,x(\phi(s))-f(s,x^*(\phi(s)))\, ds \mid \\
&\leq |x_0-x_0^*|\, e^{-\lambda t}+\beta \int_0^t s^{\beta-1}\, e^{-\lambda(t-s)}\, |f(s,x(\phi(s))-f(s,x^*(\phi(s))|\, ds
\end{aligned}$$

$$
\begin{aligned}
&\leq |x_0 - x_0^*|\ e^{-\lambda t} + \beta \int_0^t s^{\beta-1}\ |f(s, x(\phi(s)) - f(s, x^*(\phi(s))|\ ds \\
&\leq |x_0 - x_0^*|\ e^{-\lambda t} + \beta \int_0^t s^{\beta-1}\ b\ |x(\phi(s) - x^*(\phi(s))|\ ds \\
&\leq |x_0 - x_0^*|\ e^{-\lambda t} + \beta\ b\ ||x - x^*|| \int_0^t s^{\beta-1}\ ds \\
&\leq \delta\ e^{-\lambda t} + \beta\ b\ ||x - x^*||\ \frac{t^\beta}{\beta} \\
\sup_t |x(t) - x^*(t)| &\leq \delta\ e^{-\lambda t} +\ bT^\beta ||x - x^*|| \\
||x - x^*|| &\leq \delta\ e^{-\lambda t} +\ bT^\beta ||x - x^*|| \\
(1 - bT^\beta)\ ||x - x^*|| &\leq \delta\ e^{-\lambda t} \\
||x - x^*|| &\leq \frac{\delta\ e^{-\lambda t}}{(1 - bT^\beta)}\ =\ \epsilon.
\end{aligned}
$$

Definition 4.7. [6] *The solution* $x \in C(I)$ *of* (6) *depends continuously on* f *, if* $\forall \epsilon > 0,\ \exists \delta > 0$ *s.t.* $|f(t, x(\phi(t))) - f^*(t, x(\phi(t)))| < \delta \ \Rightarrow\ ||x - x^*|| < \epsilon$

Theorem 4.4. *Let the assumption of Theorem 3.2 be satisfied. Then the solution of the problem (6) depends continuously on the function* f*.*

Proof. Let $|f(s, x(\phi(s)) - f^*(s, x(\phi(s)))| < \delta$, then

$$
\begin{aligned}
|x(t) - x^*(t)| &= |\ e^{-\lambda t}\ x_0 + \beta \int_0^t s^{\beta-1}\ e^{-\lambda(t-s)}\ f(s, x(\phi(s))\ ds \\
&- e^{-\lambda t}\ x_0 - \beta \int_0^t s^{\beta-1}\ e^{-\lambda(t-s)}\ f^*(s, x^*(\phi(s))\ ds\ | \\
&= |\ \beta \int_0^t s^{\beta-1}\ e^{-\lambda(t-s)}\ (f(s, x(\phi(s)) - f^*(s, x^*(\phi(s)))\ ds\ | \\
&\leq \beta \int_0^t s^{\beta-1}\ e^{-\lambda(t-s)}\ |f(s, x(\phi(s)) - f^*(s, x^*(\phi(s)))|\ ds \\
&\leq \beta \int_0^t s^{\beta-1}\ |f(s, x(\phi(s)) - f^*(s, x^*(\phi(s)))|\ ds \\
&\leq \beta \int_0^t s^{\beta-1}\ |f(s, x(\phi(s)) - f^*(s, x(\phi(s))) \\
&+ f^*(s, x(\phi(s))) - f^*(s, x^*(\phi(s)))|\ ds \\
&\leq \beta \int_0^t s^{\beta-1}\ |f(s, x(\phi(s)) - f^*(s, x(\phi(s)))|\ ds \\
&+ \beta \int_0^t s^{\beta-1}\ |f^*(s, x(\phi(s))) - f^*(s, x^*(\phi(s)))|\ ds \\
&\leq \beta\ \delta \int_0^t s^{\beta-1}\ ds + \beta \int_0^t s^{\beta-1}\ b\ |x(\phi(s)) - x^*(\phi(s))|\ ds \\
&\leq \beta\ \delta\ \frac{t^\beta}{\beta} + \beta\ b\ ||x - x^*|| \int_0^t s^{\beta-1}\ ds
\end{aligned}
$$

