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Super Edge Magic Harmonious labeling for Certain Graphs

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ABSTRACT

Edge labelling of graphs has received a lot of attention in the last few years. Both graph theory, networks, and discrete mathematics are fields that are still interested in this area. It is yet uncertain for many graphs whether super edge magic harmonious labeling exists or not. A graph $\Gamma = (V(\Gamma), E(\Gamma))$ with $P = |V(\Gamma)|$ vertices and $q = |E(\Gamma)|$ edges, is called an edge bimagic harmonious graph if there exists a bijective mapping $\Psi: [V(\Gamma) \cup E(\Gamma)] \to \{1,2,3,\cdots, p+q\}$ such that for each edge $xy \in E(\Gamma)$, the value of the formula $[(\Psi(x) + \Psi(y)) \, mod(q) + \Psi(xy)] = K_1 \, or \, K_2$, where K_i is a constant. If there exist three constants K_1, K_2 and K_3, it is said to be edge trimagic harmonious graph. We demonstrate in this study that the wheel graph W_n and the splitting graph of odd cycle are super edge bimagic harmonious graphs. Furthermore, we point out that the sunflower graph and the double sunflower graph are super edge trimagic harmonious graphs.

1. Introduction

Everyday issues can be expressed visually through the use of graphs in numerous situations. Graph theory is relevant to all other fields, whether they are applied or pure. Labelling is one of the many fascinating issues covered in graph theory. A mapping of integers to vertices or edges in a graph according to predetermined standards is called a labeling of the graph. There are many kinds of labeling, such as graceful labeling, total edge irregularity labeling, harmonious labeling, total edge irregularity reflexive labeling, magic labeling, anti-magic labeling, Zumkeller and half Zumkeller labelinh, geometric labeling, mean labeling and irregular labeling, etc.

Rosa introduced the graph labelling in 1960 [1]. Sedlacek [2] proposed the concept of magic labelling. A magic valuation of a graph was defined in 1970 by Kotzig and Rosa [3]. Ringel and Llado [4] referred to this labelling as "edge magic" in 1996. In 2004, Babujee [5] introduced edge bimagic tagging of graphs. Harmonious labelling was obviously the result of Graham and Sloane's [6] research. Dushyant Tanna has introduced several harmonious labelling techniques [7].

Graph labelling techniques are utilized for communication network addressing system problems, fast communication in sensor networks, fault-tolerant system design using facility graphs, coding theory problems involving the creation of effective radar type codes, and mobile ad hoc network problems. [8,9].

See an up-to-date survey of graph labelling [10] for an in-depth overview of the subject of conflict.

In this paper, all graphs will be finite, undirected, and simple connected graphs, for a graph $\Gamma = (V, E)$, let $P = |V(\Gamma)|$ be the cardinality of vertices $V(\Gamma)$ and $q = |E(\Gamma)|$ be that of $E(\Gamma)$. We investigate and improve the concept of super edge bimagic and trimagic harmonious labeling of graphs. We prove that the wheel graph, and the splitting graph of cycle are super edge bimagic harmonious graphs. Furthermore, the sunflower, the double sunflower are super edge trimagic harmonious graphs.

Definition 1.1. [11] (1) A graph $\Gamma = (V(\Gamma), E(\Gamma))$ with $p = |V(\Gamma)|$ vertices and $q = |E(\Gamma)|$ edges is said to be an edge magic harmonious graph if there exists a bijective mapping $\Psi: [V(\Gamma) \cup E(\Gamma)] \to \{1,2,3,\cdots, p+q\}$ such that for each edge $xy \in E(\Gamma)$, the value of the formula $[(\Psi(x) + \Psi(y)) \, mod(q) + \Psi(xy)] = K$ where K is a constant.

(2) A graph $\Gamma = (V(\Gamma), E(\Gamma))$ with $p = |V(\Gamma)|$ vertices and $q = |E(\Gamma)|$ edges is said to be an edge bimagic harmonious graph if there exists a bijective mapping $\Psi: [V(\Gamma) \cup E(\Gamma)] \to \{1,2,3,\cdots, p+q\}$ such that for each edge $xy \in E(\Gamma)$, the value of the formula $[(\Psi(x) + \Psi(y)) \, mod(\, q) + \Psi(xy)] = K_1$ or K_2, where K_1 and K_2 are constant.

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(3) A graph $\Gamma = (V(\Gamma), E(\Gamma))$ with $p = |V(\Gamma)|$ vertices and $q = |E(\Gamma)|$ edges is said to be an edge trimagic harmonious graph if there exists a bijective mapping $\Psi: [V(\Gamma) \cup E(\Gamma)] \to \{1,2,3,\cdots, p+q\}$ such that for each edge $xy \in E(\Gamma)$, the value of the formula $[(\Psi(x) + \Psi(y))\ mod(\ q) + \Psi(xy)] = K_1$ or K_2 or K_3, where K_1 ,K_2 and K_3 are constant.

Definition 1.2. [11]. If the graph Γ has the extra property that the vertex labels are 1 to $p = |V(\Gamma)|$, then an edge bimagic harmonious labeling of the graph $\Gamma = (V(\Gamma), E(\Gamma))$ becomes a super edge bimagic harmonious labeling**.**

2. Super edge bimagic harmonious labeling of the wheel graph

Theorem 2.1. For every positive integer $n \geq 3$, the wheel graph W_n is super edge bimagic harmonious graph with bimagic harmonious numbers $k_1 = 3n + 1$ and $k_2 = 4n$ for n is even, while $k_2 = 4n + 1$ for n is odd.

Proof. Let $\{x_1, x_2, \dots, x_n\}$ be the vertices of the rim of wheel with hub vertex x_0 , the edges of W_n will be $\{x_0 x_i,\ x_i x_{i+1},\ i \in [1, n-1]\} \cup \{x_n x_1\}$. So, $p = |V(W_n)| = n + 1$ and $q = |E(W_n)| = 2n$.
We define the labeling function $\mathbf{\Psi_1} : [(V \cup E)(W_n)] \to \{1, 2, 3, \dots, 3n + 1\}$ as follows:

$$\mathbf{\Psi_1}(x_0) = n + 1 \ ,$$

$$\mathbf{\Psi_1}(x_i) = i\ , \qquad i \in [1, n],$$

$$\mathbf{\Psi_1}(x_0 x_i) = \begin{cases} 2n - i\ , & if \quad i \in [1, n-2]; \\ 3n + 1, & if \quad i = n - 1; \\ 3n\ , & if \quad i = n\ . \end{cases}$$

For n even,

$$\mathbf{\Psi_1}(x_i x_{i+1}) = \begin{cases} 3n - 2i\ , & for \quad i \in \left[1, \frac{n}{2}\right]; \\ 4n - 2i - 1, & for \quad i \in \left[\frac{n}{2} + 1\ , n - 1\right]\ ; \\ 3n - 1, & for \quad i = n\ . \end{cases}$$

For n odd,

$$\mathbf{\Psi_1}(x_i x_{i+1}) = \begin{cases} 3n - 2i\ , & for \quad i \in \left[1, \frac{n-1}{2}\right]; \\ 4n - 2i, & for \quad i \in \left[\frac{n+1}{2}\ , n - 1\right]; \\ 2n\ , & for \quad i = n\ . \end{cases}$$

To demonstrate that Ψ_1 is a super edge bimagic harmonious labeling: -

For the edges $x_0 x_i$, $\quad i \in [1, n-2]$

$$[(\mathbf{\Psi_1}(x_0) + \mathbf{\Psi_1}(x_i))\ mod(q) + \mathbf{\Psi_1}(x_0 x_i)] = [(n + 1 + i)\ mod(2n) + 2n - i]\ \ 3n + 1 = K_1.$$

For the edges $x_0 x_{n-1}$

$$[(\mathbf{\Psi_1}(x_0) + \mathbf{\Psi_1}(x_{n-1}))\ mod(q) + \mathbf{\Psi_1}(x_0 x_{n-1})] = [(n + 1 + n - 1) mod(2n) + 3n + 1] = 3n + 1 = K_1.$$

For the edges x_0x_n

$$\left[\left(\boldsymbol{\Psi}_1(x_0)+\boldsymbol{\Psi}_1(x_n)\right)mod(q)+\boldsymbol{\Psi}_1(x_0x_n)\right]=[(n+1+n)mod(2n)+3n]=3n+1=K_1.$$

For the edges x_ix_{i+1}, $i\in[1,\frac{n}{2}]$ and n is even, or $i\in[1,\frac{n-1}{2}]$ and n is odd

$$[(\boldsymbol{\Psi}_1(x_i)+\boldsymbol{\Psi}_1(x_{i+1}))\,mod(q)+\boldsymbol{\Psi}_1(x_ix_{i+1})]=[(i+i+1)\,mod(2n)+3n-2i]=3n+1=K_1.$$

For the edges x_ix_{i+1}, $i\in[\frac{n}{2}+1,n-1]$ and n is even

$$[(\boldsymbol{\Psi}_1(x_i)+\boldsymbol{\Psi}_1(x_{i+1}))\,mod(q)+\boldsymbol{\Psi}_1(x_ix_{i+1})]=[(i+i+1)\,mod(2n)+4n-2i-1]=4n=K_2.$$

For the edges x_ix_{i+1}, $i\in[\frac{n+1}{2},n-1]$ and n is odd

$$[(\boldsymbol{\Psi}_1(x_i)+\boldsymbol{\Psi}_1(x_{i+1}))\,mod(q)+\boldsymbol{\Psi}_1(x_ix_{i+1})]=[(i+i+1)\,mod(2n)+4n-2i]=4n+1=K_2.$$

For the edges x_1x_n, n is even

$$\left[\left(\boldsymbol{\Psi}_1(x_1)+\boldsymbol{\Psi}_1(x_n)\right)mod(q)+\boldsymbol{\Psi}_1(x_1x_n)\right]=[(n+1)\,mod(2n)+3n-1]=4n=K_2.$$