$$\begin{aligned}
&\leq \delta\ T^{\beta}\ + \beta\ b\ ||x-x^*||\ \frac{t^{\beta}}{\beta} \\
\sup_t |x(t)-x^*(t)| &\leq \delta\ T^{\beta}\ +\ bT^{\beta}||x-x^*|| \\
||x-x^*|| &\leq \delta\ T^{\beta}\ +\ bT^{\beta}||x-x^*|| \\
(1-bT^{\beta})\ ||x-x^*|| &\leq \delta\ T^{\beta} \\
||x-x^*|| &\leq \frac{\delta\ T^{\beta}}{(1-bT^{\beta})} = \epsilon.
\end{aligned}$$

Definition 4.8. [6] *The solution $x \in C(I)$ of* (6) *depends continuously on ϕ, if $\forall \epsilon > 0,\ \exists \delta > 0\ s.t.\ |\phi(t)-\phi^*(t)| < \delta\ \Rightarrow\ ||x-x^*|| < \epsilon$*

Theorem 4.5. *Let the assumption of Theorem 3.2 be satisfied. Then the solution of the problem (6) depends continuously on the function ϕ.*

Proof. Let $|\phi(t)-\phi^*(t)| < \delta$, then

$$\begin{aligned}
|x(t)-x^*(t)| &= |\ e^{-\lambda t}\ x_0 + \beta\ \int_0^t s^{\beta-1}\ e^{-\lambda(t-s)}\ f(s,x(\phi(s))\ ds \\
&- \ e^{-\lambda t}\ x_0 - \beta\ \int_0^t s^{\beta-1}\ e^{-\lambda(t-s)}\ f(s,x^*(\phi^*(s))\ ds\ | \\
&= |\ \beta\ \int_0^t s^{\beta-1}\ e^{-\lambda(t-s)}\ (f(s,x(\phi(s))-f(s,x^*(\phi^*(s)))\ ds\ | \\
&\leq \beta\ \int_0^t s^{\beta-1}\ e^{-\lambda(t-s)}\ |f(s,x(\phi(s))-f(s,x^*(\phi^*(s)))|\ ds \\
&\leq \beta\ \int_0^t s^{\beta-1}\ |f(s,x(\phi(s))-f(s,x^*(\phi^*(s)))|\ ds \\
&\leq \beta\ \int_0^t s^{\beta-1}\ |f(s,x(\phi(s))-f(s,x^*(\phi(s))) \\
&+ \ f(s,x^*(\phi(s)))-f(s,x^*(\phi^*(s)))|\ ds \\
&\leq \beta\ \int_0^t s^{\beta-1}\ |f(s,x(\phi(s))-f(s,x^*(\phi(s)))|\ ds \\
&+ \ \beta\ \int_0^t s^{\beta-1}\ |f(s,x^*(\phi(s)))-f(s,x^*(\phi^*(s)))|\ ds \\
&\leq \beta\ \int_0^t s^{\beta-1}\ b\ |x(\phi(s))-x^*(\phi(s))|\ ds + \beta\ \epsilon_1\ \int_0^t s^{\beta-1}\ ds \\
&\leq \beta\ b\ ||x-x^*||\ \frac{t^{\beta}}{\beta} + \beta\ \epsilon_1\ \frac{t^{\beta}}{\beta} \\
&\leq b\ ||x-x^*||\ T^{\beta}\ + \epsilon_1\ T^{\beta} \\
\sup_t |x(t)-x^*(t)| &\leq bT^{\beta}\ ||x-x^*|| + \epsilon_1\ T^{\beta} \\
||x-x^*|| &\leq bT^{\beta}\ ||x-x^*|| + \epsilon_1\ T^{\beta} \\
(1-bT^{\beta})\ ||x-x^*|| &\leq \epsilon_1\ T^{\beta} \\
||x-x^*|| &\leq \frac{\epsilon_1\ T^{\beta}}{(1-bT^{\beta})} = \epsilon.
\end{aligned}$$