For the edges x_1x_n, n is odd

$$\left[\left(\boldsymbol{\Psi}_1(x_1)+\boldsymbol{\Psi}_1(x_n)\right)mod(q)+\boldsymbol{\Psi}_1(x_1x_n)\right]=[(n+1)mod(2n)+2n]=3n+1=K_1.$$

Clearly, for each edge $xy\in E(W_n)$, the value of $[(\boldsymbol{\Psi}_1(x)+\boldsymbol{\Psi}_1(y))\,mod(q)+\boldsymbol{\Psi}_1(xy)]$ provides any one of the bimagic constants $k_1=3n+1$ and $k_2=4n$ when n is even, while $k_2=4n+1$ when n is odd. Therefore, a super edge bimagic harmonious labelling for all n is allowed by the wheel graphs W_n.

Example 2.2. In Fig. 1 we present W_{13} with super edge bimagic harmonious labeling with bimagic harmonious numbers $k_1=40$ and $k_2=53$, and W_{14} with super edge bimagic labeling with bimagic harmonious numbers $k_1=43$ and $k_2=56$, respectively.

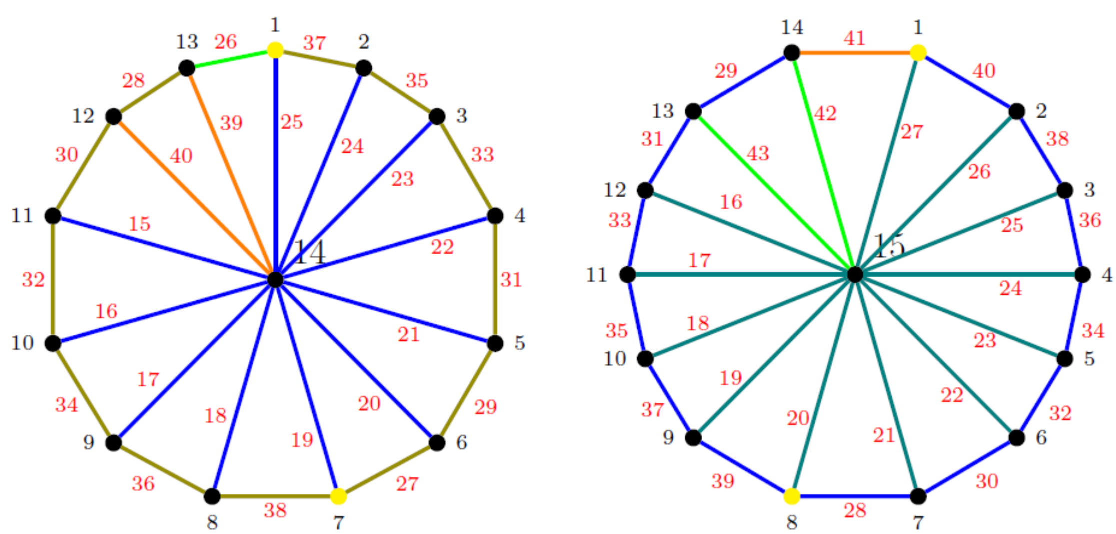

Figure 1: (a) W_{13} with $k_1=40$ and $k_2=53$ (b) W_{14} with $k_1=43$ and $k_2=56$

3. Super edge bimagic harmoniums labeling of the splitting graph $S'(C_n)$

Theorem 3.1. For any positive odd integer n, the splitting graph $S'(C_n), n\geq 3$ is super edge bimagic harmonious labeling graph with bimagic harmonious numbers $k_1=5n\ and\ k_2=6n$.

Proof. The splitting graph $S'(C_n), n \geq 3$ has vertex set $V[S'(C_n)] = \{x_i, y_i, i \in [1,n]\}$ and its edges are $E[S'(C_n)] = \{x_i x_{i+1}, x_i y_{i+1}, y x_{i+1}, i \in [1, n-1]\} \cup \{x_n x_1, y_n x_1, x_n y_1\}$. In the graph $S'(C_n)$, we have $p = 2n$ vertices and $q = 3n$ edges.

When $n \geq 3$, is odd integer, we define the labeling function $\boldsymbol{\Psi}_2 : [(V \cup E)(S'(C_n))] \to \{1, 2, 3, \ldots, 5n\}$ as follows:

$$\boldsymbol{\Psi}_2(x_i) = n + i, \quad i \in [1,n],$$

$$\boldsymbol{\Psi}_2(y_i) = i, \quad i \in [1,n],$$

$$\boldsymbol{\Psi}_2(x_i x_{i+1}) = \begin{cases} 3n - 2i - 1, & if \quad i \in \left[1, \frac{n-3}{2}\right]; \\ 6n - 2i - 1, & if \quad i \in \left[\frac{n-1}{2}, n-1\right]; \\ 5n - 1, & if \quad i = n; \end{cases}$$

$$\boldsymbol{\Psi}_2(x_i y_{i+1}) = \begin{cases} 5n - 2i - 1, & if \quad i \in [1, n-1]; \\ 4n - 1, & if \quad i = n; \end{cases}$$

$$\boldsymbol{\Psi}_2(y_i x_{i+1}) = \begin{cases} 4n - 2i - 1, & if \quad i \in [1, n-1]; \\ 3n - 1, & if \quad i = n. \end{cases}$$

To demonstrate that Ψ_2 is a super edge bimagic harmonious labeling: -

For the edges $x_i x_{i+1}$, $1 \leq i \leq \frac{n-3}{2}$

$$[(\boldsymbol{\Psi}_2(x_i) + \boldsymbol{\Psi}_2(x_{i+1}))\, mod(q) + \boldsymbol{\Psi}_2(x_i x_{i+1})] = [(2n + 2i + 1)\, mod(3n) + 3n - 2i - 1] = 5n = K_1.$$

For the edges $x_i x_{i+1}$, $\frac{n-1}{2} \leq i \leq n - 1$

$$\left[(\boldsymbol{\Psi}_2(x_i) + \boldsymbol{\Psi}_2(x_{i+1}))mod(q) + \boldsymbol{\Psi}_2(x_i x_{i+1})\right] = [(2n + 2i + 1)\, mod(3n) + 6n - 2i - 1] = 5n = K_1.$$

For the edges $x_n x_1$

$$\left[(\boldsymbol{\Psi}_2(x_n) + \boldsymbol{\Psi}_2(x_1))mod(q) + \boldsymbol{\Psi}_2(x_n x_1)\right] = [(3n + 1)mod(3n) + 5n - 1] = 5n = K_1.$$

For the edges $y_i x_{i+1}$, $1 \leq i \leq n - 1$

$$\left[(\boldsymbol{\Psi}_2(x_i) + \boldsymbol{\Psi}_2(x_{i+1}))mod(q) + \boldsymbol{\Psi}_2(y_i x_{i+1})\right] = [(n + 2i + 1)\, mod(3n) + 4n - 2i - 1] = 5n = K_1.$$

For the edges $y_n x_1$

$$\left[(\boldsymbol{\Psi}_2(y_n) + \boldsymbol{\Psi}_2(x_1))mod(q) + \boldsymbol{\Psi}_2(y_n x_1)\right] = [(2n + 1)mod(3n) + 3n - 1] = 5n = K_1.$$

For the edges $x_i y_{i+1}$, $1 \leq i \leq n$

$$\left[(\boldsymbol{\Psi}_2(x_i) + \boldsymbol{\Psi}_2(y_{i+1}))mod(q) + \boldsymbol{\Psi}_2(x_i y_{i+1})\right] = [(n + 2i + 1)\, mod(3n) + 5n - 2i - 1] = 6n = K_2.$$

For the edges $y_1 x_n$

$$\left[(\boldsymbol{\Psi}_2(y_1) + \boldsymbol{\Psi}_2(x_n))mod(q) + \boldsymbol{\Psi}_2(y_1 x_n)\right] = [(2n + 1)mod(3n) + 4n - 1] = 6n = K_2.$$

Obviously, the edge labels are all distinct and for each edge $xy \in E\left(S'(C_9)\right)$, the value of the formula $[(\Psi_2(x) + \Psi_2(y)) \, mod(q) + \Psi_2(xy)]$ provides either of the magic constants $k_1 = 5n$ or $k_2 = 6n$ when n is odd. Therefore, a super edge bimagic harmonious labelling is allowed by the splitting graph $S'(C_n)$.

Example 3.2. In Fig. 2, we present the splitting graph $S'(C_9)$ with an edge bimagic harmonious labelling and bimagic constants $k_1 = 55, and\ k_2 = 64$.

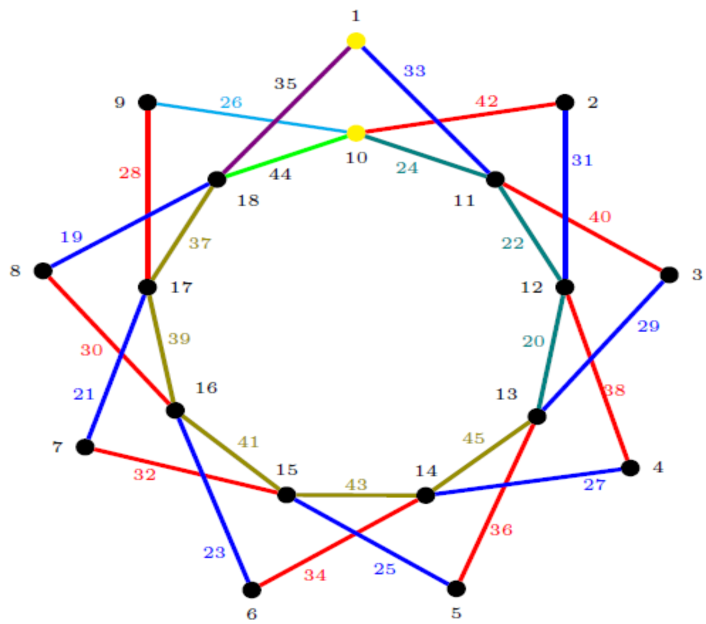

Figure 2: (a) The splitting graph $S'(C_9)$ with bimagic harmonious constants $k_1 = 55, and\ k_2 = 64$