5. Hyers-Ulam stability

Definition 5.9. *Let the solution of the tempered initial-value problem of the delay fractal differential equation (6) be exists uniquely, then the problem (6) is Hyers-Ulam stable [5] if $\forall \epsilon > 0, \exists \delta > 0$ such that for any $\delta-$approximate solution x_s of problem (6) satisfying*

$$|e^{-\lambda t}\frac{d}{dt^{\beta}}(x_s(t)e^{\lambda t}) - f(t, x_s(\phi(t)))| < \delta$$

implies

$$||x - x_s|| < \epsilon.$$

Theorem 5.6. *Let the assumption of Theorem 3.2 be satisfied.Then the problem (6) is Hyers-Ulam stable.*

Proof. Let $|e^{-\lambda t}\frac{d}{dt^{\beta}}(x_s(t)\ e^{\lambda t}) - f(t, x_s(\phi(t)))| < \delta$, then

$$-\delta \ < \ e^{-\lambda t}\frac{d}{dt^{\beta}}(x_s(t)\ e^{\lambda t}) - f(t, x_s(\phi(t))) \ < \ \delta$$

and

$$\begin{aligned}
-\delta\ e^{\lambda t} \ &< \ \frac{d}{dt^{\beta}}(x_s(t)\ e^{\lambda t}) - \ e^{\lambda t}\ f(t, x_s(\phi(t))) \ < \ \delta\ e^{\lambda t}, \\
-\delta\ e^{\lambda t} \ &< \ \frac{dt}{dt^{\beta}}\frac{d}{dt}(x_s(t)\ e^{\lambda t}) - \ e^{\lambda t}\ f(t, x_s(\phi(t))) \ < \ \delta\ e^{\lambda t}, \\
-\delta\ e^{\lambda t} \ &< \ \frac{1}{\beta\ t^{\beta-1}}\frac{d}{dt}(x_s(t)\ e^{\lambda t}) - \ e^{\lambda t}\ f(t, x_s(\phi(t))) \ < \ \delta\ e^{\lambda t}
\end{aligned}$$

also

$$-\delta\ \beta\ t^{\beta-1}\ e^{\lambda t} \ < \ \frac{d}{dt}(x_s(t)\ e^{\lambda t}) - \beta\ t^{\beta-1}\ e^{\lambda t}\ f(t, x_s(\phi(t))) \ < \ \delta\ \beta\ t^{\beta-1}\ e^{\lambda t}.$$

Integrating we get

$$-\delta\ \beta\int_0^t s^{\beta-1}e^{\lambda s}ds \ < \ x_s(t)e^{\lambda t} - x_s(0)e^0 - \beta\int_0^t s^{\beta-1}e^{\lambda s}f(s, x_s(\phi(s)))ds \ < \ \delta\ \beta\int_0^t s^{\beta-1}e^{\lambda s}ds$$

and

$$\begin{aligned}
-\delta\ \beta\ \int_0^t s^{\beta-1}e^{-\lambda(t-s)}\ ds \ &< \ x_s(t) - x_s(0)e^{-\lambda t} - \beta\ \int_0^t s^{\beta-1}e^{-\lambda(t-s)}f(s, x_s(\phi(s)))\ ds \\
&< \ \delta\ \beta\ \int_0^t s^{\beta-1}e^{-\lambda(t-s)}\ ds.
\end{aligned}$$