4. Super edge trimagic harmonious labeling of the graph SF_n

Definition 4.1. For a vertex $x \in V(\Gamma)$, the open neighborhood set $N(x)$ is the set of vertices which are adjacent to x in Γ. Duplication of an edge $e = x_i x_{i+1}$ in a graph Γ by a new vertex y_i produces a new graph Γ' such that $N(y_i) = \{x_i, x_{i+1}\}$. The sunflower graph SF_n, is defined as a graph obtained by starting with an n- cycle $C_n = \{x_1, x_2, \dots, x_n\}$ and duplicating every edge by new vertices $\{y_1, y_2, \dots, y_n\}$, with y_i connected to x_i and x_{i+1}, so

$$E(SF_n) = \{x_i y_i, \quad y_i x_{i+1}, \quad x_i x_{i+1}, \quad i \in [1, n]\} \cup \{ y_n x_1, \ x_n x_1\}.$$

Theorem 4.2.

(i) The sunflower graph SF_n is super edge trimagic harmonious graph with trimagic harmonious numbers $k_1 = 5n, k_2 = 7n\ and\ k_3 = 6n$ when $n \geq 3$ is odd integer.

(ii) The sunflower graph SF_n is super edge trimagic harmonious graph with trimagic harmonious numbers $k_1 = 5n, k_2 = 7n$ and $k_3 = 6n - 1$ when n is even integer.

Proof. The sunflower graph SF_n has $p = 2n$ vertices and $q = 3n$ edges.

Case (1): When $n \geq 3$ is odd integer, define the labeling function $\Psi_3 : [(V \cup E)(SF_n)] \rightarrow \{1, 2, 3, \dots, 5n\}$ as follows:

$$\Psi_3(x_i) = i, \qquad i \in [1, n],$$

$$\mathbf{\Psi_3}(y_i) = n + i, \qquad i \in [1,n],$$

$$\mathbf{\Psi_3}(y_i x_{i+1}) = \begin{cases} 4n - 2i - 1, & if \quad i \in [1, n-1]; \\ 5n - 1, & if \quad i = n; \end{cases}$$

$$\mathbf{\Psi_3}(x_i x_{i+1}) = \begin{cases} 5n - 2i - 1, & if \quad i \in [1, n-1]; \\ 4n - 1, & if \quad i = n; \end{cases}$$

$$\mathbf{\Psi_3}(x_i y_i) = \begin{cases} 5n - 2i, & if \quad i \in \left[1, \frac{n-1}{2}\right]; \\ 4n - 2i, & if \quad i \in \left[\frac{n+1}{2}, n-1\right]; \\ 5n, & if \quad i = n. \end{cases}$$

To demonstrate that $\mathbf{\Psi_3}$ is a super edge trimagic harmonious labeling: -

For the edges $y_i x_{i+1}$, $\quad i \in [1, n - 1]$,

$$[(\mathbf{\Psi_3}(y_i) + \mathbf{\Psi_3}(x_{i+1}))mod(q) + \mathbf{\Psi_3}(y_i x_{i+1})] = [(n + i + i + 1)mod(3n) + 4n - 2i - 1] = 5n = K_1.$$

For the edges $y_n x_1$,

$$[(\mathbf{\Psi_3}(y_n) + \mathbf{\Psi_3}(x_1))mod(q) + \mathbf{\Psi_3}(y_n x_1)] = [(2n + 1)mod(3n) + 5n - 1] = 7n = K_2.$$

For the edges $x_i x_{i+1}$, $\quad i \in [1, n - 1]$

$$[(\mathbf{\Psi_3}(x_i) + \mathbf{\Psi_3}(x_{i+1}))\, mod(q) + \mathbf{\Psi_3}(x_i x_{i+1})] = [(i + i + 1)\, mod(3n) + 5n - 2i - 1] = 5n = K_1.$$

For the edges $x_1 x_n$

$$[(\mathbf{\Psi_3}(x_1) + \mathbf{\Psi_3}(x_n))mod(q) + \mathbf{\Psi_3}(x_1 x_n)] = [(1 + n)mod(3n) + 4n - 1] = 5n = K_1.$$

For the edges $x_i y_i$, $\quad i \in [1, \frac{n-1}{2}]$,

$$[(\mathbf{\Psi_3}(x_i) + \mathbf{\Psi_3}(y_i))\, mod(q) + \mathbf{\Psi_3}(x_i y_i)] = [(i + n + i)\, mod(3n) + 5n - 2i] = 6n = K_3.$$

For the edges $x_i y_i$, $\quad i \in [\frac{n+1}{2}, n - 1]$,

$$[(\mathbf{\Psi_3}(x_i) + \mathbf{\Psi_3}(y_i))mod(q) + \mathbf{\Psi_3}(x_i y_i)] = [(i + n + i)mod(3n) + 4n - 2i] = 5n = K_1.$$

For the edge $x_n y_n$

$$[(\mathbf{\Psi_3}(x_i) + \mathbf{\Psi_3}(y_i))\, mod(q) + \mathbf{\Psi_3}(x_i y_i)] = [(n + 2n)\, mod(3n) + 5n] = 5n = K_1.$$

Case (2) : When $n \geq 4$ is even integer. The labeling function $\boldsymbol{\Psi}_3$ defined as in case (1) but with some modifications which are given by

$$\boldsymbol{\Psi}_3(x_i x_{i+1}) = \begin{cases} 5n - 2i - 1, & if \quad i \in \left[1, \frac{n}{2}\right]; \\ 6n - 2i - 2, & if \quad i \in \left[\frac{n}{2} + 1, n - 1\right]; \\ 5n - 2 & if \qquad i = n; \end{cases}$$

$$\boldsymbol{\Psi}_3(x_i y_i) = \begin{cases} 4n - 2i, & if \quad i \in [1, n-1]; \\ 5n, & if \qquad i = n. \end{cases}$$

To demonstrate that $\boldsymbol{\Psi}_3$ is a super edge trimagic harmonious labeling: -

For the edges $x_i x_{i+1}$, $\quad i \in [1, \frac{n}{2}]$,

$$\left[\left(\boldsymbol{\Psi}_3(x_i) + \boldsymbol{\Psi}_3(x_{i+1})\right) mod(q) + \boldsymbol{\Psi}_3(x_i x_{i+1})\right] = [(i + i + 1)\, mod(3n) + 5n - 2i - 1] = 5n = K_1.$$

For the edges $x_i x_{i+1}$, $\quad i \in [\frac{n}{2} + 1, n - 1]$,

$$\left[\left(\boldsymbol{\Psi}_3(x_i) + \boldsymbol{\Psi}_3(x_{i+1})\right) mod(q) + \boldsymbol{\Psi}_3(x_i x_{i+1})\right] = [(i + i + 1)\, mod(3n) + 6n - 2i - 2] = 6n - 1 = K_3.$$

For the edges $x_1 x_n$

$$\left[\left(\boldsymbol{\Psi}_3(x_1) + \boldsymbol{\Psi}_3(x_n)\right) mod(q) + \boldsymbol{\Psi}_3(x_1 x_n)\right] = [(1 + n) mod(3n) + 5n - 2] = 6n - 1 = K_3.$$

For the edges $x_i y_i$, $\quad i \in [1, n - 1]$,

$$\left[\left(\boldsymbol{\Psi}_3(x_i) + \boldsymbol{\Psi}_3(y_i)\right) mod(q) + \boldsymbol{\Psi}_3(x_i y_i)\right] = [(i + n + i) mod(3n) + 4n - 2i] = 5n = K_1.$$

For the edge $x_n y_n$

$$[(\boldsymbol{\Psi}_3(x_n) + \boldsymbol{\Psi}_3(y_n))\, mod(q) + \boldsymbol{\Psi}_3(x_n y_n)] = [(n + 2n)\, mod(3n) + 5n] = 5n = K_1.$$

Therefore, an edge trimagic harmonious labelling for all n is allowed by the graph $\boldsymbol{SF_n}$.

Example 4.3. In Fig. 3, we present $\boldsymbol{SF_9}$ with an edge trimagic harmonious labelling and trimagic constants $k_1 = 45$, $k_2 = 63$ and $k_3 = 54$, and $\boldsymbol{SF_{10}}$ with the trimagic harmonious constants are $k_1 = 50$, $k_2 = 70$ and $k_3 = 59$.