Then

$$
\begin{aligned}
|\ x_s(t) - x_s(0)\ e^{-\lambda t} - \beta\ \int_0^t s^{\beta-1}\ e^{-\lambda(t-s)}\ f(s, x_s(\phi(s)))\ ds\ | &< \delta\ \beta\ \int_0^t |s^{\beta-1}|\ |e^{-\lambda(t-s)}|\ ds, \\
|\ x_s(t) - x_s(0)\ e^{-\lambda t} - \beta\ \int_0^t s^{\beta-1}\ e^{-\lambda(t-s)}\ f(s, x_s(\phi(s)))\ ds\ | &< \delta\ \beta\ \int_0^t |s^{\beta-1}|\ ds, \\
|\ x_s(t) - x_s(0)\ e^{-\lambda t} - \beta\ \int_0^t s^{\beta-1}\ e^{-\lambda(t-s)}\ f(s, x_s(\phi(s)))\ ds\ | &< \delta\ \beta\ \frac{t^\beta}{\beta}, \\
|\ x_s(t) - x_s(0)\ e^{-\lambda t} - \beta\ \int_0^t s^{\beta-1}\ e^{-\lambda(t-s)}\ f(s, x_s(\phi(s)))\ ds\ | &< \delta\ T^\beta.
\end{aligned}
$$

Now,

$$
\begin{aligned}
|x(t) - x_s(t)| &= |\ e^{-\lambda t}\ x_0 + \beta\ \int_0^t s^{\beta-1}\ e^{-\lambda(t-s)}\ f(s, x(\phi(s)))\ ds - x_s(t)\ | \\
&= |\ e^{-\lambda t}\ x_0 + \beta\ \int_0^t s^{\beta-1}\ e^{-\lambda(t-s)}\ f(s, x(\phi(s)))\ ds - x_s(t) \\
&+ \beta\ \int_0^t s^{\beta-1}\ e^{-\lambda(t-s)}\ f(s, x_s(\phi(s)))\ ds \\
&- \beta\ \int_0^t s^{\beta-1}\ e^{-\lambda(t-s)}\ f(s, x_s(\phi(s)))\ ds\ | \\
&= |\ \beta\ \int_0^t s^{\beta-1}\ e^{-\lambda(t-s)}\ f(s, x(\phi(s)))\ ds \\
&- \beta\ \int_0^t s^{\beta-1}\ e^{-\lambda(t-s)}\ f(s, x_s(\phi(s)))\ ds \\
&- (x_s(t) - e^{-\lambda t}\ x_0 - \beta\ \int_0^t s^{\beta-1}\ e^{-\lambda(t-s)}\ f(s, x_s(\phi(s)))\ ds)\ | \\
&= |\ \beta\ \int_0^t s^{\beta-1}\ e^{-\lambda(t-s)}\ (f(s, x(\phi(s))) - f(s, x_s(\phi(s))))\ ds \\
&- (x_s(t) - e^{-\lambda t}\ x_0 - \beta\ \int_0^t s^{\beta-1}\ e^{-\lambda(t-s)}\ f(s, x_s(\phi(s)))\ ds)\ | \\
&\leq |\ \beta\ \int_0^t s^{\beta-1}\ e^{-\lambda(t-s)}\ (f(s, x(\phi(s))) - f(s, x_s(\phi(s))))\ ds\ | \\
&+ |\ x_s(t) - e^{-\lambda t}\ x_0 - \beta\ \int_0^t s^{\beta-1}\ e^{-\lambda(t-s)}\ f(s, x_s(\phi(s)))\ ds| \\
&\leq \beta\ \int_0^t s^{\beta-1}\ e^{-\lambda(t-s)}\ |f(s, x(\phi(s))) - f(s, x_s(\phi(s)))|\ ds\ + \delta\ T^\beta \\
&\leq \beta\ \int_0^t s^{\beta-1}\ |f(s, x(\phi(s))) - f(s, x_s(\phi(s)))|\ ds\ + \delta\ T^\beta \\
&\leq \beta\ \int_0^t s^{\beta-1}\ b\ |x(\phi(s)) - x_s(\phi(s))|\ ds\ + \delta\ T^\beta \\
&\leq \beta\ b\ ||x - x_s|| \int_0^t s^{\beta-1}\ ds\ + \delta\ T^\beta
\end{aligned}
$$

$$\begin{aligned}
&\leq \quad \beta\ b\ ||x-x_s||\frac{t^{\beta}}{\beta}\ + \delta\ T^{\beta} \\
&\leq \quad b\ T^{\beta}||x-x_s||\ + \delta\ T^{\beta} \\
\sup_t |x(t)-x_s(t)| &\leq \quad b\ T^{\beta}||x-x_s||\ + \delta\ T^{\beta} \\
||x-x_s|| &\leq \quad b\ T^{\beta}||x-x_s||\ + \delta\ T^{\beta} \\
(1-b\ T^{\beta})\ ||x-x_s|| &\leq \quad \delta\ T^{\beta} \\
||x-x_s|| &\leq \quad \frac{\delta\ T^{\beta}}{(1-b\ T^{\beta})} = \epsilon.
\end{aligned}$$

6. Special cases and examples

We can deduce the following particular cases in (problem 6).