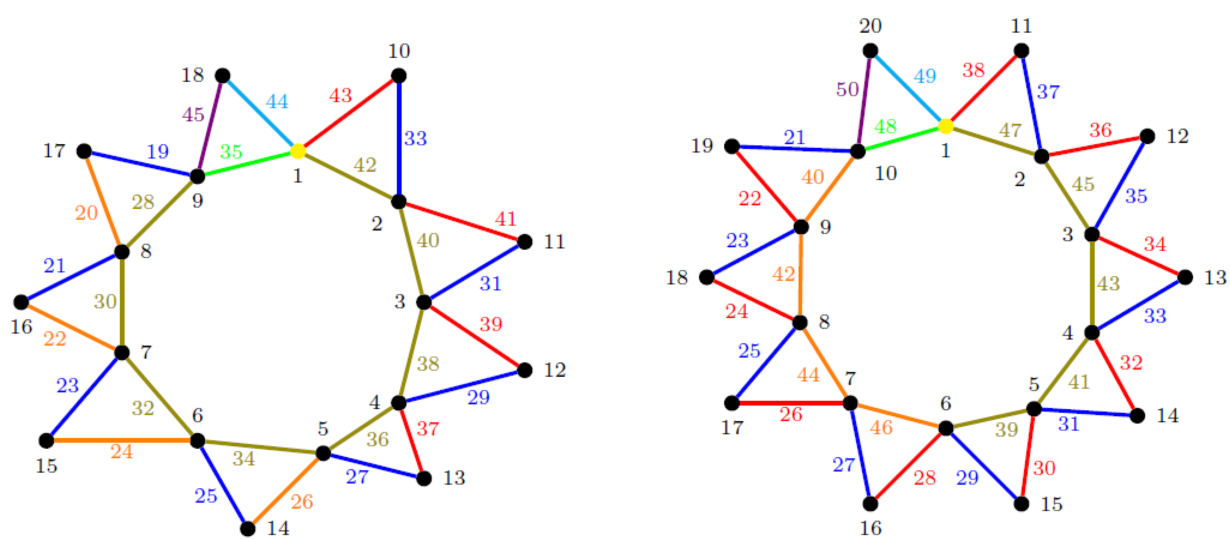

Figure 3: (a) SF_9 with $k_1 = 45$, $k_2 = 63$ and $k_3 = 54$ (b) SF_{10} with $k_1 = 50$, $k_2 = 70$ and $k_3 = 59$

5. Super edge trimagic harmonious labeling of the double sunflower graph

Definition 5.1. A double sunflower graph of order n, represented by DSF_n, is a graph that is obtained from the sunflower graph SF_n by inserting a new vertex u_i on each edge $x_i x_{i+1}$on the rim of the cycle and adding edges $y_i u_i$ for each $i \in [1,n]$.

Theorem 5.2. For every integer $n \geq 3$, the double sunflower graph DSF_n is super edge trimagic harmonious graph with trimagic numbers $k_1 = 8n,\ k_2 = 10n - 1, and\ k_3 = 11n$.

Proof. The double sun flower graph DSF_n has vertex set $\{ x_i, u_i, y_i, i \in [1,n]\}$ and edge set $E[DSF_n] = \{ x_i u_i, x_i y_i, u_i y_i, i \in [1,n]\} \cup \{u_i x_{i+1}, y_i x_{i+1}, i \in [1, n-1]\} \cup \{u_n x_1, y_n x_1\}$. So, $p = 3n$ vertices and $q = 5n$ edges.

Defined the labeling function $\mathbf{\Psi_4} : [(V \cup E)(DSF_n)] \to \{1,2,3, \ldots, 8n\}$ as follows:

$$\mathbf{\Psi_4}(x_i) = 2i - 1, \qquad i \in [1,n],$$
$$\mathbf{\Psi_4}(u_i) = 2i, \qquad i \in [1,n],$$
$$\mathbf{\Psi_4}(y_i) = 2n + i, \qquad i \in [1,n],$$
$$\mathbf{\Psi_4}(x_i y_i) = 6n - 3i + 1, \qquad i \in [1,n],$$

$$\mathbf{\Psi_4}(y_i x_{i+1}) = \begin{cases} 6n - 3i - 1, & if \quad i \in [1, n-1]; \\ 8n - 1, & if \quad i = n; \end{cases}$$

$$\mathbf{\Psi_4}(y_i u_i) = \begin{cases} 6n - 3i, & if \quad i \in [1, n-1]; \\ 8n, & if \quad i = n, \end{cases}$$

$$\mathbf{\Psi_4}(x_i u_i) = \begin{cases} 8n - 4i + 1, & for \quad i \in \left[1, \frac{n}{2}\right], n\ is\ even; \\ & or \quad i \in \left[1, \frac{n+1}{2}\right], n\ is\ odd; \\ 10n - 4i, & for \quad i \in \left[\frac{n}{2} + 1, n\right], n\ is\ even; \\ & or \quad i \in \left[\frac{n+3}{2}, n\right], n\ is\ odd, \end{cases}$$

$$\mathbf{\Psi_4}(u_i x_{i+1}) = \begin{cases} 8n - 4i - 1, & for \quad i \in \left[1, \frac{n}{2}\right], n\ is\ even; \\ & or \quad i \in \left[1, \frac{n-1}{2}\right], n\ is\ odd; \\ 10n - 4i, & for \quad i \in \left[\frac{n}{2} + 1, n - 1\right], n\ is\ even; \\ & or \quad i \in \left[\frac{n+1}{2}, n - 1\right], n\ is\ odd; \\ 8n - 2, & for \quad i = n. \end{cases}$$

To demonstrate that Ψ_3 is a super edge trimagic harmonious labeling: -

For the edges $x_i y_i$, $\quad i \in [1,n]$

$$[(\mathbf{\Psi_4}(x_i) + \mathbf{\Psi_4}(y_i)) mod(q) + \mathbf{\Psi_4}(x_i y_i)] = [(2i - 1 + 2n + i) mod(5n) + 6n - 3i + 1] = 8n = K_1.$$