• If $\lambda = 0$, then we have

$$\frac{d}{dt^{\beta}}x(t) = f(t, x(\phi(t))),\ a.e.,\ t \in (0, T],\ x(0)\ =\ x_o$$

and

$$x(t) = x_0 + \beta\ \int_0^t s^{\beta-1}\ f(s, x(\phi(s)))\ ds$$

• For $\beta = 1$, then we have

$$e^{-\lambda t}\frac{d}{dt}(x(t)\ e^{\lambda t})\ =\ f(t, x(\phi(t))),\ t \in (0, T],\ x(0)\ =\ x_o$$

and

$$x(t) = e^{-\lambda t}\ x_0 + \int_0^t e^{-\lambda(t-s)}\ f(s, x(\phi(s)))\ ds$$

• For $\lambda = 0$ and $\beta = 1$, then we have

$$\frac{d}{dt}x(t) = f(t, x(\phi(t))),\ t \in (0, T],\ x(0)\ =\ x_o$$

and

$$x(t) = x_0 + \int_0^t\ f(s, x(\phi(s)))\ ds$$

Example 1. Consider the following initial-value problem of the tempered-fractal differential equation

$$e^{-\lambda t}\frac{d}{dt^{\beta}}(x(t)\ e^{\lambda t})\ =\ 2t + \frac{1}{5}\ \frac{x(\gamma t)}{1+|x(\gamma t)|},\ \text{a.e. } t \in [0, 1] \tag{9}$$

with initial data

$$x(0) = x_0. \tag{10}$$

Then

$$f(t, x) = 2t + \frac{1}{5}\ \frac{x(t)}{1+|x(t)|},$$

$a(t) = 2t \in L_1(I)$ such that $\int_0^1 s^{\beta-1}\ |a(s)|\ ds$ exists, $b = \frac{1}{5} > 0,\ \beta = 0.9$ such that $bT^{\beta} < 1$.

We can show that all conditions of Theorem 2.1 are satisfied. Then, the initial-value problem (9) and (10) has a solution.

Example 2. Consider the following initial-value problem of the tempered-fractal differential equation

$$e^{-\lambda t}\frac{d}{dt^{\beta}}(x(t)\ e^{\lambda t}) \ = \ 2t^{1-\beta}+\frac{1}{5}\ \frac{x(\gamma t)}{1+sin^2 t},\ \text{a.e. } t\in[0,1] \tag{11}$$

with initial data

$$x(0)=x_0. \tag{12}$$

Then

$$f(t,x)=2t^{1-\beta}+\frac{1}{5}\ \frac{x(t)}{1+sin^2 t},$$

$a(t)=2t^{1-\beta}\in L_1(I)$ such that $\int_0^1 s^{\beta-1}\ |a(s)|\ ds$ exists, $b=\frac{1}{5}>0,\ \beta=0.9$ such that $bT^{\beta}<1$.
we can show that all conditions of Theorem 2.1 are satisfied. Then, the initial-value problem (11) and (12) has a solution.

7. Conclusions

This research paper focuses on investigate the existence of solutions for the delay tempered fractal differential problem (6) and properties associated with these solutions. Firstly, we examined the equivalence between the problem (6) and the integral equation (7), then we studied the existence of at least one solution $x\in C(I)$ of (7) by applying Schauder's fixed point Theorem [3]. Furthermore, we established sufficient conditions to ensure the uniqueness of the solution and its dependence on the initial data x_0, the delay function ϕ and on the function f. We also investigated the Hyers-Ulam stability of the problem (6). Finally, we discussed special cases.

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