For the edges $y_n x_1$

$$[(\mathbf{\Psi_4}(y) + \mathbf{\Psi_4}(x_1)) mod(q) + \mathbf{\Psi_4}(y_n x_1)] = [(3n + 1) mod(5n) + 8n - 1] = 11n = K_2.$$

For the edges $x_i u_i$, $i \in \left[\frac{n}{2} + 1, n\right]$, $if\ n\ is\ even, and\ i \in [\frac{n+3}{2}, n]$, $if\ n\ is\ odd$

$$[(\Psi_4(u_i) + \Psi_4(x_i))mod(q) + \Psi_4(xu_i)] = [(2i - 1 + 2i)mod(5n) + 10n - 4i] = 10n - 1 = K_3.$$

It is clear that, for each edge $xy \in E(DSF_n)$, the value of $[(\Psi_4(x) + \Psi_4(y))\ mod(q) + \Psi_4(xy)]$ provides either of the trimagic constants $k_1 = 8n$, $k_2 = 10n - 1$, or $k_3 = 11n$. Therefore, a super edge trimagic harmonious labelling for all n is allowed by the graph DSF_n.

Example 5.3. In Fig. 4, we present the double sunflower graph DSF_7 with bimagic harmonious constants k_1 = 56, k_2 = 69, and k_3= 77 and DSF_8 with an edge bimagic harmonious labelling and bimagic constants k_1= 64, k_2 = 79, and k_3 = 88.

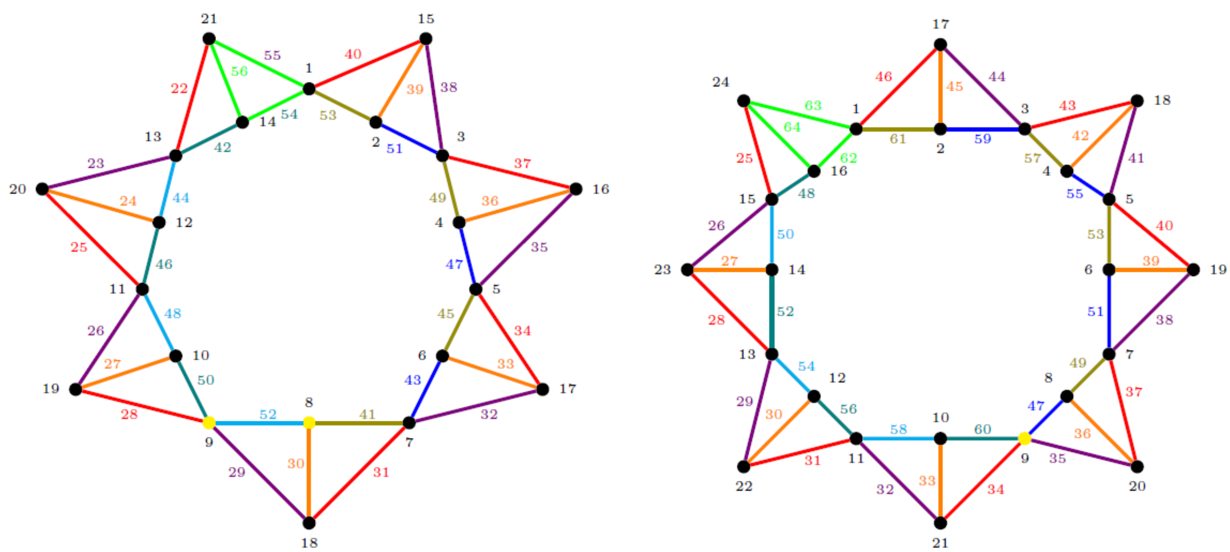

Figure 4: (a) DSF_7 with $k_1 = 56$, $k_2 = 69$ and $k_3 = 77$ (b) DSF_8 with $k_1 = 64$, $k_2 = 79$ and $k_3 = 88$

Conclusion

In the past few years, edge labeling of graphs has been studied heavily and this topics continue to be attractive in the field of graph theory and discrete mathematics. So far, many graphs are unknown if it is super edge harmonious or not. In this work, we prove that the wheel graph W_n and the splitting graph of odd cycle are super edge bimagic harmonious graphs.

Furthermore, we prove that the sunflower graph and the double sunflower graph are super edge trimagic harmonious graphs. In the following tables we summaries our results: -

Graph	p	q	k_1	k_2	k_3	Remark
Wheel graph W_n	$n+1$	$2n$	$3n+1$	$4n$	---	n even
Wheel graph W_n	$n+1$	$2n$	$3n+1$	$4n+1$	---	n odd
Splitting graph $S^{'}$ (C_n)	$2n$	$3n$	$5n$	$6n$	---	n odd
Sunflower graph SF_n	$2n$	$3n$	$5n$	$7n$	$6n$	n odd
Sunflower graph SF_n	$2n$	$3n$	$5n$	$7n$	$6n-1$	n even
Double sunflower graph DSF_n	$3n$	$5n$	$8n$	$10n-1$	$11n$	————

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